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Discrepancies in the Distribution of Prime Numbers*

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Chebyshev has noticed a certain predominance of primes of the form $4n+3$ over those of the form $4n+1$. He asserted that $\lim_{x\to\infty} \sum_{p>2} (-1)^{(p-1)/2} e^{-p/x} = -\infty$. This was unproven until today. G. H. Hardy, J. E. Littlewood and E. Landau have shown its equivalence with an analogue to the famous Riemann hypothesis, namely, $L(s, \chi_1 \bmod 4) \neq 0$, $\mathrm{Re}(s) > \frac{1}{2}$. S. Knapowski and P. Turán have given some similar (unproven) relations, e.g., $\lim_{x\to\infty} \sum_{p>2} (-1)^{(p-1)/2} \log p e^{-\log^2(p/x)} = -\infty$, which are also equivalent to the above. Using Explixit Formulas the author shows that

$$\lim_{x\to\infty} \sum_{p>2} (-1)^{(p-1)/2} \log p p^{-1/2} e^{-(\log^2 p)/x} = -\infty \qquad (*)$$

holds without any conjecture. (In addition, the order of magnitude of divergence is calculated.) It turns out that $(*)$ is only a special case (in several respects). At first, it may be enlarged into

$$\lim_{x\to\infty} \sum_{p>2} (-1)^{(p-1)/2} \log p p^{-\alpha} e^{-(\log^2 p)/x} = -\infty, \qquad 0 \leqslant \alpha \leqslant \tfrac{1}{2}.$$

Then, it can be generalised to a wider class of progressions. For example, the same is true if one sums over the primes in the classes $3n+2$ and $3n+1$, with a "$-$" and a "$+$" sign, respectively. All results of this type depend on the location of the first nontrivial zero of the corresponding L-series. D. Shanks has given some arguments for the predominance of primes in residue classes of nonquadratic type. He conjectured "If $m_1 \bmod k$ is a quadratic residue and $m_2 \bmod k$ a non-residue, then there are "more" primes congruent m_2 than congruent $m_1 \bmod k$." This indeed turns out to be true in the sense of $(*)$, not only for $k = 3, 4$, but for some higher moduli as well. Finally, numerical calculations were made to investigate the behaviour of $\Delta_3(X) := \pi(X, 2 \bmod 3) - \pi(X, 1 \bmod 3)$ in the interval $2 \leqslant X \leqslant 18, 633, 261$. No zero was found in this range. In the analogue case of $\Delta_4(X) := \pi(X, 3 \bmod 4) - \pi(X, 1 \bmod 4)$ the first sign change occurs at $X = 26, 861$.

1. PRELIMINARIES

Dirichlet's famous theorem states that every arithmetic progression $am + b$, $(a, b) = 1$, contains an infinitude of prime numbers. As usual, let

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$\pi(x)$ denote the number of primes up to x, and $\pi\ (x, b \bmod a)$ that part of it which lies in $am + b$, and ϕ, Euler's totient function. Then we have a more precise formulation of the above statement:

$$\pi(x, b \bmod a) \sim \frac{1}{\phi(a)} \pi(x) \sim \frac{1}{\phi(a)} \cdot \frac{x}{\log x}. \tag{1}$$

From this we see that the primes are "equally distributed" over the classes $b \bmod a$ in the given asymptotic sense. The error term in (1) permits discrepancies of the primes in the different classes.

Chebyshev has noticed some kind of discrepancy in the special case $a = 4$. In a letter written by Kronecker, Kummer and Weierstrass we find the following remark: "Endlich ist Herr Tschebyschef der erste Mathematiker, welcher für die Anzahl der Primzahlen bis zu einer hohen Grenze den Überschuß der Primzahlen der Form $4n + 3$ über diejenigen von der Form $4n + 1$ konstatiert und für den den asymptotischen Ausdruck $\sqrt{x}/\log x$ angegeben hat" [4].

Nowadays we have numerical data up to 3,000,000, which affirm Chebyshev's statement. The "naive" conjecture

$$\Delta(x) := \pi(x, 3 \bmod 4) - \pi(x, 1 \bmod 4) \to \infty \qquad \text{if} \quad x \to \infty$$

does not hold, as Hardy and Littlewood have shown. In 1917 they proved [7]

$$\lim_{x \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \Delta(x) = \pm \infty. \tag{2}$$

Now, Chebeshev's original statement is much more subtle and was undecided until today. It is

$$\lim_{x \to \infty} \sum_{p > 2} (-1)^{\frac{p-1}{2}} \cdot e^{-p/x} = -\infty. \tag{3}$$

Here the primes $4n + 1$ give the "+" sign and the other odd primes give the "−" sign.

Landau did not believe that Chebyshev had a (correct) proof of (3). One reason for his doubt is the fact that the truth of (3) would imply

$$L(s, \chi_1 \bmod 4) \neq 0, \qquad \operatorname{Re}(s) > \tfrac{1}{2}, \tag{4}$$

χ_1 nonprincipal character mod 4 [13]. But (4) has the same depth as the famous Riemann hypothesis

$$\zeta(s) \neq 0, \qquad \operatorname{Re}(s) > \tfrac{1}{2}, \tag{5}$$

"und dies erhöht für Ungläubige die Wahrscheinlichkeit, daß Tschebyschef sich geirrt hat, und für Gläubige den Wunsch, aus seinen Papieren den Beweis von (1) rekonstruiert zu sehen"— as Landau has noticed [13]. On the other hand Hardy and Littlewood have proved that (3) will follow from (4), so the statements (3) and (4) are fully equivalent. Actually, none of it is known to be true.

In a series of 15 papers [11, 12] Knapowski and Turán have investigated a number of questions concerning discrepancies and have considerably enlarged our knowledge in a field of prime number theory which we call "comparative prime number theory" since then. Especially in the Chebyshevian case these two mathematicians have found that in addition to (3)

$$\lim_{x \to \infty} \sum_{p>2} (-1)^{\frac{p-1}{2}} \log p \cdot e^{-\log^2(p/x)} = -\infty \tag{6}$$

and

$$\lim_{x \to \infty} \sum_{p>2} (-1)^{\frac{p-2}{2}} \cdot e^{-\log^2(p/x)} = -\infty \tag{7}$$

are equivalent to (4), too. The proof of (7) $\Leftrightarrow$ (4) was announced by Turán [10, 18], but has never been published.

In the analogue case Hardy and Littlewood [7] removed the "$\log p$" factor from

$$\lim_{x \to \infty} \sum_{p>2} (-1)^{\frac{p-1}{2}} \log p \cdot e^{-p/x} = -\infty \tag{8}$$

to reach (3) without any new information about the zeros of $L(s, \chi_1 \bmod 4)$.

