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NUMERICAL INVESTIGATION OF SEVERAL PROBLEMS IN NUMBER THEORY

by
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NOTATION

The letters p and q denote prime numbers. We use χ to denote a Dirichlet character mod q and χ_0 to denote the principal character. We have $e = \exp(1)$ and $e(\theta) = \exp(2\pi i \theta)$. Also we have

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi(n) = \#\{m,\ 1 \leq m \leq n:\ m \text{ and } n \text{ are relatively prime}\},$$

$$\pi(u) = \sum_{p \leq u} 1,$$

$$\theta(u) = \sum_{p \leq u} \log p,$$

$$\psi(u) = \sum_{n \leq u} \Lambda(n),$$

$$\psi(u, \chi) = \sum_{n \leq u} \Lambda(n)\, \chi(n).$$

and

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\, dt, \quad |\arg z| < \pi .$$

We let $N(T) = \#\{\gamma : 0 \leq \gamma \leq T\}$, where $\rho = \beta + i\gamma$ is a nontrivial zero of $\zeta(s)$, the Riemann zeta function. We use the Bessel functions

$$J_j(z) = \left(\frac{1}{2} z\right)^j \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4} z^2\right)^k}{k!\,\Gamma(j+k+1)}$$

and

$$I_j(z) = e^{-\frac{1}{2} j\pi i} J_j\left(ze^{\frac{1}{2}\pi i}\right) \quad \left(-\pi \le \arg z \le \frac{1}{2}\pi\right).$$

When we use Euler-Maclaurin summation, we make frequent use of the Bernoulli polynomials. The nth Bernoulli polynomial is by definition the unique polynomial of degree n with the property that $\int_x^{x+1} B_n(t)dt = x^n$. We make use of the fact that $B_n'(x) = nB_{n-1}(x)$. The Bernoulli polynomials which we use are

$$B_1(x) = x - \frac{1}{2},$$

$$B_2(x) = x^2 - x + \frac{1}{6},$$

$$B_3(x) = x^3 - \frac{3}{2} x^2 + \frac{1}{2} x,$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$$

and

$$B_5(x) = x^5 - \frac{5}{2} x^4 + \frac{5}{3} x^3 - \frac{x}{6}.$$

We use the standard notation

$$\hat{B}_n(x) = B_n(\{x\}),$$

where $\{x\}$ is the fractional part of x.

CHAPTER I

THE DISTRIBUTION OF arg $L(1, \chi)$

1. Statement of results. Dirichlet L-functions and other sums involving Dirichlet characters have always attracted a great deal of attention. Here we discuss the distribution of arg $L(1, \chi)$, where

$$L(1, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-1}$$

and χ is a Dirichlet character. More precisely, we examine

$$D(x) = \lim_{\substack{q \to \infty \\ q \text{ prime}}} D_q(x),$$

where

(1) $$D_q(x) = \frac{1}{q-2} \# \{ \chi \bmod q,\ \chi \neq \chi_0 : \arg L(1, \chi) \leq x \} .$$

In section 2 we show that D(x) is the distribution function of a sum of independent random variables. In sections 3 and 4, we show how to calculate the values of D(x). The following table contains typical values for D(x):

x	D(x)	Maximum possible error
$-(.5)\pi$	0.0000001894	0.000000002
$-(.4)\pi$	0.0007492671	0.000000002
$-(.3)\pi$	0.0208192351	0.000000002
$-(.2)\pi$	0.1080262718	0.000000002
$-(.1)\pi$	0.2767402886	0.000000002
0	0.5	0.0
$(.1)\pi$	0.7232597114	0.000000002
$(.2)\pi$	0.8919737282	0.000000002
$(.3)\pi$	0.9791807649	0.000000002
$(.4)\pi$	0.9992507329	0.000000002
$(.5)\pi$	0.9999998106	0.000000002

A graph of D(x) for $0 \leq x \leq (.5)\pi$ appears as Figure 1. We also show that

$$\lim_{\substack{q\to\infty \\ q \text{ prime}}} \frac{1}{q-2} \#\{\chi \bmod q,\ \chi \neq \chi_0 :\ \operatorname{Re} L(1,\chi) < 0\} = 0.378 \cdot 10^{-6} + \epsilon,$$

where $|\epsilon| \leq 4 \cdot 10^{-9}$. We obtain these results using computer programs (Appendix A) based on Theorem 1.16.

P. D. T. A. Elliott [8], [9], [10] has studied the distribution of $L(1,\chi)$ and $\arg L(1,\chi)$, but has not obtained precise numerical results. Our computational approach is similar to that used by Rice [21].

It is interesting to observe that although $L(1,\chi)$ has negative real part for about one in every three million characters, no such character is known. We have computed $L(1,\chi)$ for all characters with prime modulus ≤ 1300 using the formulas in section 5. The smallest real part which we found was 0.1886 and the greatest argument which we found was $1.3865 = (0.4413)\pi$.

