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ON THE REMAINDER TERM OF THE PRIME NUMBER FORMULA V. EFFECTIVE MEAN VALUE THEOREMS

by

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1. In part I—IV of this series [7], [8], [9], [10] we investigated extremely large values and the sign of the various forms of the remainder term in the prime number formula. Let

$$\Delta_1(x) \stackrel{\text{def}}{=} \pi(x) - \operatorname{li} x \stackrel{\text{def}}{=} \sum_{p \leqq x} 1 - \int_0^x \frac{dr}{\log r}$$

(1)
$$\Delta_2(x) \stackrel{\text{def}}{=} \Pi(x) - \operatorname{li} x \stackrel{\text{def}}{=} \sum_{\nu \geqq 1} \frac{1}{\nu} \pi(x^{1/\nu}) - \operatorname{li} x$$

$$\Delta_3(x) \stackrel{\text{def}}{=} \theta(x) - x \stackrel{\text{def}}{=} \sum_{p \leqq x} \log p - x$$

$$\Delta_4(x) \stackrel{\text{def}}{=} \psi(x) - x \stackrel{\text{def}}{=} \sum_{n \leqq x} \Lambda(n) - x.$$

Now we turn to the problem: how behaves $\Delta_i(x)$ in the average; more precisely, we want to give lower estimates for

(1.2)
$$D_i(Y) \stackrel{\text{def}}{=} \frac{1}{Y} \int_2^Y |\Delta_i(x)|\, dx.$$

The possible lower bounds depend naturally strongly on the distribution of the zeros of $\zeta(s)$; e.g. on the truth of the Riemann hypothesis. Similarly to our investigations in part I—IV it is more difficult to prove effective lower bounds than ineffective ones.

The problem was earlier studied by S. KNAPOWSKI. He proved [3] in 1959 using TURÁN's method that

(1.3)
$$\int_1^Y \frac{|\Delta(x)|}{x} > Y^{\beta_0} \exp\left(-14 \frac{\log Y}{\sqrt{\log\log Y}}\right)$$

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for $Y \geqq \max(c_1, \operatorname{expexp}(60 \log^2 |\varrho_0|))$, where $\varrho_0 = \beta_0 + i\gamma_0$ is an arbitrary zero of $\zeta(s)$ and c_1 is an explicitly calculable constant. Somewhat later [4] he showed that for $Y > c_2$

$$(1.4) \qquad D_4(Y) \geqq \frac{1}{Y} \int_{Y\exp(-\log^{3/4} Y)}^{Y} |\Delta_4(x)|\,dx > \sqrt{Y} \exp\left(-\frac{2\log Y}{\log\log Y}\right)$$

with explicitly calculable c_2.

2. In the following let c_ν denote explicitly calculable absolute constants, $\log_\nu Y$ the ν-times iterated logarithm function, i.e. $\log_1 Y = \log Y$, $\log_{\nu+1} = \log(\log_\nu Y)$. With the notations (1.1)–(1.2) we shall prove

THEOREM I. *If* $\zeta(\beta_1 + i\gamma_1) = 0$ $(\beta_1 \geqq \frac{1}{2},\ \gamma_1 > 0)$ *and* $Y > \max(c_3, e^{\gamma_1})$ *then*

$$(2.1) \qquad D_i(Y) \geqq \frac{1}{Y} \int_{Y\exp(-5\sqrt{\log Y})}^{Y} |\Delta_i(x)|\,dx > Y^{\beta_1} \exp\left(-2\sqrt{\log Y}\log_2^2 Y\right)$$

for $1 \leqq i \leqq 4$.