So, up to know, there exist some interesting but unproved conjectures of type (3).

Numerical calculations by electronic computers could give a better insight into those irregularities. The present tables show that the first sign change of $\Delta(x)$ occurs at $x = 26, 861$, so we have

$$\Delta(26, 861) = -1 \qquad [14, 16].$$

Extended calculations by Shanks give a very good insight into the deviations of $\Delta(x)$ in the range from 1 to $3 \cdot 10^6$, see [16]. In that range Δ varies between is maximum, 256, and its minimum, −8, but a certain preponderance to the positive side is obvious: The calculations show that $\Delta(n) > 0$ for 99.84% of the $n \leqslant 3 \cdot 10^6$.

Shanks notices: "This detailed description makes it highly plausible that the predominantly positive character of $\Delta(n)$ in this range of n is not merely a passing fancy of the integers ⋯ but a permanent phenomenon for which a sufficient number-theoretic cause should be assigned and of which a more precise formulation is desirable."

One kind of number-theoretic cause is given by Shanks, but his arguments are more heuristic. However, there seems to be a connection between the above kind of irregularities of the primes and the "quadratic" and "non-quadratic" residue classes mod k. More precisely, the numerical calculations confirm the following conjecture (Shanks):

"If l_1 mod k is a quadratic residue and l_2 mod k a non-quadratic residue, then we have "more" primes in l_2 mod k than in l_1 mod k."[1]

It seemed to be very difficult to give statements about the regularity or irregularity of primes in residue classes which are (both) of quadratic or nonquadratic type. The situation is similar in the case of different types, which do not contain the class 1 mod k, as Knapowski and Turán noticed [12, V]: "We wish to emphasize once more that we have not been able to prove a similar result in case where exactly one of the l_j's is a quadratic residue and none of l_1, l_2 is $\equiv 1 \pmod{k}$. The simplest case in which our present method fails is that of $k = 5$, $l_1 = 2$ (or 3), $l_2 = 4$."

Precisely this will be investigated (as an example) in Section 4. Our method gives a result analogous to Theorem 1 below.

The following chapters are a contribution to both demands expressed by Shanks.

THEOREM 1. *We have*

$$\lim_{x\to\infty} \sum_{p>2} (-1)^{\frac{p-2}{2}} \frac{\log p}{\sqrt{p}} e^{-(\log^2 p)/x} = -\infty. \tag{9}$$

The magnitude of divergence is given by

$$\tfrac{1}{4}\sqrt{\pi x} + O(1).$$

Note 1. This is true without any conjecture!

Note 2. Compare the weight-factor "$e^{-z/x}$" Chebyshev) with that in (9) "$e^{-(\log^2 z)/x}$." Both are of the same quality, i.e., they monotonically decrease to the constant function 1, if x increases to infinity. Looking at (2) the divergence of (9) is surprising, because the factor '$(\log z)/\sqrt{z}$" suggests convergence.

Further investigations give

[1] See the note at the end of this section.

THEOREM 2. *We have without any further conjecture*

$$\lim_{x\to\infty} \sum_{p>2} (-1)^{\frac{p-1}{2}} \frac{\log p}{p^{\alpha}} e^{-(\log^2 p)/x} = -\infty \tag{10}$$

for all α in $0 \leqslant \alpha < \frac{1}{2}$. The magnitude of divergence is given by

$$\sim \tfrac{1}{2} \sqrt{\pi x} \cdot e^{\frac{x}{16}(1-2\alpha)^2}.$$

Theorems 4 and 5 give similar statements for the progressions $3m+1$ and $3m+2$. The question arises how the nontrivial zeros of $L(s, \chi_1 \bmod 4)$ and other L-series influence the discrepancies of the primes. In the Chebyshevian case the "first" zero in the critical strip is

$$\rho_{1,2} = \tfrac{1}{2} \pm i \cdot 6.02 \cdots,$$

and in the case $L(s, \chi_1 \bmod 3)$ we have $\rho_{1/2} = \frac{1}{2} \pm i \cdot 4,....$

These zeros are not too close to the real axis, so the arguments to prove Theorem 1 (and others) work well. In all similar cases only the height of the *first* zero decides whether corresponding theorms hold or not. It is *not* important that the real part equals $\frac{1}{2}$ (Riemann–Piltz conjecture)! The same peculiarity was noticed by Knapowski and Turán. They worked with the so called "*Haselgrove Condition*" [11, 12]. In any case we emphasize that the position of that first zero conceals *one further* number-theoretic cause for the statements (9) and (10).

In sections 2 and 3 we investigate the situation with the modules $k = 4$ and $k = 3$. So all essential cases $\phi(k) = 2$ are treated.

Section 3.2 gives some numerical data for the corresponding $\Delta(n)$ in the range $2 \leqslant n \leqslant 18{,}633{,}261$. We recognize the same behaviour as in the Chebyshevian case, although no sign change occurs within this range. (Even with calculations up to 3.5×10^7 a sign change was not found.)

The situation in the cases $k = 3, 4$ is comparatively clear as all characters are real. The next size of moduli $k = 5$ and $k = 8$, is treated in Section 4, so we have dealt with all essential cases $\phi(k) = 4$. The nonreal characters in the case $k = 5$ render the investigations more difficult. Nevertheless, we get results similar to Theorems 1 and 2. The case $k = 8$, however, is quite analogous to the case $k = 4$. The investigations show that in general the situation becomes more and more complicated with greater k. On the one hand this comes from the nonreal characters, and on the other hand we do not know the position of the first zero in the critical strip, besides some exeptions of lower k's.

Let us come back to Shanks' remarks concerning the predominantly positive character of $\Delta(n)$. It is clear that Theorems 1 and 2 give at least a more precise formulation of the preponderance of the primes in $4m+3$ over those in $4m+1$ (and analogue cases). As far as I know there exits no other statement of this kind which holds without any (unproven) conjecture.

Note. With regard to Shanks' conjecture concerning the quadratic residues and nonresidues, we refer to the paper "Quadratic Residues and the Distribution of Prime Numbers," by the author and J. Pintz. In this paper we affirm Shanks' conjecture in the sense of Theorem 1. In the general case this remains true under a hypothesis on the nontrivial zeros of the L-series, which is weaker than the Riemann–Piltz hypothesis [1].

