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ON THE REMAINDER TERM OF THE PRIME NUMBER FORMULA VI. INEFFECTIVE MEAN VALUE THEOREMS

by

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In part V [6] we gave effective lower bounds for the mean value of $|\Delta_i(x)|$, where

$$\Delta_1(x) \overset{\text{def}}{=} \pi(x) - \operatorname{li} x \overset{\text{def}}{=} \sum_{p \leqq x} 1 - \int_0^x \frac{dr}{\log r}$$

(1.1)
$$\Delta_2(x) \overset{\text{def}}{=} \Pi(x) - \operatorname{li} x \overset{\text{def}}{=} \sum_{\nu \geqq 1} \frac{1}{\nu} \pi(x^{1/\nu}) - \operatorname{li} x$$

$$\Delta_3(x) \overset{\text{def}}{=} \theta(x) - x \overset{\text{def}}{=} \sum_{p \leqq x} \log p - x$$

$$\Delta_4(x) \overset{\text{def}}{=} \psi(x) - x \overset{\text{def}}{=} \sum_{e \leqq x} \Lambda(n) - x.$$

With the notation

(1.2)
$$D_i(Y) \overset{\text{def}}{=} \frac{1}{Y} \int_2^Y |\Delta_i(x)|\, dx$$

we proved that if $\zeta(\beta_1 + i\gamma_1) = 0$ $(\beta_1 \geqq \frac{1}{2},\ \gamma_1 > 0)$ and $Y > \max(c_0, e^{\gamma_1})$ then

(1.3)
$$D_i(Y) \geqq \frac{1}{Y} \int_{Y \exp(-5\sqrt{\log Y})}^{Y} |\Delta_i(x)|\, dx > Y^{\beta_1} \exp\left(-2\sqrt{\log Y} \log_2^2 Y\right)$$

where $\log_2 Y = \log\log Y$ and c_0 further $c_1, c_2, c_3, \ldots$ are explicitly calculable positive constants.

Taking $\beta_1 + i\gamma_1 = 1/2 + i \cdot 14.13\ldots$, the first zero of $\zeta(s)$, (1.3) implies for $Y > c_1$

(1.4)
$$D_i(Y) \geqq \frac{1}{Y} \int_{Y \exp(-5\sqrt{\log Y})}^{Y} |\Delta_i(x)|\, dx > \sqrt{Y} \exp\left(-2\sqrt{\log Y} \log_2^2 Y\right)$$

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which improved the estimate $D_4(Y) > \sqrt{Y}\exp(-2\log Y\log_2^{-1} Y)$ of S. KNAPOWSKI [5].

In the present work we shall prove the ineffective improvement of (1.4) which is as follows.

THEOREM 1. *For $Y > Y_0$ (an ineffective constant) we have*

(1.5) $$D_1(Y) > \frac{1}{Y}\int_{Y\exp(-5\sqrt{\log Y})}^{Y} |\Delta_1(x)|\,dx > 0.62\frac{\sqrt{Y}}{\log Y}$$

(1.6) $$D_2(Y) > \frac{1}{Y}\int_{Y\exp(-5\sqrt{\log Y})}^{Y} |\Delta_2(x)|\,dx > 9\cdot 10^{-5}\frac{\sqrt{Y}}{\log Y}$$

(1.7) $$D_3(Y) > \frac{1}{Y}\int_{Y\exp(-5\sqrt{\log Y})}^{Y} |\Delta_3(x)|\,dx > 0.62\sqrt{Y}$$

(1.8) $$D_4(Y) > \frac{1}{Y}\int_{Y\exp(-5\sqrt{\log Y})}^{Y} |\Delta_4(x)|\,dx > 10^{-4}\sqrt{Y}.$$