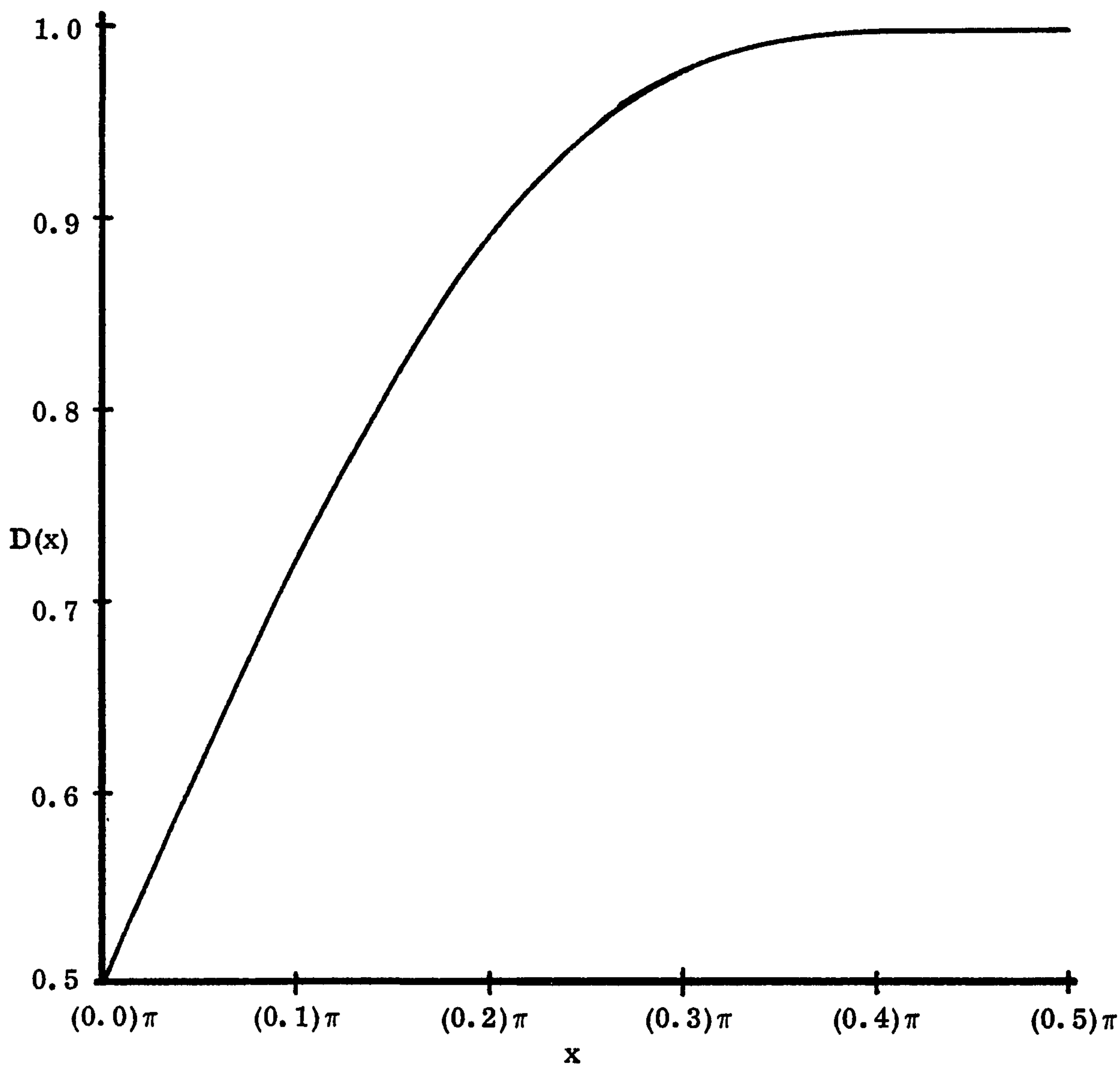

Fig. 1 - A Graph of D(x) for $0 \leq x \leq (0.5)\pi$

<u>2. Reduction to random variables.</u> We wish to reduce the problem of finding $D(x)$ to the problem of determining the distribution of a random variable. Since $L(1, \chi) = \prod_p (1-\chi(p)/p)^{-1}$ for $\chi \neq \chi_0$, we have $\arg L(1, \chi) = \sum_p \arg(1-\chi(p)/p)^{-1}$. As $\chi(2)$, $\chi(3)$, $\chi(5)$, ... tend to be independently uniformly distributed on $|z| = 1$, we consider the possibility that the random variable $\mathfrak{Y} = \sum \arg(1-e(\theta_p)/p)^{-1}$, where the θ_p are independent and uniformly distributed in $[0, 1)$, has a distribution similar to that of $\arg L(1, \chi)$. Using a standard result from probability theory (see Petrov [20, p. 266]), we have $P(\mathfrak{Y}$ converges$)$ $= 1$, since $E(\arg(1-e(\theta_p)/p)^{-1}) = 0$ and

$$\sum_{n=1}^{\infty} E(\arg(1-e(\theta_p)/p)^{-1})^2 << \sum_{n=1}^{\infty} \frac{1}{p^2} << 1.$$

The relation between these two distributions is given in

<u>Theorem 1.1.</u> Let $\mathfrak{Y}$ be as above. Then $D(x)$ is the distribution function of $\mathfrak{Y}$. In addition, $D(x)$ is differentiable, $D(-x) + D(x) = 1$ and for $x \geq \frac{\pi}{2}$

$$e^{-e^{c_1 e^x}} \leq 1-D(x) \leq e^{-e^{c_2 e^x}}$$

Here c_1 and c_2 are positive absolute constants.

Since the summands $\arg(1-e(\theta_p)/p)^{-1}$ are symmetrically distributed, we have $F(-x) + F(x) = 1$, where $F(x)$ is the distribution function of $\mathfrak{Y}$.

From Montgomery [16, Theorem 6.2], we have

<u>Lemma 1.2.</u> Let $s(\chi) = \sum_{n=M+1}^{M+N} a_n \chi(n)$, where χ is a character modulo q. Then

$$\sum_{\chi} |s(\chi)|^2 \leq \varphi(q) \left(1 + \left[\frac{N-1}{q}\right]\right) \sum_{\substack{M+1 \\ (n,q)=1}}^{M+N} |a_n|^2.$$

If $N \leq q$, then this holds with equality.

We now prove

<u>Lemma 1.3.</u> **For all sufficiently large prime q,**

$$\sum_{\chi \neq \chi_0} \left| \log L(1,\chi) - \sum_{p \leq A} \log(1-\chi(p)/p)^{-1} \right| < \frac{2(q-2)}{(A \log A)^{\frac{1}{2}}}$$

uniformly for $e^{631} \leq A \leq q(\log q)^{-7}$.

<u>Proof</u>. Since $L(1,\chi) = \prod_p (1-\chi(p)/p)^{-1}$ for $\chi \neq \chi_0$, we have

$$(2) \quad \left| \log L(1,\chi) - \sum_{p \leq A} \log(1-\chi(p)/p)^{-1} \right| = \left| \sum_{p>A} \log(1-\chi(p)/p)^{-1} \right|.$$

As $\log(1-z)^{-1} = \sum_{k=1}^{\infty} z^k/k$ for $|z| < 1$, the above is

$$\left| \sum_{p>A} \sum_{k=1}^{\infty} \chi(p)^k/kp^k \right| =$$

$$\left| \sum_{n>A} \Lambda(n)\chi(n)/n \log n - \sum_{p \leq A} \sum_{k > \log A/\log p} \chi(p)^k/kp^k \right|$$

$$\leq \left| \sum_{A<n\leq q} \Lambda(n)\chi(n)/n \log n \right| + \left| \sum_{q<n<U} \Lambda(n)\chi(n)/n \log n \right|$$

$$+ \left| \sum_{U \leq n} \Lambda(n)\chi(n)/n \log n \right| + \left| \sum_{p \leq A} \sum_{k > \log A/\log p} \chi(p)^k/kp^k \right|$$

$$= T_1(\chi) + T_2(\chi) + T_3(\chi) + T_4(\chi).$$

Using the Lemma 1.2, we obtain

(3) $$\sum_{\chi\neq\chi_0} |T_1(\chi)| \le \Big(\sum_{\chi\neq\chi_0} 1\Big)^{\frac{1}{2}}\Big(\sum_{\chi\neq\chi_0} \Big|\sum_{A<n\le q} \Lambda(n)\chi(n)/n\log n\Big|^2\Big)^{\frac{1}{2}}$$

$$\le (q-2)^{\frac{1}{2}}(q-1)^{\frac{1}{2}}\Big(\sum_{A<n<q} |\Lambda(n)/n\log n|^2\Big)^{\frac{1}{2}}.$$

If $A > 1$, then

(4) $$\sum_{A<n<q} \Lambda^2(n)/n^2\log^2 n < \frac{1}{\log A}\sum_{n>A}\frac{\Lambda(n)}{n^2} < \frac{2}{\log A}\int_A^\infty \frac{\psi(u)\,du}{u^3}.$$

For $17 \le u$, we have $|\psi(u)-u| < u\sqrt{8/17\pi}\ Y^{\frac{1}{2}}\exp(-Y)$, where

$Y = \sqrt{\log u/9.645908801}$, as shown by Schoenfeld [24, Theorem 11]. Since $Y^{\frac{1}{2}}\exp(-Y)$ attains its maximum at $Y = \frac{1}{2}$, we have $\psi(u) < (1.166)u$ for $17 \le u$. As $\psi(u) < u$ for $u < 19$, we obtain $\psi(u) < (1.166)u$ for $0 < u$. Using this inequality in (4), we obtain

(5) $$\sum_{A<n<q} \frac{\Lambda^2(n)}{n^2\log^2 n} < \frac{2.332}{A\log A}.$$

Combining (3) and (5), we have

(6) $$\sum_{\chi\neq\chi_0} |T_1(\chi)| < \frac{(1.55)q}{(A\log A)^{\frac{1}{2}}}.$$

We observe that

$$|T_2(\chi)| = \Big|\int_q^U \frac{d\psi(u,\chi)}{u\log u}\Big| << \frac{|\psi(q,\chi)|}{q\log q} + \frac{|\psi(U,\chi)|}{U\log U} + \int_q^U \frac{|\psi(u,\chi)|\,du}{u^2\log u}.$$

Vaughn [28, Theorem 2] has shown that if $q \geq 1$ and $u \geq 2$, then

$$\sum_{\chi} \max_{y \leq u} |\psi(y,\chi)| << ul^3 + u^{3/4} q^{5/8} l^{23/8} + u^{1/2} q l^{7/2},$$

where $l = \log(u\,q)$. Hence,

$$\sum_{\chi \neq \chi_0} |T_2(\chi)| << q^{1/2}(\log q)^{5/2} + (1+q^{5/8}U^{-\frac{1}{4}} + q\,U^{-\frac{1}{2}})(\log U)^{5/2}$$

$$+ (\log U)^3 + q^{5/8}U^{-1/4}(\log U)^{23/8} + qU^{-\frac{1}{2}}(\log U)^{7/2}.$$