If we choose $\beta_1 + i\gamma_1 = \frac{1}{2} + i \cdot 14.13\ldots$ as the first non-trivial zero of $\zeta(s)$ we get

COROLLARY I. *For* $Y > c_4$ *one has*

$$(2.2) \qquad D_i(Y) \geqq \frac{1}{Y} \int_{Y\exp(-5\sqrt{\log Y})}^{Y} |\Delta_i(x)|\,dx > \sqrt{Y} \exp\left(-2\sqrt{\log Y}\log_2^2 Y\right) \quad (1 \leqq i \leqq 4).$$

Further Theorem I trivially implies

COROLLARY II. *If* $\zeta(\beta_1 + i\gamma_1) = 0$ $(\beta_1 \geqq \frac{1}{2},\ \gamma_1 > 0)$, $Y > \max(c_3, e^{\gamma_1})$ *then*

$$(2.3) \qquad \max_{Y\exp(-5\sqrt{\log Y}) \leqq x \leqq Y} |\Delta_i(x)| > Y^{\beta_1} \exp\left(-2\sqrt{\log Y}\log_2^2 Y\right) \quad (1 \leqq i \leqq 4).$$

From this we get

COROLLARY III. *For* $Y > c_4$ *one has*

$$(2.4) \qquad \max_{Y\exp(-5\sqrt{\log Y}) \leqq x \leqq Y} |\Delta_i(x)| \geqq \sqrt{Y} \exp\left(-2\sqrt{\log Y}\log_2^2 Y\right) \quad (1 \leqq i \leqq 4).$$

The important problem of finding effective Ω-type estimates for $\Delta_4(x)$ dependent on an arbitrarily given zeta-zero was raised by LITTLEWOOD in 1937 [5]. This problem was solved by Turán [13] in 1950, who proved the inequality

$$(2.5) \qquad \max_{1 \leqq x \leqq Y} |\Delta_4(x)| > \frac{Y^{\beta_1}}{|\varrho_1|^{\frac{10\log Y}{\log_2 Y}}} \exp\left(-c_5 \frac{\log Y \log_3 Y}{\log_2 Y}\right)$$

for $Y > \max(c_6, \exp(|\varrho_1|^{60}))$. Corollary II clearly improves this in respect of the lower bound and the localization, too.

We note that with some modification of the proof it is possible to show the sharper inequality

$$(2.6) \qquad D_i(Y) \geqq \frac{1}{Y} \int_{Y\exp(-6(\log Y)^{1/3}\log_2^{4/3} Y)}^{Y} |\Delta_i(x)|\,dx > Y^{\beta_1} \exp\left(-18(\log Y)^{1/3}\log_2^{4/3} Y\right)$$

which gives the corresponding improvements in the corollaries, too. But we prefer to prove the weaker form given in Theorem I because the proof for this is more simple and transparent and we shall only indicate the necessary modifications for the proof of (2.6) in the last section.

Our main tool in the course of proof will be Turán's method, more precisely a special case of the second main theorem of the powersum theory which we state here in its continuous form as follows.

LEMMA. *For real* $a, d > 0$, $b_j > 0$ $(1 \leqq j \leqq n)$ *and for arbitrary complex numbers* α_j *one has*

$$(2.7) \qquad \max_{a \leqq t \leqq a+d} \frac{\left|\sum_{j=1}^{n} b_j e^{\alpha_j t}\right|}{|e^{\alpha_1 t}|} \geqq \left(\frac{1}{8e\left(\frac{a+d}{d}\right)}\right)^n \min_{1 \leqq j \leqq n} b_j.$$

The original powersum theorem has been proved by V. T. SÓS—TURÁN [12]. For the continuous version see e.g. the Appendix of [7].

In part VI of this series [11] we shall prove the following ineffective improvement of Corollary I.

THEOREM II. *For* $Y > Y_i$ *(ineffective constants)*

$$(2.8) \qquad D_i(Y) > c_7 \frac{\sqrt{Y}}{\log Y} \quad \textit{for} \quad i = 1, 2$$

and

$$(2.9) \qquad D_i(Y) > c_8 \sqrt{Y} \quad \textit{for} \quad i = 3, 4.$$

A result of CRAMÉR [1] shows that if the Riemann hypothesis is true then these inequalities are optimal, apart from the values of the constants.