2. Chebyshev's Case

The equality (9),

$$\lim_{x\to\infty} \sum_{p>2} (-1)^{\frac{p-1}{2}} \frac{\log p}{\sqrt{p}} e^{-(\log^2 p)/x} = -\infty,$$

is the final form of a special Explixit Formula. This formula is to be derived first. From [2] we take the general Explicit Formula:

$$\begin{aligned}\lim_{T\to\infty} &\sum_{\substack{\rho=\sigma+i\gamma\\ |\gamma|<T}} M(\rho)\\ &= \varepsilon_0\{M(0)+M(1)\} + F(0)\cdot \log f/\pi\\ &\quad - \sum_{p,n} \log p \cdot p^{-n/2}\{\chi(p^n)\cdot F(\log p^n) + \chi(p^{-n})\cdot F(\log p^{-n})\}\\ &\quad - C\cdot F(0) + \textbf{v.p.}\int_{-\infty}^{\infty} \frac{F(x)\cdot e^{(\frac{3}{2}-\delta)|x|} - F(0)}{1-e^{2|x|}}\,dx. \end{aligned} \tag{11}$$

It holds for a special class of functions F and their Mellin-transform M. The sum on the left runs over the zeros ρ of $L(s,\chi)$ in the critical strip, arranged in order of ascending ordinates. On the right we have

$$\begin{aligned} \varepsilon_0 = \varepsilon_0(\chi) &= 1 \quad &&\text{for}\quad \chi=\chi_0, \quad \chi_0 \text{ principal character},\\ &= 0 &&\phantom{\text{for}}\quad \chi\neq\chi_0,\\ \delta = \delta(\chi) &= 1 &&\text{for}\quad \chi(=1) = -1,\\ &= 0 &&\phantom{\text{for}}\quad \chi(-1)=1, \end{aligned}$$

$f=f(\chi)$, conductor of χ, $C=0.577\ldots$, Euler's constant. The (double-) sum runs over the primes p and the natural numbers n, and the abbreviation **v.p.**

indicates the (Hadamard-) principal part of the integral if the function F is discontinuous at zero.

The investigations in Sections 2.1, 2.2 and 2.3 are essentially based on formula (11). They work as paradigms to similar problems and questions. In addition, one more example is treated in Section 3. Historically, it has not had the same attention, but has the same significance as the one in this chapter. Some other (more complicated) examples (Section 4) show which types of conclusion are possible in the general case.

2.1. *The Explixit Formula with* $F(x) = e^{-x^2/4y}$

Insert the function $F(x) = \mathrm{e}^{-x^2/4y}$, with the real parameter y, into (11). In the special case $L(s, \chi) = L(s, \chi_1 \bmod 4)$, we obtain

$$2\sqrt{\pi y} \sum_{\rho(\chi_1)}{}^{*} \mathrm{e}^{y(\rho - 1/2)^2} = \log\frac{4}{\pi} - 2 \sum_{p^n, p \neq 2} \chi_1(p^n) \cdot \frac{\log p}{\sqrt{p^n}} \cdot \mathrm{e}^{-(\log^2 p^n)/4y} - C + 2\int_0^{\infty} \frac{\mathrm{e}^{-x^2/4y + x/2} - 1}{1 - \mathrm{e}^{2x}}\, dx. \qquad (12)$$

The asterisk means the ordered summation. The present choice of F has given a very simple Explicit Formula, because the terms

$$\chi_1(p^n) \cdot F(\log p^n) \qquad \text{and} \qquad \chi_1(p^{-n}) \cdot F(\log p^{-n})$$

are pairwise identical. Now, we isolate the sum in question (see (9)) and make y large. The other terms will be determined or estimated. To compare with Theorem 1 note that $\chi_1(p)$ equals $(-1)^{\frac{p-1}{2}}$, p odd prime, and is zero if $p = 2$. We get

$$\begin{aligned}\sum_{p>2} (-1)^{\frac{p-1}{2}} \cdot \frac{\log p}{\sqrt{p}} \cdot \mathrm{e}^{-(\log^2 p)/4y} &= -\sqrt{\pi y} \sum_{\rho(\chi_1)}{}^{*} \mathrm{e}^{y(\rho - \frac{1}{2})^2} + \frac{1}{2}\log\frac{4}{\pi} - \frac{1}{2}C + \int_0^{\infty} \frac{\mathrm{e}^{-x^2/4y + x/2} - 1}{1 - \mathrm{e}^{2x}}\, dx \\ &\quad - \sum_{p^n, p>2, n \geqslant 2} \chi_1(p^n) \cdot \frac{\log p}{\sqrt{pn}} \cdot \mathrm{e}^{-(\log^2 p^n)/4y}. \end{aligned} \qquad (13)$$

2.1.1. *The behaviour of* $\sum^*$ *if* y *is large.* We use the following well-known facts about the zeros ρ of $L(s, \chi_1 \bmod 4)$ in the critical strip:

(i) The first zero $\rho = \sigma + i\gamma$, $0 < \sigma < 1$, has the ordinate

$$\gamma = 6.02... \qquad [6, 17].$$

(ii) The number of zeros of $L(s, \chi_1)$ in the rectangle $0 < \sigma < 1$, $T \leqslant |\gamma| \leqslant T + 1$ is of order

$$O(\log T) \qquad \text{if} \quad T \to \infty \quad [15].$$

With this we obtain

LEMMA 1.

$$\lim_{y \to \infty} \sqrt{\pi y} \sum_{\rho(\chi_1)}{}^{*} e^{y(\rho - 1/2)^2} = 0.$$

The fact (ii) generally holds for the Dirichlet series $L(s, \chi)$, so Lemma 1 always holds (in an analogue sense), if the "first" zero is far enough away from the real axis. (Note Section 5.)