We now take $U = \exp(q^{1/7})$, so that

(7) $$\sum_{\chi \neq \chi_0} |T_2(\chi)| << q^{1/2}(\log q)^{5/2}.$$

It is well known (see Davenport [6 ,pp. 135-136]) that $\psi(u,\chi) <<$ $u \exp(-c(\log u)^{1/2})$ for $\chi \neq \chi_0 \pmod q$, provided that $q \leq (\log u)^7$. Hence,

$$|T_3(\chi)| = \left| \int_U^\infty \frac{d\psi(u,\chi)}{u \log u} \right| << \frac{|\psi(U,\chi)|}{U \log U} + \int_U^\infty \frac{|\psi(u,\chi)|\,du}{u^2 \log u}$$

$$<< \exp(-\tfrac{1}{2} c (\log U)^{1/2}) << q^{-1},$$

so that

(8) $$\sum_{\chi \neq \chi_0} |T_3(\chi)| << 1.$$

As for $T_4(\chi)$, we note that, for $p \leq A$,

$$\left| \sum_{k>\log A/\log p} \frac{\chi(p)^k}{kp^k} \right| \leq \sum_{k>\log A/\log p} \frac{1}{p^k} \leq \min(p^{-2}, A^{-1}) \frac{1}{1-\frac{1}{p}}$$

$$\leq 2 \min(p^{-2}, A^{-1}).$$

Hence,

$$|T_4(\chi)| \leq 2A^{-1} \sum_{p\leq A^{1/2}} 1 + 2 \sum_{A>p>A^{1/2}} p^{-2}.$$

From Rosser and Schoenfeld's work [23, Corollary 2], we have $\pi(x) < (1.25506)x/\log x$ for $1 < x$. This gives us

$$|T_4(\chi)| \leq 2A^{-1} \pi(A^{1/2}) + 2 \int_{A^{1/2}}^{\infty} \frac{d\pi(u)}{u^2}$$

$$\leq 2A^{-1} \pi(A^{1/2}) + 2 \left(\frac{-\pi(A^{1/2})}{A} + 2 \int_{A^{1/2}}^{\infty} \frac{\pi(u)\, du}{u^3} \right)$$

$$\leq 5.02024 \int_{A^{1/2}}^{\infty} \frac{du}{u^2 \log u} \leq \frac{5.02024}{\log A^{1/2}} \frac{1}{A^{1/2}}$$

$$\leq \frac{10.04048}{A^{1/2} \log A},$$

so that

$$(9) \qquad \sum_{\chi\neq\chi_0} |T_4(\chi)| \leq \frac{(10.04048)q}{A^{1/2} \log A}.$$

This completes the proof of Lemma 1.3, since by combining (2), (6), (7), (8), and (9), we obtain our desired result.

Our next step is to relate this lemma to the distribution of arg $L(1,\chi)$.

Lemma 1.4. Let

$$D_q(x) = \frac{1}{q-2}\,{}^{\#}\{\chi\neq\chi_0: \arg L(1,\chi) \leq x\}$$

and

$$D_q(x,A) = \frac{1}{q-2}\,{}^{\#}\{\chi\neq\chi_0: \sum_{p\leq A} \arg(1-\chi(p)/p)^{-1} \leq x\}$$

for prime q. If q is sufficiently large and $e^{631} \leq A \leq q(\log q)^{-7}$, then

$$D_q(x-\delta, A) - \frac{2}{\delta(A\log A)^{1/2}} < D_q(x) < D_q(x+\delta, A) + \frac{2}{\delta(A\log A)^{1/2}}$$

for all $\delta > 0$.

Proof. Let N denote the number of $\chi\neq\chi_0 \pmod q$ such that

$\sum_{p\leq A} \arg(1-\chi(p)/p)^{-1} \leq x-\delta$ but $\arg L(1,\chi) > x$. Then $D_q(x-\delta, A) \leq D_q(x) + \frac{N}{q-2}$.

But by Lemma 1.3,

$$N \leq \frac{2(q-2)}{\delta(A\log A)^{1/2}}.$$

This gives the lower bound for $D_q(x)$, and the upper bound is proved similarly.

Finally, we relate $D_q(x,A)$ to the distribution of a random variable. We first need

Lemma 1.5 (Weyl's criterion). Let $\mu_1, \mu_2, \ldots$ be probability measures on I^R. Then the following are equivalent:

(a) the μ_k tend weakly to Lebesgue measure on I^R;

(b) for any Riemann-integrable function f defined on I^R,

$$\lim_{k\to\infty} \int_{I^R} f(\underline{\theta})\, d\mu_k(\underline{\theta}) = \int_{I^R} f(\underline{\theta})\, d\underline{\theta} \;;$$

(c) for any $\underline{h} \in \mathbb{Z}^R$, $\underline{h} \neq 0$,

$$\lim_{k\to\infty} \int_{I^R} e(\underline{h}\cdot\underline{\theta})\, d\mu_k(\underline{\theta}) = 0.$$

Proof. This lemma is a special case of the continuity theorem for R-dimensional characteristic functions. Billingsley [3, p. 329-335], proves that for probability measures μ_k and μ on $\mathbb{R}^R$, the following are equivalent:

(i) μ_k converges weakly to μ;

(ii) $\lim_k \int_{\mathbb{R}^R} q\, d\mu_k = \int_{\mathbb{R}^R} q\, d\mu$ for bounded continuous q;

(iii) $\lim_k \mu_k(A) = \mu(A)$ for all Borel sets A with $\mu(\partial A) = 0$.

Since the measures in the lemma are zero outside the unit cube, we see immediately that (a) → (ii) → (c) and (b) → (ii) → (a). By the definition of Riemann integration, we have (a)→(iii)→(b). Finally, we have (c)→(ii)→(a) using the Weierstrass approximation theorem. Using this lemma, we prove

Lemma 1.6. Let F(x, A) be the distribution function of the random variable $\Psi_A = \sum_{p\leq A} \arg(1-e(\theta_p)/p)^{-1}$, where the θ_p are independent and uniformly distributed in [0, 1). Given any $\epsilon > 0$, there is a $q_0(\epsilon)$ such that if $q > q_0(\epsilon)$, then

$$|D_q(x, A) - F(x, A)| < \epsilon$$

for all x.

Proof. The function $F(x, A)$ is continuous; thus, by compactness and monotonicity, it suffices to establish the inequality for any fixed x. We use Lemma 1.5 with $R = \pi(A)$; the primes $p \le A$ index the coordinates of our vectors. Let

$$f(\underline{\theta}) = \begin{cases} 1 & \text{if } \sum_{p \le A} \arg(1-e(\theta_p)/p)^{-1} \le x, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$F(x, A) = \int_{I^R} f(\underline{\theta})\, d\underline{\theta}$$

and

$$D_q(x, A) = \int_{I^R} f(\underline{\theta})\, d\mu_q(\underline{\theta}),$$

where μ_q is the probability measure which has point masses of weight $\frac{1}{q-2}$ at each of the points $\left(\frac{\arg\chi(p)}{2\pi}\right)$ for $\chi \ne \chi_0 \pmod q$. Thus, the desired result follows from (b). We establish (c), which is to say

$$\lim_{\substack{q \to \infty \\ q \text{ prime}}} \int_{I^R} e(\underline{h} \cdot \underline{\theta})\, d\mu_q(\underline{\theta}) = 0$$

for $\underline{h} \in \mathbb{Z}^R$, $\underline{h} \ne \underline{0}$. If $\underline{\theta} = \left(\frac{\arg\chi(p)}{2\pi}\right)$, then

$$\underline{h} \cdot \underline{\theta} = \sum_{p \le A} h_p \frac{\arg\chi(p)}{2\pi} = \frac{1}{2\pi} \arg \prod_{p \le A} \chi(p)^{h_p}.$$

Put $a/b = \prod_{p \leq A} p^{h_p}$ in lowest terms; here a and b depend only on A and $\underline{h}$. Then $e(\underline{h} \cdot \underline{\theta}) = \chi(a)\,\bar{\chi}(b)$, and the integral above is

$$\frac{1}{q-2} \sum_{\chi \neq \chi_0} \chi(a)\bar{\chi}(b) = \begin{cases} 1 & \text{if } a \equiv b \pmod{q}, \\ -\frac{1}{q-2} & \text{otherwise.} \end{cases}$$

If q is large and $a \equiv b \pmod{q}$, then a = b. But (a, b) = 1, so that a = b = 1. But then $\underline{h} = \underline{0}$, contrary to supposition. Hence,

$$\int_{I^R} e(\underline{h} \cdot \underline{\theta})\, d\mu_q(\underline{\theta}) = -\frac{1}{q-2}$$

for all large primes q and (c) is established. This completes the proof of Lemma 1.6.