3. We shall give the detailed proof for the simplest case $i = 4$ and later only indicate the slight changes in the cases $1 \leqq i \leqq 3$.

Let k be a real number to be chosen later with

$$(3.1) \qquad 1 - \frac{1}{\log Y} \leqq k \leqq 1.$$

Let λ be the unique positive real number satisfying

$$\lambda^2+2\lambda = \log Y. \tag{3.2}$$

Further let

$$\mu = k\lambda^2 \tag{3.3}$$

$$L = \log_2 Y \tag{3.4}$$

$$M = \sqrt{\log Y}. \tag{3.5}$$

Then we have

$$\lambda \leqq M \tag{3.6}$$

$$\log Y-1 \leqq k(\lambda^2+2\lambda) = \mu+2k\lambda \leqq \log Y \tag{3.7}$$

and

$$\mu-2k\lambda \geqq \log Y-1-4\lambda \geqq \log Y-5M. \tag{3.8}$$

Writing simply $\Delta(x)$ instead of $\Delta_4(x)$, we shall start with the formula

$$\int_1^\infty \Delta(x)\frac{d}{dx}(x^{-s})dx = \frac{\zeta'}{\zeta}(s)+\frac{s}{s-1} = H(s) \tag{3.9}$$

which can be proved easily by partial integration. Using the well-known formula

$$\frac{1}{2\pi i}\int_{(2)} e^{As^2+Bs}ds = \frac{1}{2\sqrt{\pi A}}\exp\left(-\frac{B^2}{4A}\right) \tag{3.10}$$

which is valid for real $A>0$ and arbitrary complex B we get from (3.9)

$$U = \frac{1}{2\pi i}\int_{(2)} H(s+i\gamma_1)e^{ks^2+\mu s}ds =$$

$$= \frac{1}{2\pi i}\int_{(2)}\int_1^\infty \Delta(x)\frac{d}{dx}(x^{-s-i\gamma_1}e^{ks^2+\mu s})dxds =$$

$$= \int_1^\infty \Delta(x)\frac{d}{dx}\left\{x^{-i\gamma_1}\frac{1}{2\pi i}\int_{(2)} e^{ks^2+(\mu-\log x)s}ds\right\}dx = \tag{3.11}$$

$$= \int_1^\infty \Delta(x)\frac{d}{dx}\left\{x^{-i\gamma_1}\frac{1}{2\sqrt{\pi k}}\exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\right\}dx =$$

$$= \frac{1}{2\sqrt{\pi k}}\int_1^\infty \frac{\Delta(x)}{x}x^{-i\gamma_1}\exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\left(i\gamma_1+\frac{\mu-\log x}{2k}\right)dx.$$

We split the integral U in (3.11) into three parts:

$$U_1 = \int_1^{e^{\mu-2k\lambda}}, \quad U_2 = \int_{e^{\mu-2k\lambda}}^{e^{\mu+2k\lambda}}, \quad U_3 = \int_{e^{\mu+2k\lambda}}^{\infty}. \tag{3.12}$$

Using $\Delta(x)=O(x)$, $\gamma_1<e^M$ we obtain

$$|U_3| < c_9\gamma_1\int_{e^{\mu+2k\lambda}}^{\infty}\exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\left(\frac{\log x-\mu}{2k}-1\right)dx =$$

$$= c_9\gamma_1\int_{2k\lambda}^{\infty}\exp\left(-\frac{r^2}{4k}\right)\left(\frac{r}{2k}-1\right)e^{r+\mu}dr = \tag{3.13}$$

$$= c_9\gamma_1 e^{\mu+2k\lambda-\frac{4k^2\lambda^2}{4k}} = O(e^{3M})$$

and analogously we get

$$U_1 = O(e^{3M}). \tag{3.14}$$

For the main part U_2 of the integral U we have by $\gamma_1<e^M$, (3.1) and (3.5)–(3.8)

$$|U_2| \leqq \frac{1}{2}\left(\gamma_1+\frac{2k\lambda}{2k}\right)\int_{e^{\mu-2k\lambda}}^{e^{\mu+2k\lambda}}\frac{|\Delta(x)|}{x}dx \leqq$$

$$\leqq e^M\frac{e^{5M}}{Y}\int_{Ye^{-5M}}^{Y}|\Delta(x)|. \tag{3.15}$$

Now (3.13)–(3.15) imply

$$D(Y) \geqq \frac{1}{Y}\int_{Ye^{-5M}}^{Y}|\Delta(x)|dx \geqq e^{-6M}|U|+o(1). \tag{3.16}$$