2.1.2. *The behaviour of the integral.* In this case the integral is particularly simple, and we get

LEMMA 2.

$$\lim_{y \to \infty} \int_0^{\infty} \frac{e^{-x/4y + x/2} - 1}{1 - e^{2x}} \, dx = O(1) \qquad (= -0.25432...).$$

2.1.3. *The higher prime powers.* Use

$$\sum_{p, m} \frac{\log p}{p^{ms}} = -\frac{\zeta'}{\zeta}(s) < \infty, \qquad \text{for } \mathrm{Re}(s) > 1 \quad [9],$$

to get

LEMMA 3.

$$\lim_{y \to \infty} \left| \sum_{\substack{p^n, p > 2 \\ n \geqslant 3}} \chi_1(p^n) \cdot \frac{\log p}{\sqrt{p^n}} \cdot e^{-(\log^2 p^n)/4y} \right| \leqslant -3 \frac{\zeta'}{\zeta}\left(\frac{3}{2}\right) < \infty.$$

This leaves the squares of the primes to be examined. With $\sum_{p<x} \log p/p = \log x + O(1)$, for large x [15], we get

LEMMA 4.

$$\lim_{y \to \infty} \sum_{p > 2} \log p/p \cdot e^{-(\log^2 p)/y} = +\infty. \tag{14}$$

Lemmas 1 to 4 and (13) immediately give the first part of

THEOREM 1. *We have*

$$\lim_{x\to\infty} \sum_{p>2} (-1)^{\frac{p-1}{2}} \log p/\sqrt{p} \cdot e^{-(\log^2 p)/x} = -\infty.$$

The magnitude of divergence is given by

$$\tfrac{1}{4}\sqrt{\pi x} + O(1).$$

2.1.4. *The exact order of the series in Theorem* 1. The second part of Theorem 1 is proved by partial summation (e.g., [9, Theorem A]), which gives [5]

LEMMA 5.

$$\lim_{y\to\infty} \sum_{p>2} \log p/p \cdot e^{-(\log^2 p)/y} = \tfrac{1}{2}\sqrt{\pi y} + O(1).$$

For further details of the proofs of Lemmas 1–5 see [3].

2.2. *An Explicit Formula with* $F(x) = e^{-x^2/4y + (\frac{1}{2} - \alpha)x}$

The function $F(x) = e^{-x^2/4y}$ has given a very simple Explixit Formula. We now change this function F to the function

$$F(x) = e^{-x^2/4y + (\frac{1}{2} - \alpha)x}$$

with a new parameter α. This essentially gives an additional p-term in the Explicit Formula, which is weak enough to conserve the statements of the lemmas and of Theorem 1. But it is interesting to observe its effect on the order of divergence of the series in (9). We are still looking at the case $L(s, \chi_1 \bmod 4)$. The function

$$F(x) = e^{-x^2/4y + (\frac{1}{2} - \alpha)x}$$

is defined on the whole real axis, the parameters are $y \geqslant 0$, real, and α. The latter is at first arbitrarily real. From (11) we get

$$\begin{aligned} &\sum_{\rho(\chi_1)}{}^{*} 2\sqrt{\pi y}\, e^{y(\rho-\alpha)^2} \\ &\quad = \log 4/\pi - \sum_{p,n} \log p \cdot p^{-n\alpha} \cdot e^{-(\log^2 p^n)/4y} \cdot \chi_1(p^n) \\ &\quad\quad - \sum_{p,n} \log p \cdot p^{-n(1-\alpha)} \cdot e^{-(\log^2 p^n)/4y} \cdot \chi_1(p^{-n}) \\ &\quad\quad -C + \int_0^\infty \frac{e^{-x^2/4y + \alpha x} - 1}{1 - e^{2x}}\, dx \\ &\quad\quad + \int_0^\infty \frac{e^{-x^2/4y + (1-\alpha)x} - 1}{1 - e^{2x}}\, dx. \end{aligned} \tag{15}$$

Note. If $\alpha = \frac{1}{2}$ some sums and integrals coincide and we have the simpler formula (12) from Section 2.1. We again investigate (15) with some sufficiently large y. For the sake of symmetry now the parameter α shall be within the range

$$0 \leqslant \alpha < \tfrac{1}{2}.$$

For, our formula (15) remains invariant by substituting

$$\alpha \to 1 - \alpha,$$

because the zero $\rho_a = \sigma + i\gamma_a$ always has a "twin,"

$$1 - \bar{\rho}_a = 1 - \sigma + i\gamma_a, \qquad \text{if} \quad \sigma \neq \tfrac{1}{2}.$$

In addition, if χ is purely real, then

$$\bar{\rho}_b = 1 - \sigma + i\gamma_a \qquad \text{and} \qquad \bar{\rho}_b = 1 - \sigma - i\gamma_a$$

are also zeros of $L(s, \chi)$.

2.2.1. *The behaviour of* $\sum^*$.

LEMMA 6. *Let* $-5 \leqslant \alpha \leqslant 6$, *then*

$$\lim_{y \to \infty} \sum_{\rho(\chi_1)}^{*} 2\sqrt{\pi y} \cdot e^{y(\rho - \alpha)^2} = 0.$$

Proof. At first we have

$$\begin{aligned} &|2\sqrt{\pi y} \cdot e^{y\{(\sigma-\alpha)^2 - \gamma^2 + 2i(\sigma-\alpha)\}}| \\ &\quad \leqslant 2\sqrt{\pi y}\, e^{-y\{\gamma^2 - (\sigma-\alpha)^2\}} \\ &\quad = e^{-\varepsilon y} \cdot 2\sqrt{\pi y}\, e^{-y\{\gamma^2 - (\sigma-\alpha)^2 - \varepsilon\}}. \end{aligned}$$

Take $\varepsilon = 10^{-2}$, then $\{\gamma^2 - (\sigma - \alpha)^2 - \varepsilon\} > 0$, for γ, σ and α are restricted by

$$\gamma > 6.02, \qquad 0 < \sigma < 1, \qquad -5 \leqslant \alpha \leqslant 6.$$

Now use the density statement (ii) (see 2.1.1.):

$$\left| \sum_{\rho}^{*} 2\sqrt{\pi y}\, e^{y(\rho-\alpha)^2} \right| = O(e^{-\varepsilon y} \log y).$$

2.2.2. *The integrals.* In this section we work with α in $0 \leqslant \alpha \leqslant 1$, for this choice simplifies the proof of

LEMMA 7. *Let* $0 \leqslant \alpha \leqslant 1$ *then*

$$\text{(i)} \quad -\frac{2}{2-\alpha} \leqslant \lim_{y\to\infty} \int_0^\infty \frac{e^{-x^2/4y+\alpha x}-1}{1-e^{2x}}\,dx \leqslant 0 \quad \text{and}$$

$$\text{(ii)} \quad -\frac{2}{1-\alpha} \leqslant \lim_{y\to\infty} \int_0^\infty \frac{e^{-x^2/4y+(1-\alpha)x}-1}{1-e^{2x}}\,dx \leqslant 0.$$

Proof. If $\alpha = 0$ then we have the equality sign on the right of (i), because the integral vanishes if $y \to \infty$. This is easy to see: The integrand is zero if $x = 0$. Let $\varepsilon > 0$, then

$$\lim_{y\to\infty} \int_\varepsilon^\infty \frac{e^{-x^2/4y}-1}{1-e^{2x}} = 0.$$

Now, if $\alpha > 0$, the exponent $-x^2/4y + \alpha x$ is positive as long as $x < 4\alpha y$, so we have

$$0 \leqslant \int_0^{4\alpha y} \frac{e^{-x^2/4y+\alpha x}-1}{e^{2x}-1}\,dx \leqslant \int_0^{4\alpha y} \frac{e^{-x^2/4y+\alpha x}}{e^{2x}}\,dx \leqslant \frac{2}{2-\alpha}.$$

The tail

$$\int_{4\alpha y}^\infty \frac{e^{-x^2/4y+\alpha x}-1}{e^{2x}-1}\,dx$$

vanishes if $y \to \infty$. The inequality (ii) holds by substituting $\alpha \to 1-\alpha$.