Combining Lemma 1.2, Lemma 1.4, and Lemma 1.6, we obtain

Lemma 1.7. If $A \geq e^{631}$, then

(10)

$$F(x-\delta, A) - \frac{2}{\delta (A \log A)^{1/2}} < D(x) < F(x+\delta, A) + \frac{2}{\delta (A \log A)^{1/2}}$$

for any x and any $\delta > 0$.

We can now complete the proof of the theorem. Taking $\delta = A^{-1/4}$ and letting $A \to \infty$ in (10), we obtain $D(x) = \lim_{A \to \infty} F(x, A) = F(x)$, where F(x) is the distribution function of $\mathbf{Y}$, at every point x where the limit exists and F(x) is continuous. We have shown that $\mathbf{Y}$ converges almost surely.

Thus, F(x) exists for all x. In order to show that F(x) is continuous, we examine the characteristic function

$$\hat{F}(\lambda) = \int_{-\infty}^{\infty} e(-\lambda x) dF(x) = \int_{I^\infty} e(-\lambda \Upsilon) d\underline{\theta} = \prod_p G(p, 2\pi\lambda),$$

where

$$G(p,\mu) = \int_0^1 e^{i\mu \arg(1-e(\theta)/p)} d\theta .$$

In order to bound $\hat{F}(\lambda)$, we need

Lemma 1.8. Let

$$B(p,\mu_0) = \sqrt{\frac{\mu_0}{p}\left(6-\sqrt{36-\frac{24\pi p}{\mu_0}-\frac{12\pi^2}{\mu_0^2}}\right)} .$$

If $\mu \geq \mu_0 \geq \pi$ and $\frac{2}{\pi} B(p,\mu_0)\sqrt{\frac{p}{\mu_0}} \leq 1$, then

$$\left|G(p,\mu)\right| \leq \frac{2}{\pi} B(p,\mu_0) \sqrt{\frac{p}{\mu}} .$$

Proof. We first show that if $-1 \leq p \sin \frac{\pi}{2\mu} \leq 1$ and $-1 \leq p \sin (\sin^{-1} \frac{1}{p} - \frac{\pi}{\mu})$ ≤ 1, then

$$|G(p,\mu)| \leq \frac{2}{\pi}\left(\frac{\pi}{2} - \sin^{-1}\left(p \sin\left(\sin^{-1}\frac{1}{p} - \frac{\pi}{\mu}\right)\right) - \sin^{-1} p \sin\frac{\pi}{2\mu}\right) .$$

We have

$$G(p,\mu) = \int_0^1 e^{i\mu \arg(1-e(\theta)/p)} d\theta = \int_{-\infty}^{\infty} e^{i\mu t} h_p(t)\, dt,$$

where $h_p(t)$ is the density function of $\arg(1-e(\theta)/p)$. If we let $H_p(t)$ be the distribution function of $\arg(1-e(\theta)/p)$, then

(11)

$$H_p(t) = \begin{cases} 1 & \text{if } t \geq \tan^{-1}\frac{1}{\sqrt{p^2-1}}\,, \\ \frac{1}{2} + \frac{\sin^{-1} p \sin t}{\pi} & \text{if } -\tan^{-1}\frac{1}{\sqrt{p^2-1}} \leq t \leq \tan^{-1}\frac{1}{\sqrt{p^2-1}}\,, \\ 0 & \text{if } t \leq -\tan^{-1}\frac{1}{\sqrt{p^2-1}}\,. \end{cases}$$

Hence,

$$h_p(t) = \begin{cases} \frac{p \cos t}{\pi\sqrt{1-p^2\sin^2 t}} & \text{if } -\tan^{-1}\frac{1}{\sqrt{p^2-1}} \leq t \leq \tan^{-1}\frac{1}{\sqrt{p^2-1}}\,, \\ 0 & \text{otherwise.} \end{cases}$$

Using this result, we obtain

$$G(p,\mu) = \int_{-\infty}^{\infty} e^{i\mu t} h_p(t)dt = -\int_{-\infty}^{\infty} e^{i\mu t} h_p(t + \frac{\pi}{\mu})\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{i\mu t}(h_p(t) - h_p(t + \frac{\pi}{\mu}))\, dt.$$

This gives us

(12)

$$|G(p,\mu)| \leq \frac{2}{\pi}\left(\frac{\pi}{2} - \sin^{-1}\left(p \sin\left(\sin^{-1}\frac{1}{p} - \frac{\pi}{\mu}\right)\right) - \sin^{-1} p \sin \frac{\pi}{2\mu}\right).$$

We next show that for all $\mu \geq \mu_0 \geq \pi$

(13) $$\sin^{-1} p \sin\left(\sin^{-1}\frac{1}{p} - \frac{\pi}{\mu}\right) \geq \frac{\pi}{2} - B(p,\mu_0)\sqrt{\frac{p}{\mu}}\,.$$

It is sufficient to demonstrate that

$$\cos\frac{\pi}{\mu} - p\sin\frac{\pi}{\mu} - \cos B(p,\mu)\sqrt{\frac{p}{\mu}} \geq 0.$$

This inequality follows from the fact that

$$\cos\frac{\pi}{\mu} \geq 1 - \frac{\left(\frac{\pi}{\mu}\right)^2}{2!} + \frac{\left(\frac{\pi}{\mu}\right)^4}{4!} - \frac{\left(\frac{\pi}{\mu}\right)^6}{6!},$$

$$p\sin\frac{\pi}{\mu} \leq p\left(\frac{\pi}{\mu} - \frac{\left(\frac{\pi}{\mu}\right)^3}{3!} + \frac{\left(\frac{\pi}{\mu}\right)^5}{5!}\right)$$

and

$$\cos B(p,\mu)\sqrt{\frac{p}{\mu}} \leq 1 - \frac{B^2(p,\mu)p}{2!\mu} + \frac{B^4(p,\mu)p^2}{4!\mu^2} - \frac{B^6(p,\mu)p^3}{6!\mu^3} + \frac{B^8(p,\mu)p^4}{8!\mu^4}.$$

Combining (12) and (13), we complete the proof of the lemma.

To show that F(x) has a continuous derivative, it is sufficient to show that

$$\int_{-\infty}^{\infty} |\hat{F}(\lambda)|\, d\lambda < \infty \text{ (see Billingsley [3, p.301])}.$$

This follows from the fact that $|G(p,\mu)| \leq 1$ and $|G(p,\mu)| << \mu^{-\frac{1}{2}}$ for $\mu \geq \mu_0 \geq \pi$. Thus, F(x) is continuous and hence, D(x) = F(x) for all x (remark: this also has an elementary proof). We now determine upper and lower bounds for 1-D(x) = P($\Upsilon \geq x$). The upper bound is standard. Let $\lambda > 0$.

Since $e^{\lambda \mathbf{Y}} \geq e^{\lambda x}$ for $\mathbf{Y} \geq x$, we have $P(\mathbf{Y} \geq x) \leq e^{-\lambda x} E(e^{\lambda \mathbf{Y}})$.