Assuming the Riemann hypothesis, CRAMÉR [1] showed for $Y > c_2$

(1.9) $$\frac{1}{Y}\int_{2}^{Y} \Delta_4^2(x)\,dx < c_3 Y,$$

which implies

(1.10) $$\frac{1}{Y}\int_{2}^{Y} |\Delta_4(x)|\,dx < \sqrt{c_3}\sqrt{Y}$$

and applying our Lemma 1 and 2 mentioned later, from this one gets also

(1.11) $$D_3(Y) \leqq c_4\sqrt{Y}$$

(1.12) $$D_2(Y) \leqq c_5\frac{\sqrt{Y}}{\log Y}$$

(1.13) $$D_1(Y) \leqq c_6\frac{\sqrt{Y}}{\log Y}.$$

This shows that if the Riemann hypothesis is true then all the inequalities (1.5)—(1.8) are best possible apart from the values of the constants. Using some numerical computations Cramér's result shows that with the constant 0.16 instead of 10^{-4} and $9\cdot 10^{-5}$ the inequalities (1.6) and (1.8) are already false for every $Y > Y_0$ and analogously one cannot substitute 0.62 by 0.83 in (1.5) and (1.7) if the Riemann hypothesis is true.

On the other hand if the Riemann hypothesis is false then (1.5)—(1.8) are not optimal since in this case (1.3) gives a better lower bound. This means also that

in the course of proof of the Theorem 1 we can restrict ourselves for the case when the Riemann hypothesis is supposed to be true. In this case we can prove the following effective result which also furnishes a better localization than Theorem 1.

THEOREM 2. *If the Riemann hypothesis is true then for $Y > c_7$*

(1.14) $$\frac{1}{Y}\int_{10^{-3}Y}^{Y} \Delta_1(x)\log x\,dx < -0.62\sqrt{Y}$$

(1.15) $$\frac{1}{Y}\int_{Y/6}^{Y} |\Delta_2(x)|\,dx > 9\cdot 10^{-5}\frac{\sqrt{Y}}{\log Y}$$

(1.16) $$\frac{1}{Y}\int_{10^{-3}Y}^{Y} \Delta_3(x)\,dx < -0.62\sqrt{Y}.$$

(1.17) $$\frac{1}{Y}\int_{Y/6}^{Y} |\Delta_4(x)|\,dx > 10^{-4}\sqrt{Y}.$$

We want to note that analogously to (1.14) and (1.16) it would be possible to prove

(1.18) $$\frac{1}{(5Y/6)}\int_{Y/6}^{Y} \Delta_1(x)\log x\,dx < -0.68\sqrt{Y}$$

(1.19) $$\frac{1}{(5Y/6)}\int_{Y/6}^{Y} \Delta_3(x)\,dx < -0.68\sqrt{Y}$$

or

(1.20) $$\frac{1}{(Y/10)}\int_{9Y/10}^{Y} \Delta_1(x)\log x\,dx < -\frac{\sqrt{Y}}{9}$$

(1.21) $$\frac{1}{(Y/10)}\int_{9Y/10}^{Y} \Delta_3(x)\,dx < -\frac{\sqrt{Y}}{9}.$$

Theorem 2 and Cramér's results (1.10)—(1.13) together give that if the Riemann hypothesis is true then we know the exact order of magnitude of $D_i(Y)$; we have namely the effective

THEOREM 3. *If the Riemann hypothesis is true then for $Y > c_8$*

(1.22) $$c_9\frac{\sqrt{Y}}{\log Y} < D_i(Y) < c_{10}\frac{\sqrt{Y}}{\log Y}\quad (i = 1, 2)$$

(1.23) $$c_{11}\sqrt{Y} < D_i(Y) < c_{12}\sqrt{Y}\quad (i = 3, 4).$$

2. For the proof of Theorem 2 we shall need the following lemmata.

LEMMA 1.

(2.1) $$\Delta_1(x) = \Delta_2(x) - (1+o(1))\frac{\sqrt{x}}{\log x}$$

(2.2) $$\Delta_3(x) = \Delta_4(x) - (1+o(1))\sqrt{x}.$$

The proof follows from the prime number theorem.

LEMMA 2. *If the Riemann hypothesis is true then*

(2.3) $$\Delta_2(x)\log x = \Delta_4(x) + o(\sqrt{x}).$$

For the proof see INGHAM [3], p. 104.