2.2.3. *The higher prime-powers.* If divergent, the sum

$$\sum_{\substack{p>2\\ n=v}} \log p \cdot p^{-v\alpha} \cdot e^{-(\log^2 p^v)/4y} \cdot \chi_1(p^v)$$

is of higher order than

$$\sum_{\substack{p>2\\ n=v}} \log p \cdot p^{-v(1-\alpha)} \cdot e^{-(\log^2 p^v)/4y} \cdot \chi_1(p^v),$$

because the sums only differ in the exponent of p. As long as $0 \leqslant \alpha < \frac{1}{2}$, we have $-v(1-\alpha) < -v \cdot \alpha$. As $\chi_1(p^2) = 1$, we definitely know that in (15)

$$\sum_{\substack{p>2\\ n=2}} \log p \cdot p^{-2\alpha} \cdot e^{-(\log^2 p)/y}$$

increases to infinity with larger order. This order of magnitude is given by

LEMMA 8. *Let α in $0 \leqslant \alpha < \frac{1}{2}$. If $y \to \infty$ we have asymptotically*

$$\sum_{\substack{p>2\\n=2}} \log p \cdot p^{-2\alpha} \cdot e^{-(\log^2 p)/y} \sim \sqrt{\pi y} \cdot e^{y(1-2\alpha)^2/4}.$$

Proof. By partial summation.
Now, with Lemmas 6, 7 and 8, (15) gives

$$\sum_{p>2} \log p \cdot p^{-\alpha} \cdot e^{-(\log^2 p)/4y} \cdot \chi_1(p)$$
$$= -\sqrt{\pi y}\, e^{y(1-2\alpha)^2/4} - \sum_{\substack{p>2\\n\geqslant 3}} \log p \cdot p^{-n\alpha} \cdot e^{-(\log^2 p^n)/4y} \cdot \chi_1(p^n)$$
$$+ o(\sqrt{\pi y}\, e^{y(1-2\alpha)^2/4}).$$

Concerning the sum $\sum_{p>2,n\geqslant 3}$, which is left, we first take $\frac{1}{3} < \alpha < \frac{1}{2}$, then, comparing with

$$\sum_{p,m} \frac{\log p}{p^{ms}} = -\frac{\zeta'}{\zeta}(s) < \infty, \qquad \mathrm{Re}(s) > 1,$$

we get

$$\left| \lim_{y\to\infty} \sum_{\substack{p>2\\n\geqslant 3}} \log p \cdot p^{-n\alpha} \cdot e^{-(\log^2 p^n)/4y} \cdot \chi_1(p^n) \right| < \infty.$$

If α becomes smaller, then some new divergent series in $\sum_{p>2,n\geqslant 3}$ occur, but only a limited number, as long as $\alpha > 0$. We know them by Lemma 8. Their magnitude is given by the maximal term "$p^{-2\alpha}$" in the preceding step.

If $\alpha = 0$ then $\sum_{p>2,n\geqslant 2}$ dominates for $y \to \infty$, too. One has to compare the exponents

$$-(\log^2 p)/4y \text{ (case } n=1), \qquad -(\log^2 p)/y \text{ (case } n=2),$$

$$-(\log^2 p)\Big/\frac{4}{9}y \text{ (case } n=3),\ldots,$$

$$-(\log^2 p)\Big/\frac{4}{b^2}y \text{ (case } n=b),\ldots.$$

Altogether, this gives

THEOREM 2. *Let α in $0 \leqslant \alpha < \frac{1}{2}$, then*

$$\lim_{x\to\infty} \sum_{p>2} (-1)^{\frac{p-1}{2}} \frac{\log p}{p^{\alpha}} \cdot e^{-(\log^2 p)/4y} = -\infty. \tag{17}$$

The order of divergence is given by

$$\sim \tfrac{1}{2}\sqrt{\pi x}\, \mathrm{e}^{x(1-2\alpha)^2/16}.$$

This theorem shows that for $x \to \infty$

$$S(x, \alpha) := \sum_{p>2} (-1)^{\frac{p-1}{2}} \cdot \frac{\log p}{p^{\alpha}} \cdot \mathrm{e}^{-(\log^2 p)/x}$$

behaves "reasonably," i.e., $S(x, \alpha)$ increases faster than $S(x, \alpha+\varepsilon)$, $\varepsilon > 0$. This is due to the more convergence forcing (monotonic) factors $p^{-\varepsilon}$, by which $S(x, \alpha+\varepsilon)$ differs from $S(x, \alpha)$.

2.2.4. *Relations to the Chebyshev series.* The just mentioned quality could have far-reaching importance. One is lead to proceed from (3) to (8) directly by pointwise multiplication with

$$g(p) = \mathrm{e}^{-p/x}/\mathrm{e}^{-(\log^2 p)/x}.$$

Unfortunately $g(p)$ is monotonically decreasing in p, so that we cannot conclude

$$S(x, \alpha) \to \infty \Rightarrow T(x) \to \infty$$

with $T(x) := \sum_{p>2} (-1)^{(p-1)/2} \log p \cdot \mathrm{e}^{-p/x}$. More insight could be derived if one uses the fact that the parameter x in $S(x, \alpha)$ and x in $T(x)$ may increase independently from each other. So let us take $S(h(x), \alpha)$ and $T(x)$, with $h(x) \to \infty$ if $x \to \infty$, and

$$\tilde{g}(p) = \mathrm{e}^{-p/x}/\mathrm{e}^{-(\log^2 p)/h(x)}.$$

Then we could derive some statements of the following type: "If (8) remains bounded, i.e., if (4) is wrong, then

$$\sum_{A(x) \leqslant p \leqslant B(x)} (-1)^{(p-1)/2} \cdot \log p \cdot \mathrm{e}^{-p/x}$$

increases to $+\infty$. The bounds $A(x)$, $B(x)$ also increase to infinity and depend on the choice of $h(x)$."