We write

$$E(e^{\lambda \mathbf{Y}}) = \prod_p \int_0^1 e^{\lambda \arg(1-e(\theta)/p)^{-1}} d\theta$$

$$= \prod_p \int_0^1 e^{\lambda \tan^{-1} \frac{\sin 2\pi\theta}{p-\cos 2\pi\theta}} d\theta = \prod_p E(p, \lambda).$$

Since $\tan^{-1} \dfrac{\sin 2\pi\theta}{p-\cos 2\pi\theta} \leq \sin^{-1} \dfrac{1}{p}$, we have

$$E(p,\lambda) \leq e^{\lambda \sin^{-1} \frac{1}{p}}.$$

In addition, we have

$$E(p,\lambda) = \int_0^1 \left(1+\lambda \tan^{-1} \frac{\sin 2\pi\theta}{p-\cos 2\pi\theta} + \frac{(\lambda \tan^{-1} \frac{\sin 2\pi\theta}{p-\cos 2\pi\theta})^2}{2!} + \dots \right) d\theta$$

$$\leq e^{\frac{\lambda^2}{2} (\sin^{-1} \frac{1}{p})^2}.$$

Combining the above results, we obtain

$$E(e^{\lambda \mathbf{Y}}) \leq e^{\lambda \sum_{p \leq \lambda} \sin^{-1} \frac{1}{p}} e^{\frac{\lambda^2}{2} \sum_{p>\lambda} (\sin^{-1} \frac{1}{p})^2}.$$

We have

$$\sum_{p \leq \lambda} \sin^{-1} \frac{1}{p} = \sum_{p \leq \lambda} \frac{1}{p} + O(1) = \log \log \lambda + O(1)$$

and

$$\sum_{p>\lambda} (\sin^{-1} \frac{1}{p})^2 = O\left(\sum_{p>\lambda} \frac{1}{p^2}\right) = O\left(\frac{1}{\lambda \log \lambda}\right).$$

Hence,

$$P(\mathbf{Y} \geq x) \leq e^{-\lambda x + \lambda \log\log \lambda + O(\lambda)}.$$

Thus, we can choose $\lambda = e^{c_4 e^x}$ so that $P(\mathbf{Y} \geq x) \leq e^{-e^{c_2 e^x}}$.

We now obtain the lower bound. From (11), we have $\arg(1-e(\theta)/p)^{-1} \geq \sin^{-1}\frac{1}{p} - t_p$ with probability

$$\frac{1}{2} - \frac{\sin^{-1} p \sin(\sin^{-1}\frac{1}{p} - t_p)}{\pi}.$$

From (13), we have

$$\sin^{-1} p \sin(\sin^{-1}\frac{1}{p} - t_p) \geq \frac{\pi}{2} - B(p, \pi p)\sqrt{\frac{pt_p}{\pi}}$$

for $t_p \leq \frac{1}{p}$. Since $B(p,\pi p) \geq \sqrt{3}$, we see that $\arg(1-e(\theta)/p)^{-1} \geq \sin^{-1}\frac{1}{p} - t_p$ with probability $\geq \sqrt{\frac{3pt_p}{\pi}}$. Write

$$\mathbf{Y} = \sum_{p \leq T} \arg(1-e(\theta)/p)^{-1} + \sum_{p > T} \arg(1-e(\theta)/p)^{-1} = \mathbf{Y}_1 + \mathbf{Y}_2.$$

Then $P(\mathbf{Y} \geq x) \geq P(\mathbf{Y}_1 \geq x)\, P(\mathbf{Y}_2 \geq 0) \geq \frac{1}{2} P(\mathbf{Y}_1 \geq x)$. But we can choose $T = e^{c_3 e^x}$ so that

$$\mathbf{Y}_1 \geq \sum_{p \leq T} \sin^{-1}\frac{1}{p} - \frac{1}{T} = \log\log T + O(1) \geq x$$

with probability

$$\geq \prod_{p \leq T} \sqrt{\frac{3p}{\pi T}} = \left(\frac{3}{\pi T}\right)^{\frac{\pi(T)}{2}} e^{\frac{1}{2}\theta(T)} \geq 2e^{-e^{c_1 e^x}}.$$

Thus,

$$P(\mathbf{Y} \geq x) \geq e^{-e^{c_1 e^x}},$$

which completes the proof of the theorem.

3. Approximation of F(x, A) by an infinite sum. In the previous section, we reduced the problem of determining D(x) to the problem of determining the values of $F(x \pm \delta, A)$. In this section, we establish the following

Theorem 1.9. Let $G(p,\mu) = \sum_{k=0}^{\infty} \prod_{j=0}^{k-1} (j^2 - \frac{\mu^2}{4})/(k!)^2 p^{2k}$, where $\prod_{j=0}^{-1} (j^2 - \frac{\mu^2}{4}) = 1$. If $0 \leq x \leq \frac{3\pi}{2}$, $0 < h \leq \frac{1}{4\pi}$, $A = e^{647.5336}$ and $R = \pi(A)$, then

$$F(x,A) = \frac{1}{2} + h \sum_{n=-\infty}^{\infty} \left(\frac{e(nhx)-1}{2\pi i n h} \prod_{p \leq A} G(p, 2\pi n h) \right).$$

To calculate F(x, A), we write

$$F(x,A) = \frac{1}{2} + \int_{-\infty}^{\infty} \frac{e(\lambda x)-1}{2\pi i \lambda} \hat{F}(\lambda, A) d\lambda,$$

where

$$\hat{F}(\lambda, A) = \int_{-\infty}^{\infty} e(-\lambda x)\, dF(x,A);$$

see Kawata [13, p.128] with a change of variables. Here

$$\hat{F}(\lambda, A) = \int_{I^R} e(-\lambda \mathbf{Y}_A) d\underline{\theta} = \prod_{p \leq A} \int_0^1 e(-\lambda \arg(1-e(\theta)/p)^{-1}) d\theta$$

$$= \prod_{p \leq A} G(p, 2\pi\lambda),$$

where

$$G(p,\mu) = \int_0^1 e^{i\mu \arg(1-e(\theta)/p)} d\theta .$$

But $e^{i \arg z} = \dfrac{z^{\frac{1}{2}}}{\bar{z}^{\frac{1}{2}}}$, so that

$$e^{i\mu \arg(1-e(\theta)/p)} = \left(\frac{1-e(\theta)/p}{1-e(-\theta)/p}\right)^{\mu/2} .$$

Put

$$(1-e(\theta)/p)^{\mu/2} = \sum_{k=0}^{\infty} a_k e(k\theta),$$

where

$$a_k = (-1)^k \binom{\frac{\mu}{2}}{k} / p^k$$

and

$$(1-e(-\theta)/p)^{-\mu/2} = \sum_{l=0}^{\infty} b_l \, e(-l\theta),$$

where

$$b_l = (-1)^l \binom{-\frac{\mu}{2}}{l} / p^l .$$

Then the integral is

$$\int_0^1 (\sum_k a_k e(k\theta))(\sum_l b_l e(-l\theta)) d\theta = \sum_{k=0}^{\infty} a_k b_k .$$

But

$$\binom{\frac{\mu}{2}}{k}\binom{-\frac{\mu}{2}}{k} = \frac{1}{(k!)^2}\prod_{j=0}^{k-1}\left(j^2 - \frac{\mu^2}{4}\right),$$

so that

$$G(p,\mu) = \sum_{k=0}^{\infty}\prod_{j=0}^{k-1}\left(j^2 - \frac{\mu^2}{4}\right)/(k!)^2\, p^{2k}.$$

In order to write $F(x, A)$ as a sum, we use a method introduced by Rice [22]. From Weiss and Stein [31, Corollary 2.6], we take

Lemma 1.10 (Poisson summation formula). Let $\hat{f}(t) = \int_{-\infty}^{\infty} e(-t\lambda)f(\lambda)d\lambda$. If $f(\lambda) = \int_{-\infty}^{\infty}\hat{f}(t)e(t\lambda)dt$, $|f(\lambda)| \leq A(1+|\lambda|)^{-1-\delta}$ and $|\hat{f}(t)| \leq A(1+|t|)^{-1-\delta}$ for some $\delta > 0$, then

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{m=-\infty}^{\infty}\hat{f}(m).$$

If $f(\lambda)$ is integrable and of bounded variation over every finite interval, and $f(\lambda) = \frac{1}{2}(f(\lambda+0)+f(\lambda-0))$ for all λ, then $f(\lambda) = \int_{-\infty}^{\infty}\hat{f}(t)e(t\lambda)dt$ (see Kawata [13, Theorem 4.34]). Thus, we have

Corollary 1.11. If f satisfies the above conditions, and $\hat{f}(t) = 0$ for $|t| \geq \epsilon > 0$, then

$$\int_{-\infty}^{\infty} f(\lambda)d\lambda = h\sum_{n=-\infty}^{\infty} f(nh)$$

for $0 < h \leq 1/\epsilon$.