4. Now we shall give a lower estimate for the integral U by an appropriate choice of k satisfying (3.1). Shifting the line of integration in (3.11) to $\sigma=-1$ we obtain

$$U = \sum_\varrho e^{k(\varrho-i\gamma_1)^2+\mu(\varrho-i\gamma_1)}+\frac{1}{2\pi i}\int_{(-1)} H(s+i\gamma_1)e^{ks^2+\mu s}ds. \tag{4.1}$$

Making use of the estimation

$$H(-1+i(t+\gamma_1)) = O(\log(|t+\gamma_1|+2)) \tag{4.2}$$

we get for the integral I in (4.1)

$$I = O(\log\gamma_1\cdot e^{-\mu}) = o(1). \tag{4.3}$$

The contribution of non-trivial zeros with $|\gamma-\gamma_1|\geqq\lambda$ to the infinite powersum in (4.1) is

$$O\left(\sum_{n=0}^{\infty}\log(\gamma_1+\lambda+n)e^{-k(\lambda+n)^2+\mu}\right) = O(\log(\gamma_1+\lambda)) = O(M). \tag{4.4}$$

The number of remaining non-trivial zeros with $|\gamma-\gamma_1|\leqq\lambda$ is

$$1 \leqq n \leqq \lambda\log(\gamma_1+\lambda) \leqq \lambda\log(2\log Y) \leqq M(L+1) \tag{4.5}$$

(see e.g. [6], where the relation $N(T+h)-N(T-h) \ll h\log T$ is proved for $\operatorname{tg} 1 \leqq h \leqq T-4$). Now using the Lemma, i.e. Turán's second main theorem, we get the existence of a k, satisfying (3.1) for which by (4.5), (3.3), (3.4) and (3.8)

(4.6)
$$\left|\sum_{|\gamma-\gamma_1|\leqq\lambda} e^{\{(\varrho-i\gamma_1)^2+\lambda^2(\varrho-i\gamma_1)\}k}\right| \geqq e^{k(\beta_1^2+\lambda^2\beta_1)}\left(\frac{1}{8e\log Y}\right)^n > e^{\mu\beta_1-\frac{4}{3}ML^2} > Y^{\beta_1}e^{-5M\beta_1-\frac{4}{3}ML^2}.$$

Thus (4.1), (4.3), (4.4) and (4.6) imply

(4.7)
$$|U| > Y^{\beta_1}e^{-\frac{3}{2}ML^2},$$

which, together with (3.16), proves Theorem I.

5. Now we shall sketch the slight changes needed in the course of proof of the cases $i=1, 2, 3$ (see (1.1)).

Owing to the prime number theorem of KOROBOV—VINOGRADOV (see e.g. [2] p. 419) we have

(5.1)
$$\Delta_3(x) = \Delta_4(x)-\sqrt{x}+O(\sqrt{x}\exp(-\log^{3/5-\varepsilon}x))$$

and so in the case $i=3$ it is enough to prove

(5.2)
$$\frac{1}{Y}\int_{Y\exp(-5\sqrt{\log Y})}^{Y} |\Delta_4(x)-\sqrt{x}|\,dx > Y^{\beta_1}\exp\left(-2\sqrt{\log Y}\log_2^2 Y\right).$$

In this case writing $\Delta(x)=\Delta_4(x)-\sqrt{x}$ and starting with the formula