This (rather heuristic) statement becomes interesting in connection with a theorem of Knapowski and Turán [12, VII]: "Denoting by c explicitly calculable positive numerical constants, there exist U_1, U_2, U_3, U_4 numbers for $T > c$ with

$$\log\log\log T \leqslant U_2 \cdot \mathrm{e}^{-\log^{15/16} U_2} \leqslant U_1 < U_2 \leqslant T,$$
$$\log\log\log T \leqslant U_4 \cdot \mathrm{e}^{-\log^{15/16} U_4} \leqslant U_3 < U_4 \leqslant T$$

such that

$$\sum_{U_1 < p < U_2} (-1)^{(p-1)/2} \log p > \sqrt{U_2}$$

and

$$\sum_{U_3 < p < U_4} (-1)^{(p-1)/2} \log p < -\sqrt{U_4}$$

hold".

Note. We do not try here to remove the "$\log p$" factor in (10). The same "heuristical" arguments as above give rise to the presumption that the behaviour of the series does not change if the $\log p$ factor is absent. (One could try to manage this with the same methods used by Landau [13, II] or by Hardy and Littlewood [7].)

Remark. The results of Sections 2.1, and 2.2. are published under the author's former name of "Besenfelder" in Crelles Journal, see [3].

3. The Primes in the Progressions $3n+1$ and $3n+2$

In this section we investigate an analogue case to the original one of Chebyshev, namely, the primes in the progressions $3n+1$ and $3n+2$. For this purpose we take the function

$$\begin{aligned} \chi_1(m) &= +1 \quad &&\text{if} \quad m \equiv 1 \bmod 3, \\ &= -1 \quad &&\text{if} \quad m \equiv 2 \bmod 3, \\ &= 0 \quad &&\text{if} \quad m \equiv 0 \bmod 3, \end{aligned}$$

so χ_1 has a different meaning here than in the preceding sections. Now, if we sum the consecutive values of χ_1 over the primes only, up to a finite bound, we notice a similar behaviour as in the case mod 4.

Our investigations are split into two parts, a theoretical one, 3.1, and a practical one, 3.2. In the latter, numerical data are given, based on computer calculations.

3.1. *Theoretical Investigations*

The same methods as used before can be used to give statements about the discrepancies of the primes in the progressions $3n+1$, $3n+2$. They correspond to those already given. So, essentially all moduli with $\phi(k) = 2$ are settled. (The case $k = 6$, $\phi(k) = 2$, is contained in $k = 3$, for the Dirichlet character χ_1 mod 6 is generated by χ_1 mod 3, it is imprimitive.) The main Explicit Formula, which can be deduced in the same way as (12), is

$$\begin{aligned}2\sqrt{\pi y}\sum_{\rho(\chi_1)}{}^{*}\, e^{y(\rho-\alpha)^2} &= \log\frac{3}{\pi}-\sum_{p,n}\log p\cdot p^{-n\alpha}\cdot e^{-\log^2 p^n/4y}\cdot\chi_1(p^n)\\ &\quad-\sum_{p,n}\log p\cdot p^{-n(1-\alpha)}\cdot e^{-\log^2 p^n/4y}\cdot\chi_1(p^{-n})\\ &\quad-C+\int_0^\infty\frac{e^{-x^2/4y+\alpha x}-1}{1-e^{2x}}\,dx\\ &\quad+\int_0^\infty\frac{e^{-x^2/4y+(1-\alpha)x}-1}{1-e^{2x}}\,dx. \qquad (18)\end{aligned}$$

This differs from (12) by the zeros $\rho(\chi_1)$ and the conductor f (=3). So the constant $\log 3/\pi$ is negative here ($=-0.046117\ldots$). Note that the sum $\sum_{p,n}$ does not run over $p=3$. We state

THEOREM 3. *Let χ_1 mod 3 be given as above, then*

$$\lim_{x\to\infty}\sum_p \chi_1(p)\cdot\log p/\sqrt{p}\cdot e^{-\log^2 p/x}=-\infty. \qquad (19)$$

The order of magnitude of divergence is given by

$$\tfrac{1}{4}\sqrt{\pi x}+O(1).$$

For the proof, we only have to look whether the first zero of $L(s,\chi_1 \bmod 3)$ in the critical strip is far enough away from the real axis. We refer to the calculations of Davies and Haselgrove [6], or of Spira [17], which give

$$\rho_{1,2}(\chi_1 \bmod 3)=\tfrac{1}{2}\pm i\cdot 8,\ldots.$$

This also proves

THEOREM 4. *Let χ_1 mod 3 be as above, then*

$$\lim_{x\to\infty}\sum_p \chi_1(p)\log p/p^{\alpha}\cdot e^{-\log^2 p/x}=-\infty \qquad (20)$$

for all α in $0\leqslant\alpha<\frac{1}{2}$. The order of magnitude of divergence is given by

$$\sim\tfrac{1}{2}\sqrt{\pi x}\cdot e^{\frac{x}{16}(1-2\alpha)^2}.$$

3.2. *Computer Data*

Chebyshev, who asserted a preponderance of the primes congruent 3 mod 4, was not able to use such fast calculating machines as those available to us. So his upper bounds of calculations do not exceed some ten thousands. Nowadays it is possible to reach the millions (or even billions) within a short time. So the first sign change of

$$\pi(x, 3 \bmod 4) - \pi(x, 1 \bmod 4)$$

has been calculated and repeatedly checked. It is $x = 26,861$. If we look at $k = 8$, the first sign change occurs a little "later" and we have

$$\pi(588,067,889, 5 \bmod 8) - \pi(588,067,889, 1 \bmod 8) = -1$$

for the first time, see [8].

To get a better insight into the numerical behaviour of

$$\Delta_3(x) := \pi(x, 2 \bmod 3) - \pi(x, 1 \bmod 3)$$

I have calculated some values by a computer of the "Computer Gesellschaft Konstanz," the TR 440. The range of computation is the interval

$$2 \leqslant x \leqslant 18,633,261. \tag{I}$$

The upper bound is chosen to be a number divisible by 3 (for technical purposes only). Δ_3 never changed its sign within this range, nor became zero. The data show its maximal value at the prime, $p = 15,662,597$. We have

$$\Delta_3(15,662,597) = +465.$$

The outcome shows the following: Every new extremal value, together with the point x (=p, prime), was printed. These values carry a " – " sign, i.e.,

$$\sum_{p \leqslant x} \chi_1(p), \qquad \chi_1 \bmod 3,$$

was calculated. From 10^6 onwards, steps of size 33,333 were printed as intermediate values, so some (eventually) important information could be preserved. So we have the values of the sum

$$\sum_{p \leqslant x} \chi_1(p)$$

at the intermediate values x. From this we notice, for example, that $\sum_{x_1 \leqslant p \leqslant x_2} \chi_1(p)$ may be large on "small" intervals $[x_1, x_2]$: we have

$$\sum_{p \leqslant 174569} \chi_1(p) = -69$$

but

$$\sum_{174569 < p \leqslant 200\,001} \chi_1(p) = +62.$$

As the intermediate value 200,001 is not prime (and rather chosen at random), it can happen that the sum increases if the upper bound of this inverval varies. The following table gives a first survey.