LEMMA 3 (see Ingham [3], Theorem 28).

(2.4) $$\Delta_0(x) \overset{\text{def}}{=} \int_0^x \Delta_4(t)\,dt = -\sum_{\varrho} \frac{x^{\varrho+1}}{\varrho(\varrho+1)} + O(x)$$

where (as in the following always) $\varrho = \beta + i\gamma$ *runs through the non-trivial zeros of* $\zeta(s)$.

LEMMA 4 (see DE LA VALLÉE POUSSIN [7], p. 13).

(2.5) $$\sum_{\varrho} \frac{1}{|\varrho|^2} < 0.0464.$$

From Lemmata 3 and 4 we get immediately

LEMMA 5. *The Riemann hypothesis implies for* $x > c_{13}$

(2.6) $$|\Delta_0(x)| < 0.0464\, x^{3/2}.$$

(2.1)—(2.3) and (2.6) together give

$$\int_{10^{-3}Y}^{Y} \Delta_1(x)\log x\,dx = \int_{10^{-3}Y}^{Y} \Delta_3(x)\,dx + o(Y^{3/2}) =$$

(2.7) $$= \int_{10^{-3}Y}^{Y} \Delta_4(x)\,dx - \int_{10^{-3}Y}^{Y} \sqrt{x}\,dx + o(Y^{3/2}) <$$

$$< Y^{3/2}\left\{0.0464\left(1+10^{-\frac{9}{2}}\right) - \frac{2}{3}\left(1-10^{-\frac{9}{2}}\right) + o(1)\right\}$$

which proves (1.14) and (1.16).

3. In view of (2.3) (1.17) implies (1.15). (1.17) will be the immediate consequence of

THEOREM 4. *If the Riemann hypothesis is true then for* $Y > c_{14}$ *there exist*

(3.1) $$x', x'' \in [Y/6, Y]$$

with

(3.2) $$\Delta_0(x') < -8\cdot 10^{-4}(x')^{3/2} < -5\cdot 10^{-5} Y^{3/2}$$

and

(3.3) $$\Delta_0(x'') > 8\cdot 10^{-4}(x'')^{3/2} > 5\cdot 10^{-5} Y^{3/2}.$$

By Lemma 3 we have

(3.4) $$\frac{\Delta_0(e^v)}{(e^v)^{3/2}} = -\sum_{\varrho} \frac{e^{i\gamma v}}{\varrho(\varrho+1)} + o(1) \overset{\text{def}}{=} -G(v) + o(1).$$

So to prove Theorem 4 it will be sufficient to show that for every H there exist

(3.5) $$v', v'' \in \left[H - \frac{\log 6}{2},\; H + \frac{\log 6}{2}\right]$$

with

(3.6) $$G(v') > 8.1\cdot 10^{-4}, \quad G(v'') < -8.1\cdot 10^{-4}.$$

In the following let γ_1 and γ_2 denote the imaginary parts of the first two zeros ϱ_1 and ϱ_2, resp., of $\zeta(s)$ in the upper half-plane for which we shall use

(3.7) $$14 < \gamma_1 < 14.14, \quad \gamma_2 > 21$$

(see e.g. GRAM [2]).

In the proof an idea of Ingham [4], the use of the Fejér-kernel will be of importance, which satisfies for every real u the relation

(3.8) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 e^{iuy}\,dy = \begin{cases} 1-|u| & \text{for } |u| < 1 \\ 0 & \text{for } |u| \geq 1. \end{cases}$$

Using the properties of the Fejér-kernel we shall estimate the following weighted mean values of $G(v)$:

(3.9) $$I_1(\omega) \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-4\pi}^{4\pi} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 G\left(\omega + \frac{y}{\gamma_2}\right) dy$$

(3.10) $$I_2(\omega) \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 G\left(\omega + \frac{y}{\gamma_2}\right) dy.$$