We need to show that if $0 \leq x \leq \frac{3\pi}{2}$, $|t| \geq 4\pi$ and $f(\lambda) = \frac{e(\lambda x)-1}{2\pi i\lambda}\hat{F}(\lambda, A)$, then $\hat{f}(t) = 0$. In demonstrating this, we make use of the following result (see Whittaker and Watson [32, p. 115]).

Lemma 1.12 (Jordan). Let Γ be the semicircle of radius r and center at the origin which lies above the real axis. If $Q(z) \to 0$ uniformly with respect to $\arg z$ as $|z| \to \infty$ for $0 \leq \arg z \leq \pi$ and $Q(z)$ is analytic when $|z| > c$ and

$0 \le \arg z \le \pi$, then $\lim_{r\to\infty} \int_\Gamma e^{imz} Q(z)dz = 0$ for all positive integers m.

Using these two preliminary lemmas, we prove

Lemma 1.13. Let $f(z) = \frac{e(zx)-1}{2\pi iz} \hat{F}(z,A)$. If $0 \le x \le \frac{3\pi}{2}$ and

$$0 < h \le \frac{1}{\sum_{p\le A} \frac{1}{\sqrt{p^2-1}} + \frac{1}{2\pi} + x},$$

then $\int_{-\infty}^{\infty} f(\lambda)d\lambda = h \sum_{-\infty}^{\infty} f(nh)$.

Proof. By Corollary 1.11, it is sufficient to show that $\hat{f}(t) = 0$ for $|t| \ge \sum_{p\le A} \frac{1}{\sqrt{p^2-1}} + \frac{1}{2\pi} + x$, since $|G(p,\mu)| \le 1$ and $|G(p,\mu)| \ll \mu^{\frac{1}{2}}$ for $\mu \ge \mu_0 \ge \pi$. We have

$$\hat{f}(t) = \int_{-\infty}^{\infty} e(-tz)f(z)dz$$

$$= \lim_{r\to\infty} \int_\Gamma e(-tz)f(z)dz.$$

From Lemma 1.12, we know that the above limit equals zero if

$e(-tz - \frac{z}{2\pi})f(z) \to 0$ uniformly with respect to arg z as $|z| \to \infty$ for

$0 \le \arg z \le \pi$. For $t \le 0$, $z = r\cos\theta + ir\sin\theta$ and $0 \le \theta \le \pi$, we have

(14)

$$\left|e(-tz - \frac{z}{2\pi})\, f(z)\right| = |\exp(-2\pi izt - iz)| \left|\frac{e(zx)-1}{r}\right| \prod_{p\le A} |G(p, 2\pi z)|$$

$$\le \left|\exp\big((2\pi t+1)r\sin\theta\big)\right| \left|\exp\big((-2\pi t-1)ir\cos\theta\big)\right| \frac{|\exp(2\pi ixr\cos\theta)||\exp(2\pi xr\sin\theta)|+1}{r} \prod_{p\le A} |G(p, 2\pi z)|$$

$$\le \frac{2\exp(r\sin\theta\,(1+2\pi x+2\pi t))}{r} \prod_{p\le A} |G(p, 2\pi z)|.$$

In addition,

(15)
$$G(p, 2\pi z) = \int_0^1 e^{2\pi iz \arg\left(1 - \frac{e(u)}{p}\right)} du$$

$$= \int_0^1 e^{2\pi iz \tan^{-1} \frac{\sin 2\pi u}{p-\cos 2\pi u}} du$$

and for $0 \le u \le 1$

(16)
$$\left|\tan^{-1} \frac{\sin 2\pi u}{p-\cos 2\pi u}\right| \le \tan^{-1} \frac{1}{\sqrt{p^2-1}} \le \frac{1}{\sqrt{p^2-1}}.$$

Combining (15) and (16), we obtain

(17)
$$|G(p, 2\pi z)| \le \max_{0 \le t \le 1} \left| e^{2\pi iz \tan^{-1} \frac{\sin 2\pi t}{p-\cos 2\pi t}} \right|$$

$$\le \exp\left(\frac{2\pi r \sin\theta}{\sqrt{p^2-1}}\right).$$

Using (17) in (14), we obtain

$$e(-tz - \frac{z}{2\pi})f(z) \le \frac{2\exp\left(2\pi r\sin\theta\left(\sum_{p<A} \frac{1}{\sqrt{p^2-1}} + \frac{1}{2\pi} + x + t\right)\right)}{r}.$$

Thus, $\left|e(-tz - \frac{z}{2\pi})f(z)\right| \to 0$ uniformly with respect to arg z as $|z| \to \infty$ for $0 \le \arg z \le \pi$ if $t \le -\left(\sum_{p \le A} \frac{1}{\sqrt{p^2-1}} + \frac{1}{2\pi} + x\right)$, which shows that $\hat{f}(t) = 0$ for these t. For $t \ge 0$, we have

$$\int_{-\infty}^{\infty} e(-tz)f(z)dz = -\int_{\infty}^{\infty} e(tz)\frac{e(-zx)-1}{2\pi iz}\prod_{p\le A} G(p, 2\pi z)dz,$$

and proceeding as before, we obtain $\hat{f}(t) = 0$ for $t \geq \sum_{p\le A} \frac{1}{\sqrt{p^2-1}} + \frac{1}{2\pi} + x$.

This completes the proof of Lemma 1.13. Finally, we need

Lemma 1.14. If $0 \leq x \leq \frac{3\pi}{2}$ and $A = e^{647.5336}$, then

$$\frac{1}{4\pi} \leq \frac{1}{\sum_{p\le A} \frac{1}{\sqrt{p^2-1}} + \frac{1}{2\pi} + x}.$$

Proof. Put

$$\sum_{p\le A} \frac{1}{\sqrt{p^2-1}} = \sum_{p\le 1010} \frac{1}{\sqrt{p^2-1}} + \sum_{1010<p\le A} \frac{1}{\sqrt{p^2-1}} = S_1 + S_2.$$

We obtain $S_1 \leq 2.30316$ by direct calculation. We have

$$S_2 \leq \frac{1010}{\sqrt{(1010)^2-1}} \sum_{1010<p\le A} \frac{1}{p} \leq 1.0000005 \sum_{1010<p\le A} \frac{1}{p}.$$

Using Lebesgue-Stieltjes integration, we write the last quantity as

$$1.0000005\left(\frac{\theta(A)}{A \log A} - \frac{\theta(1010)}{1010 \log 1010} + \int_{1010}^{A^+} \frac{(\log u+1)\,\theta(u)du}{u^2\log^2 u}\right).$$

Schoenfeld [24, p.360] has shown that $\theta(u) \leq (1.000081)u$ for $u \geq 2$ and we have calculated that $\theta(1010) \geq 963.149658$. Thus, the above expression is less than 4.55. Hence,

$$\sum_{p\le A} \frac{1}{\sqrt{p^2-1}} \leq 6.8532$$

and

$$\frac{1}{4\pi} < \frac{1}{\sum\limits_{p\leq A} \frac{1}{p^2-1} + \frac{1}{2\pi} + x}$$

for $0 \leq x \leq \frac{3\pi}{2}$, which completes the proof of the lemma.

The theorem follows directly from Lemma 1.13 and Lemma 1.14.