TABLE I

x	$-\sum \chi_1$	$-$Min
5	2	2
59	4	4
599	8	8
5, 393	16	16
26, 993	32	32
69, 959	50	50
99, 999	23	
122, 963	51	51
138, 239	64	64
174, 569	69	69
200, 001	7	
250, 949	70	70
774, 047	128	128
783, 779	137	137
999, 999	135	
3, 978, 707	256	256
6, 788, 429	300	300
7, 291, 259	364	364
8, 966, 649	150	
10, 000, 017	190	
10, 200, 009	162	
11, 033, 331	91	364
13, 310, 819	365	365
15, 662, 597	465	465
17, 366, 607	192	
18, 633, 261	310	

Further information is contained in a paper which shall be published elsewhere. (It contains numerical data not only for this case but also for some other moduli. By request it can be obtained as a preprint from the author.)

As can be seen, the sum $\sum_p \chi_1(p)$ has never changed its sign, and has never even reached zero. It may be that the primes 2 and 5 at the very beginning (so we have the "starting value" -2) overpower the whole range up to 10^7. One should compare this with the two other cases mod 4 and mod 8:

Take $\sum_{p \leqslant x} (-1)^{(p-1)/2}$ with its "starting value" -1. The x in question is of order 10^4.

Take $\sum_{p\equiv 1 \bmod 8}^{x} 1 - \sum_{p\equiv 5 \bmod 8}^{x} 1$ with its "starting value" -2. The x in question is of order 10^9.

4. THE HIGHER MODULI, e.g., $k = 5$, $k = 8$

The previous chapters settle all cases $\phi(k) = 2$. Here we try to adapt the methods to the moduli of the next bigger size, $\phi(k) = 4$. Essentially $k = 5$ and $k = 8$ are to be examined. In the latter case only real characters exist. There is only one quadratic residue class here, which is marked by a small circle. Following Shanks' conjecture (p. 4), we can expect that there are "less" primes in 1 mod 8 than in each of the given classes. This is indeed true (in the sense of Theorem 1), as the table shows. One only has to "read" it with an appropriate Explicit Formula! Take

$$\begin{aligned}\sum_p \log p/\sqrt{p}\cdot e^{-(\log^2 p)/4y}\cdot\chi(p) + \sum_p \log p/p\cdot e^{-(\log^2 p)/y}\cdot\chi^2(p)\\ &= O(1) \qquad\qquad \text{if} \quad \chi\neq\chi_0,\\ &= 4\sqrt{\pi y}\cdot e^{y/4} + O(1) \qquad \text{if} \quad \chi=\chi_0.\end{aligned}\tag{21}$$

This formula corresponds to (12) if $\chi \neq \chi_0$. The terms which are bounded for large y are summarized in $O(1)$ (they differ only slightly from those in (12)). The first zeros of $L(s, \chi \bmod 8)$ are not too close to the real axis.

If $\chi = \chi_0$ (principal character) this L-series $L(s, \chi_0 \bmod 8)$ is essentially the Riemann zeta function, so the terms $M(0)$ and $M(1)$ occur (compare with (11)). They give the exponentially increasing term

$$4\sqrt{\pi y}\, e^{y/4}.$$

To get a statement about the discrepancies we have to take (21) and Table II. Let $\chi \neq \chi_0$, then

$$\chi^2 = 1$$

TABLE II

	1 mod 8	3 mod 8	5 mod 8	7 mod 8	Other
χ_0	1	1	1	1	0
χ_1	1	-1	1	-1	0
χ_2	1	1	-1	-1	0
χ_3	1	-1	-1	1	0

if $p \equiv 1, 3, 5, 7 \bmod 8$. The order of magnitude of

$$\sum \log p/p \cdot \mathrm{e}^{-\log^2 p/y}$$

is known by Lemma 5: $\frac{1}{2}\sqrt{\pi y} + O(1)$, if $y \to \infty$. So (21) changes into

$$\sum_p \log p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p)/4y} \cdot \chi(p) = -\tfrac{1}{2}\sqrt{\pi y} + O(1) \tag{22}$$

if y is large. Take this formula simultaneously for χ_i and χ_j, $i, j = 1, 2, 3$, $i \neq j$, and add. This gives

THEOREM 5. *We have*

$$\lim_{x\to\infty} \sum_{p\equiv 1 \bmod 8} \log p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p)/x} - \sum_{p\equiv 3 \bmod 8} \log p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p)/x} = -\infty, \tag{23}$$

$$\lim_{x\to\infty} \sum_{p\equiv 1 \bmod 8} \mathrm{lop}\, p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p)/x} - \sum_{p\equiv 5 \bmod 8} \log p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p)/x} = -\infty, \tag{24}$$

$$\lim_{x\to\infty} \sum_{p\equiv 1 \bmod 8} \log p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p)/x} - \sum_{p\equiv 7 \bmod 8} \log p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p)/x} = -\infty. \tag{25}$$

The order of magnitude of divergence is given by $-\frac{1}{4}\sqrt{\pi y} + O(1)$, *respectively.*

So we have got a quantitative comparison of $p \equiv 1 \bmod 8$ and $p \equiv 3, 5, 7 \bmod 8$. If one would like to compare the quadratic nonresidues, one has to subtract the formulas. This leads to

THEOREM 6. *We have*

$$\lim_{x\to\infty} \sum_{p\equiv 3 \bmod 8} - \sum_{p\equiv 5 \bmod 8} \log p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p)/x} = O(1), \tag{26}$$

$$\lim_{x\to\infty} \sum_{p\equiv 3 \bmod 8} - \sum_{p\equiv 7 \bmod 8} \log p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p)/x} = O(1), \tag{27}$$

$$\lim_{x\to\infty} \sum_{p\equiv 5 \bmod 8} - \sum_{p\equiv 7 \bmod 8} \log p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p)/x} = O(1). \tag{28}$$

We do not intend to find the finer discrepancies here. To do this one has to investigate the bounded terms, which are in $O(1)$. Conserning the module $k = 5$, we have the following table. Here, the characters χ_2 and χ_3 are not purely real, and this renders the investigations more complicated. The quadratic residues are marked as before.