(5.3)
$$\int_1^\infty \Delta(x)\frac{d}{dx}(x^{-s})\,dx = \frac{\zeta'}{\zeta}(s)+\frac{s}{s-1}+\frac{s}{s-\frac{1}{2}} \overset{\text{def}}{=} H(s)$$

(instead of (3.9)) we get (3.10)–(3.16) without any change. Further instead of (4.1) we now have

(5.4)
$$U = \sum_\varrho e^{k(\varrho-i\gamma_1)^2+\mu(\varrho-i\gamma_1)}+\frac{1}{2}e^{k\left(\frac{1}{2}-i\gamma_1\right)^2+\mu\left(\frac{1}{2}-i\gamma_1\right)}+ +\frac{1}{2\pi i}\int_{(-1)} H(s+i\gamma_1)e^{ks^2+\mu s}\,ds.$$

But adding the term $\frac{1}{2}e^{\left\{\left(\frac{1}{2}-i\gamma_1\right)^2+\lambda^2(\varrho-i\gamma_1)\right\}k}$ to the powersum in (4.6) the bound $M(L+1)$ in (4.5) holds also and the final estimate in (4.6) remains valid apart from a factor 1/2 for the slightly modified powersum. Thus (4.7) remains true without any change and this with (3.16) gives Theorem I for case $i=3$.

In the case $i=2$ it is clearly sufficient to prove the theorem for

(5.5)
$$\Delta(x) = \Delta_2(x)+\operatorname{li} 2 = \Pi(x)-\int_2^x\frac{dt}{\log t}$$

instead of $\Delta_2(x)$. In this case the modified definition of $H(s)$ and the starting formula will be

(5.6)
$$\int_2^\infty \Delta(x)\frac{d}{dx}(x^{-s}\log x)\,dx = \frac{\zeta'}{\zeta}(s)+\frac{2^{1-s}}{s-1} \overset{\text{def}}{=} H(s)$$

instead of (3.9). Now we get instead of (3.11) with easy calculation

(5.7)
$$U \overset{\text{def}}{=} \frac{1}{2\pi i}\int_{(2)} H(s+i\gamma_1)e^{ks^2+\mu s}\,ds = = \frac{1}{2\sqrt{\pi k}}\int_2^\infty \frac{\Delta(x)\log x}{x}\, x^{-i\gamma_1}\exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\left(-i\gamma_1+\frac{1}{\log x}+\frac{\mu-\log x}{2k}\right)dx$$

and since $\Delta(x)\log x=O(x)$ (3.10)—(3.16) hold again. Since Section 4 needs no change at all, this proves Theorem I for $i=2$.

Finally, in the case $i=1$, defining $\Delta(x)$ as

(5.8)
$$\Delta(x) = \Pi(x)-\int_2^x\frac{dt}{\log t}-\frac{1}{2}\int_2^{\sqrt{x}}\frac{dt}{\log t}$$

we have similarly to (5.1)

(5.9)
$$\Delta_1(x) = \Delta(x)+O(\sqrt{x}\exp(-\log^{3/5-\varepsilon}x))$$

and so it is sufficient to prove Theorem I for the $\Delta(x)$ in (5.8). Now the suitable modification of (3.9) and so the definition of $H(s)$ is

(5.10)
$$\int_2^\infty \Delta(x)\frac{d}{dx}(x^{-s}\log x)\,dx = \frac{\zeta'}{\zeta}(s)+\frac{2^{1-s}}{s-1}+\frac{2^{-\frac{1}{2}-s}}{s-\frac{1}{2}} \overset{\text{def}}{=} H(s).$$

Instead of (3.11) we have now (5.7) and (3.10)–(3.16) remain true (owing to $\Delta(x)\log x=O(x)$).

Proceeding in Section 4 as in the case $i=3$ further on we have again (5.4) instead of (4.1), otherwise everything remains unchanged (apart from the factor 1/2 in (4.6)) and we get the proof for the last case $i=1$.