<u>4. The calculation of F(x,A).</u> In Theorem 1.9 we reduced the problem of evaluating $F(x,A)$ for $0 \leq x \leq \frac{3\pi}{2}$ to that of evaluating

$$\frac{1}{2} + h \sum_{n=-\infty}^{\infty} \frac{e(nhx)-1}{2\pi inh} \prod_{p\leq A} G(p, 2\pi nh)$$

when $0 < h \leq \frac{1}{4\pi}$ and $A = e^{647.5336}$. For $n = 0$, we have

$$\lim_{n\to 0} h \frac{e(nhx)-1}{2\pi inh} \prod_{p\leq A} G(p, 2\pi nh) = hx.$$

For $n \geq 0$, we have

$$\frac{e(nhx)-1}{2\pi inh} \prod_{p\leq A} G(p, 2\pi nh) + \frac{e(-nhx)-1}{-2\pi inh} \prod_{p\leq A} G(p, -2\pi nh)$$

$$= \frac{2 \sin 2\pi nhx}{2\pi nh} \prod_{p\leq A} G(p, 2\pi nh).$$

Using the above equalities, we obtain

$$F(x,A) = \frac{1}{2} + hx + 2h \sum_{n=1}^{\infty} \frac{\sin 2\pi nhx}{2\pi nh} \prod_{p\leq A} G(p, 2\pi nh).$$

We now take $h = \frac{1}{4\pi}$. We first truncate the above sum. By taking $\mu_0 = 800$ in Lemma 1.8, we obtain

$$\left|\frac{1}{2\pi}\sum_{n=1601}^{\infty}\frac{\sin nx/2}{n/2}\prod_{p\le A} G(p,n/2)\right| \le \frac{1}{2\pi}\sum_{n=1601}^{\infty}\frac{\prod_{p\le 241}\frac{2}{\pi}B(p,800)\sqrt{\frac{p}{n/2}}}{n/2}$$

$$\le \frac{(800)^{53/2}}{2\pi}\int_{800}^{\infty}\frac{\prod_{p\le 241}\frac{2}{\pi}\sqrt{6-\sqrt{36-\frac{24\pi p}{800}-\frac{12\pi^2}{800^2}}}\,d\mu}{\mu^{55/2}}$$

$$\le 0.5\cdot 10^{-16}.$$

Using this, we obtain

$$F(x,A) = \frac{1}{2}+\frac{x}{4\pi}+\frac{1}{2\pi}\sum_{n=1}^{1600}\frac{\sin nx/2}{n/2}\prod_{p\le A} G(p,n/2)+\epsilon,$$

with $|\epsilon| \le 0.5\cdot 10^{-16}$. To further truncate our sum, we need to compute $G(p,\mu)$. The method we use is justified by

Lemma 1.15. Let $L = \left[\frac{\mu}{\sqrt{2}}\right]$ and $f(p,\mu,N) = \frac{\left(\frac{\mu}{2}\right)^{2(N+1)}e^{2(N+1)}}{2\pi(N+1)^{2N+3}p^{2(N+1)}}$.

If we choose

$$N = \begin{cases} L & \text{if } f(p,\mu,L) = 10^{-16}, \\ \left[\dfrac{8\log 10+\left(\left[\frac{\mu}{\sqrt{2}}\right]+1\right)\log\left(\dfrac{\mu e}{2\left(\left[\frac{\mu}{\sqrt{2}}\right]+1\right)}\right)-\frac{1}{2}\log 2\pi\left(\left[\frac{\mu}{\sqrt{2}}\right]+1\right)}{\log p}\right] & \text{if } f(p,\mu,L) > 10^{-16}, \\ \text{greatest integer } k \text{ such that} \left(\dfrac{\mu e}{2(k+1)p}\right)^{2(k+1)}\dfrac{1}{2\pi(k+1)} < 10^{-16} & \\ & \text{if } f(p,\mu,L) < 10^{-16}, \end{cases}$$

then

$$G(p,\mu) = \sum_{k=0}^{N} \frac{\prod_{j=0}^{k-1}\left(j^2 - \frac{\mu^2}{4}\right)}{(k!)^2 p^{2k}} + R_N,$$

where $\left|R_N\right| \le \frac{4}{3} \cdot 10^{-16}$.

Proof. Put

$$u_k = \frac{\prod_{j=0}^{k-1}\left(j^2 - \frac{\mu^2}{4}\right)}{(k!)^2 p^{2k}} = \frac{-1}{k^2}\left(\frac{\mu^2}{4}\right)\left(1 - \frac{\mu^2}{4}\right)\left(1 - \frac{\mu^2}{4(2)^2}\right)\cdots\left(1 - \frac{\mu^2}{4(k-1)^2}\right)\frac{1}{p^{2k}}$$

and

$$T_N = \max\left(\frac{\mu^2}{4(N+1)^2 p^2}\left(\frac{1}{1 - \frac{\mu^2}{4(N+1)^2 p^2}}\right), \frac{1}{p^2}\left(\frac{1}{1 - \frac{1}{p^2}}\right)\right).$$

We first show that

(18) $$u_{N+1}(1 - \mathrm{sgn}(u_{N+1})T_N \le R_N \le u_{N+1}(1 + \mathrm{sgn}(u_{N+1})T_N).$$

We have

$$R_N = u_{N+1}\left(1 + \frac{(N+1)^2}{(N+2)^2}\left(1 - \frac{\mu^2}{4(N+1)^2}\right)\frac{1}{p^2} + \cdots\right).$$

If we put

$$S_N = \frac{(N+1)^2}{(N+2)^2}\left|1-\frac{\mu^2}{4(N+1)^2}\right|\frac{1}{p^2} + \frac{(N+1)^2}{(N+3)^2}\left|1-\frac{\mu^2}{4(N+1)^2}\right|\left|1-\frac{\mu^2}{4(N+2)^2}\right|\frac{1}{p^4} + \ldots,$$

then

$$u_{N+1}(1-S_N) \le R_N \le u_{N+1}(1+S_N) \quad (u_{N+1} \ge 0)$$

and

$$u_{N+1}(1+S_N) \le R_N \le u_{N+1}(1-S_N) \quad (u_{N+1} < 0).$$

Obviously,

$$S_N \le \left|1-\frac{\mu^2}{4(N+1)^2}\right|\frac{1}{p^2} + \left|1-\frac{\mu^2}{4(N+2)^2}\right|\frac{1}{p^4} + \ldots .$$

Less obviously, we have, for all $M \ge N$,

$$\left|1-\frac{\mu^2}{4(M+1)^2}\right| \le \max\left(\frac{\mu^2}{4(M+1)^2}, 1\right) \le \begin{cases} \dfrac{\mu^2}{4(N+1)^2} & \text{if } N+1 \le \dfrac{\mu}{2}, \\ 1 & \text{otherwise}. \end{cases}$$

This is certainly true if $M \le \frac{\mu}{2}$, since

$$\left|1-\frac{\mu^2}{4M^2}\right| = \frac{\mu^2}{4M^2} - 1 \le \frac{\mu^2}{4M^2}.$$

When $\frac{\mu}{2} \le M \le \frac{\mu}{\sqrt{2}}$, we have

$$\left|1-\frac{\mu^2}{4M^2}\right| \le 1-\frac{\mu^2}{4(\mu/\sqrt{2})^2} \le \frac{\mu^2}{4(\mu/\sqrt{2})^2} \le \frac{\mu^2}{4M^2}.$$