TABLE III

	1 mod 5	2 mod 5	3 mod 5	4 mod 5	Other
χ_0	1	1	1	1	0
χ_1	1	-1	-1	1	0
χ_2	1	i	$-i$	-1	0
χ_3	1	$-i$	i	-1	0

The Explicit Formulas for the two real characters do not much differ from (21). However, if we take those for χ_2 and χ_3, we have, in general,

$$\chi_{2,3}(p^n) \neq \chi_{2,3}(p^{-n}).$$

So the terms in the sum over the prime-powers do not coincide here. If $\chi = \chi_{2,3}$ we have to work with

$$\sum_{p^n} \log p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p^n)/4y}(\chi(p^n) + \chi(p^{-n})) = O(1) \qquad \text{if} \quad y \to \infty. \quad (29)$$

Take the Explixit Formulas for χ_0 and for χ_1 and subtract them. Then the table gives

$$\sqrt{\pi y}\, \mathrm{e}^{y/4} + O(1) = \sum_{p\equiv 2} + \sum_{p\equiv 3} \log p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p)/4y}. \qquad (30)$$

However, the sum of the two gives

$$\sqrt{\pi y}\, \mathrm{e}^{y/4} + O(1) = \sum_{p\equiv 1} + \sum_{p\equiv 4} \log p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p)/4y}$$
$$+ \sum_{p\equiv 2} + \sum_{p\equiv 3} \log p/p \cdot \mathrm{e}^{-(\log^2 p)/y}. \qquad (31)$$

The single formula for χ_2 is

$$O(1) = \sum_{p\equiv 1} - \sum_{p\equiv 4} \log p/\sqrt{p} \cdot \mathrm{e}^{-(\log^2 p)/4y}$$
$$+ \sum_{p\equiv 1} - \sum_{p\equiv 2} - \sum_{p\equiv 3} + \sum_{p\equiv 4} \log p/p \cdot \mathrm{e}^{-(\log^2 p)/y}. \qquad (32)$$

The sum and the difference, respectively, of (31) and (32) give the following information:

$$\sum_{p\equiv 1} \log p/\sqrt{p} \cdot e^{-(\log^2 p)/4y}$$
$$= \tfrac{1}{2} \sqrt{\pi y}\, \mathrm{e}^{y/4} - \sum_{p\equiv 2} - \sum_{p\equiv 3} \log p/p \cdot \mathrm{e}^{-(\log^2 p)/y} + O(1) \qquad (33)$$

and

$$\sum_{p\equiv 4} \log p/\sqrt{p} \cdot e^{-(\log^2 p)/4y}$$
$$= \tfrac{1}{2}\sqrt{\pi y}\, e^{y/4} - \sum_{p\equiv 1} - \sum_{p\equiv 4} \log p/p \cdot e^{-(\log^2 p)/y} + O(1). \tag{34}$$

Both formulas show that $\sum_{p\equiv 1}$ and $\sum_{p\equiv 4}$ as well, if $y\to\infty$, increase slower than $\frac{1}{2}\sqrt{\pi y}\, e^{y/4}$. For we have

$$\lim_{y\to\infty} \sum_{p\equiv j} \log p/p \cdot e^{-(\log^2 p)/y} = +\infty$$

for $j = 1, 2, 3, 4$. However, (30) says that

$$\liminf_{y\to\infty} \Bigg(\sum_{p\equiv {2 \atop 3}?} \log p/\sqrt{p} \cdot e^{-(\log^2 p)/4y} - \tfrac{1}{2}\sqrt{\pi y}\, e^{y/4} \Bigg) > -\infty$$

holds for *at least* one of the two sums. Altogether we state

THEOREM 7. *For at least one of the two classes* 2 mod 5, 3 mod 5 *we have*

$$\lim_{y\to\infty} \Bigg(\sum_{p\equiv {2 \atop 3}?} - \sum_{p\equiv 4} \log p/\sqrt{p} \cdot e^{-(\log^2 p)/4y} \Bigg) = +\infty. \tag{35}$$

This theorem also holds if one changes $p\equiv 4$ into $p \equiv 1 \bmod 5$.

Finally we note that such tables (Tables II and III) show which comparisons can be made. But our method always depends on the location of the first zero of $L(s,\chi)$ in the critical strip. For some small k these zeros all have the desired property.

5. Concluding Remarks

5.1. *The Zeros with "Small" Ordinates*

We have learned the importance of the location of the first zero for the use of the Explicit Formulas. Now the conditions for this use is to be examined more exactly.

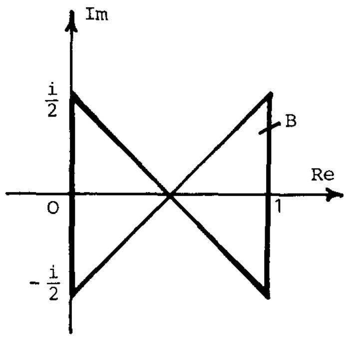
First we reformulate the "Haselgrove condition," which was referred to in the papers of Knapowski and Turán:

($\mathscr{HC}$) "There is a number $A(k)$ in $0 < A(k) \leqslant 1$ such that

$$L(s,\chi) \neq 0, \qquad s = \sigma + it,$$

in the rectangle $0 < \sigma < 1$, $|t| \leqslant A(k)$."

The results in this paper can be obtained if no nonreal zero of the given L-series is in the "bow-tie" B.



The difference between this and $(\mathscr{HC})$ is mainly, that real zeros are allowed in B. The condition is:

(B) "There is no nonreal zero of $L(s, \chi \bmod k)$, $s = \sigma + it$, in the region

$$|\sigma - \tfrac{1}{2}| < |t|."$$

Of course, from the present state of knowledge, it is important to find an L-series which vanishes in this region B.

5.2. *Some Further Problems*

We refer to the long list of problems given by Knapowski and Turán [11, I] and would like to add the following:

1. If k is large, do theorems like 1, 2, 3 still hold? If not, what are the reasons for the failure?

2. Let F from 2.1 be restricted on $(0, \infty)$ only. Then Table III immediately applies. How does the $\sum^*$ behave if y increases?

3. Prove the following conjecture:

"$L(s, x)$ never vanishes in the "bow-tie" B."

4. Are there some other weight-functions (see Section 1) which give statements similar to Theorem 1? In the author's opinion all these points are important for further investigations.

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