6. Now we shall sketch the changes (in the case $i=4$) which lead to the stronger estimate (2.6). Instead of (3.1), (3.2) and (3.5) we write with $L=\log_2 Y$

$$\left(1-\frac{1}{\log Y}\right)(\log Y)^{-\frac{1}{3}}L^{\frac{8}{3}} \leqq k \leqq (\log Y)^{-\frac{1}{3}}L^{\frac{8}{3}} \tag{6.1}$$

$$\lambda^2+2\lambda = (\log Y)^{\frac{4}{3}}L^{-\frac{8}{3}} \tag{6.2}$$

$$M = (\log Y)^{\frac{1}{3}}L^{\frac{4}{3}}. \tag{6.3}$$

Then instead of (3.6)−(3.8) we have

$$\lambda \leqq (\log Y)^{2/3}L^{-4/3} \tag{6.4}$$

and

$$\log Y-5M \leqq \mu-2k\lambda < \mu+2k\lambda \leqq \log Y. \tag{6.5}$$

Starting from the original zero $\beta_1+i\gamma_1$ one can construct a zero $\beta_1'+i\gamma_1'$ with

$$\beta_1' \geqq \beta_1, \quad \gamma_1 \leqq \gamma_1' \leqq \gamma_1+\frac{\log^2 Y}{2} \leqq \frac{2}{3}\log^2 Y \tag{6.6}$$

for which the domain

$$|t-\gamma_1'| \leqq \log Y \quad \sigma \geqq \beta_1'+\frac{1}{\log Y} \tag{6.7}$$

is zero-free (for this procedure see [10], Section 3). Further on we shall always write β_1' and γ_1' instead of β_1 and γ_1, respectively.

With this change all the estimates (3.11)—(3.16) remain true. (The only change is within formula (3.13) where an additional factor $k^{-1/2}$ occurs, but owing to $\gamma_1' k^{-1/2}=O(e^M)$ this means no change in the final estimate.)

In Section 4 the formulae (4.1)−(4.4) are unchanged valid, but owing to (6.6)−(6.7) we can now estimate the contribution of zeros with

$$\lambda \geqq |\gamma-\gamma_1'| \geqq D \overset{\text{def}}{=} 3(\log Y)^{1/3}L^{-2/3}. \tag{6.8}$$

This contribution is by (4.5) and (6.6)—(6.8)

$$\begin{aligned} &\leqq M(L+1)e^{k\beta_1'^2+o(1)+\mu\beta_1'+1}e^{-kD^2} \leqq \\ &\leqq Y^{\beta_1}\exp\left(-8(\log Y)^{1/3}L^{4/3}\right). \end{aligned} \tag{6.9}$$

Thus the number of remaining zeros with $|\gamma-\gamma_1'|<D$ is smaller than in (4.5), namely we have

$$1 \leqq n \leqq D\log(\gamma_1+D) \leqq 2DL = 6(\log Y)^{1/3}L^{1/3}. \tag{6.10}$$

The above estimate improves (4.6) to

$$\left|\sum_{|\gamma-\gamma_1'|<D}\cdots\right| \geqq Y^{\beta_1}\exp\left(-5M\beta_1-\frac{19}{3}(\log Y)^{1/3}L^{4/3}\right) = Y^{\beta_1}e^{-M\left(5\beta_1+\frac{19}{3}\right)} \tag{6.11}$$

and (4.7) to

$$|U| > Y^{\beta_1}e^{-\frac{23}{2}M} \tag{6.12}$$

which by (6.3) together with (3.16) proves the sharper form (2.6) of Theorem I.

NOTE (Added in the proof). We want to use the possibility to correct some slight errors in Parts I, III and IV of this series [7, 9, 10].

In [9], p. 350, in

formula (7.2)	for	$1/\lambda$	read	$1/\mu$,
line 7	for	$[\lambda/2]$	read	$\mu/2$,
formula (7.5)	for	$1/\lambda$	read	$1/\mu$.

In [7], p. 353, (13.1), in [9], p. 352, (11.1) further in [10], p. 37, (7.1) for $\zeta(z)$ read $\zeta(z)-1$.

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