If we can assure the existence of

(3.11) $$\omega', \omega'' \in \left[H - \frac{\log 6}{2} + \frac{4\pi}{\gamma_2},\; H + \frac{\log 6}{2} - \frac{4\pi}{\gamma_2}\right]$$

with

(3.12) $$I_1(\omega')>8.1\cdot 10^{-4},\quad I_1(\omega'')<-8.1\cdot 10^{-4}$$

then in view of (3.8) we get the existence of v', v'' with (3.5)—(3.6) and so the theorem will be proved. Further to satisfy (3.12) it is enough to find ω', ω'' with (3.11) for which

(3.13) $$I_2(\omega')>3.2\cdot 10^{-3},\quad I_2(\omega'')<-3.2\cdot 10^{-3}$$

since by (2.5), (3.4) and (3.7)

(3.14) $$|I_1(\omega)-I_2(\omega)|<0.0464\frac{1}{2\pi}2\int_{4\pi}^{\infty}\frac{1-\cos y}{(y^2/2)}\,dy<\frac{0.0464}{2\pi^2}<2.39\cdot 10^{-3}.$$

But (3.4), (3.8) and (3.10) imply

(3.15) $$I_2(\omega)=\sum_{\varrho}\frac{e^{i\gamma\omega}}{\varrho(\varrho+1)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 e^{iy\frac{\gamma}{\gamma_2}}\,dy= \sum_{|\gamma|<\gamma_2}\frac{e^{i\gamma\omega}}{\varrho(\varrho+1)}\left(1-\frac{|\gamma|}{\gamma_2}\right)=2\left(1-\frac{\gamma_1}{\gamma_2}\right)\mathrm{Re}\,\frac{e^{i\gamma_1\omega}}{\varrho(\varrho+1)}$$

and this obviously assumes positive and negative values with absolute value

(3.16) $$2\left(1-\frac{\gamma_1}{\gamma_2}\right)\frac{1}{|\varrho_1||\varrho_1+1|}>3.2\cdot 10^{-3}$$

in every closed interval of length $\frac{2\pi}{\gamma_1}<\log 6-\frac{8\pi}{\gamma_2}$. Q. E. D.

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ON THE EXPLICIT FORMULA OF RIEMANN — VON MANGOLDT

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1. Introduction

The classical formula of Riemann — von Mangoldt, which connects primes and zeros of the zeta-function, reads in the most important case as follows

(1.1) $$\Psi(x)=x-\sum_{\varrho,\,|\gamma|\le T}\frac{x^{\varrho}}{\varrho}+O\left(\frac{x}{T}\log^2 x\right),$$

where $2\le T\le x$,

$$\Psi(x)=\sum_{n\le x}\Lambda(n)=\sum_{p^k\le x}\log p,$$

and $\varrho=\beta+i\gamma$ denotes the non-trivial zeros of the Riemann zeta-function (see PRACHAR [4], p. 229). As the order of magnitude of the error term plays an important role in the prime number theorem on may ask whether (1.1) can be proved with a better error term estimation. Using mean value theorems for Dirichlet polynomials (see HUXLEY [2], §19) and zero density results, we will derive a slight improvement.

THEOREM. *Let* $x\ge 2$, $\log^6 x\le T\le\log^{-2}x$. *Then there exists a*

$$\tau\in\left(\frac{T}{2},\ \frac{3T}{2}\right)$$

with

$$\Psi(x)=x-\sum_{\varrho,\,|\gamma|\le\tau}\frac{x^{\varrho}}{\varrho}+O\left(\frac{x}{T}(\log x)^{\frac12}\left(\frac{\log x}{\log T}\right)^{\frac14}\left(\frac{\log x}{\log B}\right)^{\frac12}\right)$$

where

$$B=\frac{x}{T}\left(\frac{\log x}{\log T}\right)^{\frac12}.$$

In particular, for $0<c<\frac{1}{2}$, $x^c\le T\le x^{1-c}$, we have the bound

$$O_c\left(\frac{x}{T}(\log x)^{\frac12}\right)$$

for the error term.

In the following all constants implied by the symbols $O(\ )$ and $\ll$ — are absolute.

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