Finally, if $M \geq \frac{\mu}{2\sqrt{2}}$, then

$$\left|1 - \frac{\mu^2}{4M^2}\right| \leq 1.$$

From the above, we have

$$S_N \leq \begin{cases} \frac{\mu^2}{4(N+1)^2 p^2} + \left(\frac{\mu^2}{4(N+1)^2 p^2}\right)^2 + \ldots & \text{if } N+1 \leq \frac{\mu}{2}, \\ \frac{1}{p^2} + \frac{1}{p^4} + \ldots & \text{otherwise,} \end{cases}$$

so that (18) is true. We have

$$|u_{N+1}| \leq \begin{cases} \frac{1}{(N+1)^2}\left(\frac{\mu^2}{4}\right)\left(\frac{\mu^2}{4}\right)\left(\frac{\mu^2}{4(2)^2}\right)\cdots\left(\frac{\mu^2}{4N^2}\right)\frac{1}{p^{2(N+1)}} \\ \leq \frac{\left(\frac{\mu}{2}\right)^{2(N+1)}}{((N+1)!)^2 p^{2(N+1)}} \quad \text{if } N \leq \frac{\mu}{\sqrt{2}}, \\ \frac{1}{(N+1)^2}\left(\frac{\mu^2}{4}\right)\left(\frac{\mu^2}{4}\right)\left(\frac{\mu^2}{4(2)^2}\right)\cdots\left(\frac{\mu^2}{4([\frac{\mu}{2}])^2}\right)\frac{1}{p^{2(N+1)}} \\ \leq \frac{\left(\frac{\mu}{2}\right)^{2([\frac{\mu}{2}]+1)}}{(([\frac{\mu}{2}]+1)!)^2 p^{2(N+1)}} \quad \text{if } N > \frac{\mu}{\sqrt{2}}. \end{cases}$$

Since $(M-1)! = M^{M-\frac{1}{2}} e^{-M} (2\pi)^{\frac{1}{2}} e^{s/12M}$ for $0 \le s \le 1$ (see Whittaker and Watson [32, p. 253]), we have $M! \ge M^{M+\frac{1}{2}} e^{-M} (2\pi)^{\frac{1}{2}}$ and thus,

$$|u_{N+1}| \le \begin{cases} \dfrac{\left(\frac{\mu}{2}\right)^{2(N+1)} e^{2(N+1)}}{2\pi (N+1)^{2N+3} p^{2(N+1)}} & \text{if } N \le \dfrac{\mu}{\sqrt{2}} \\[2ex] \dfrac{\left(\frac{\mu}{2}\right)^{2\left(\left[\frac{\mu}{\sqrt{2}}\right]+1\right)} e^{2\left(\left[\frac{\mu}{\sqrt{2}}\right]+1\right)}}{2\pi\left(\left[\frac{\mu}{\sqrt{2}}\right]+1\right)^{2\left[\frac{\mu}{\sqrt{2}}\right]+3} p^{2(N+1)}} & \text{if } N > \dfrac{\mu}{\sqrt{2}} . \end{cases}$$

If we choose N according to our algorithm, then

$$N \ge \left[\frac{\mu e}{2p}\right], \; N+1 \ge \frac{\mu e}{2p} \text{ and } T_N \le \frac{1}{3} .$$

Thus, our algorithm insures that $|R_N| \le \frac{4}{3} \cdot 10^{-16}$.

Using (12) to bound $|G(p,\mu)|$ for $p \le 101$, and Lemma 1.15 to compute $|G(p,\mu)|$ for $101 < p \le 1871$, we obtain

$$\left| \frac{1}{2\pi} \sum_{n=401}^{1600} \frac{\sin nx/2}{n/2} \prod_{p \le A} G(p, n/2) \right| \le 0.44249 \cdot 10^{-16}.$$

From this, we obtain

$$F(x,A) = \frac{1}{2} + \frac{x}{4\pi} + \frac{1}{2\pi} \sum_{n=1}^{400} \frac{\sin xn/2}{n/2} \prod_{p \le A} G(p, n/2) + \epsilon ,$$

where $|\epsilon| \le 10^{-16}$.

Since we cannot compute $G(p,\mu)$ for all $p \leq A$, we need

Theorem 1.16. Let

$$G(p,\mu) = \sum_{k=0}^{\infty} u_k,\ S_L(p,\mu) = \sum_{k=1}^{5} u_k + u_6(1-\mathrm{sgn}(u_6)(2.1\cdot 10^{-10}))$$

and

$$S_H(p,\mu) = \sum_{k=1}^{5} u_k + u_6(1+\mathrm{sgn}(u_6)(2.1\cdot 10^{-10})).$$

Let

$$B_L(p,\mu) = \sum_{m=1}^{36} \underline{b}(m,\mu)p^{-2m}$$

$$= S_L(p,\mu) - \frac{S_L(p,\mu)^2}{2} + \frac{S_L(p,\mu)^3}{3} - \frac{S_L(p,\mu)^4}{4} + \frac{S_L(p,\mu)^5}{5} - \frac{1.00000001497\ S_L(p,\mu)^6}{6}$$

and

$$B_H(p,\mu) = \sum_{m=1}^{36} \bar{b}(m,\mu)p^{-2m}$$

$$= S_H(p,\mu) - \frac{S_H(p,\mu)^2}{2} + \frac{S_H(p,\mu)^3}{3} - \frac{S_H(p,\mu)^4}{4} + \frac{S_H(p,\mu)^5}{5} - \frac{S_H(p,\mu)^6}{6}.$$

Let V = 1155901,

$$I_L(n) = \frac{0.998697 - \frac{\theta(V)}{V}}{V^{2n-1}\log V} - \frac{0.998697 - \frac{\theta(A)}{A}}{A^{2n-1}\log A}$$

$$+ 0.998697(E_1((2n-1)\log V) - E_1((2n-1)\log A))$$

and

$$I_H(n) = \frac{1 - \frac{\theta(V)}{V}}{V^{2n-1}\log V} + \frac{0.000081}{(10^{11})^{2n-1}\log 10^{11}} + E_1((2n-1)\log V)$$

$$+ 0.000081\, E_1((2n-1)\log 10^{11}).$$

Let

$$J_L(n,\mu) = \begin{cases} I_L(n) & \text{if } \underline{b}(n,\mu) \geq 0, \\ I_H(n) & \text{if } \underline{b}(n,\mu) < 0, \end{cases}$$

$$J_H(n,\mu) = \begin{cases} I_H(n) & \text{if } \bar{b}(n,\mu) \geq 0, \\ I_L(n) & \text{if } \bar{b}(n,\mu) < 0, \end{cases}$$

$$Q_L(\mu) = \sum_{n=1}^{36} \underline{b}(n,\mu)\, J_L(n,\mu)$$

and

$$Q_H(\mu) = \sum_{n=1}^{36} \bar{b}(n,\mu)\, J_H(n,\mu).$$

Let

$$R_L(\mu) = \begin{cases} \exp(Q_L(\mu)) & \text{if } \dfrac{\sin x\mu}{\mu} \prod\limits_{2 \le p \le V} G(p,\mu) \ge 0, \\ \exp(Q_H(\mu)) & \text{otherwise} \end{cases}$$

and

$$R_H(\mu) = \begin{cases} \exp(Q_H(\mu)) & \text{if } \dfrac{\sin x\mu}{\mu} \prod\limits_{2 \le p \le V} G(p,\mu) \ge 0, \\ \exp(Q_L(\mu)) & \text{otherwise.} \end{cases}$$

If $0 \le x \le \frac{3\pi}{2}$, then

$$\frac{1}{2} + \frac{x}{4\pi} + \frac{1}{2\pi} \sum_{n=1}^{400} \frac{\sin nx/2}{n/2} \prod_{2 \le p \le V} G(p, n/2) R_L(n/2) - \epsilon \le F(x, A)$$

$$\le \frac{1}{2} + \frac{x}{4\pi} + \frac{1}{2\pi} \sum_{n=1}^{400} \frac{\sin nx/2}{n/2} \prod_{2 \le p \le V} G(p, n/2)\, R_H(n/2) + \epsilon ,$$

where $0 \le \epsilon \le 10^{-16}$. To prove Theorem 1.16, we need the following auxilliary results.

Lemma 1.17. If $0 \le y < 1$ and $M \in \mathbb{Z}^+$, then

$$-\sum_{m=1}^{M} \frac{y^m}{m} - \frac{y^{M+1}}{M+1}\left(\frac{1}{1-y}\right) \le \log(1-y) \le -\sum_{m=1}^{M} \frac{y^m}{m} - \frac{y^{M+1}}{M+1}\left(\frac{1}{1-(y/2)}\right).$$

Proof. For $0 \le y < 1$, we have

$$\log(1-y) = -\sum_{m=1}^{M} \frac{y^m}{m} - \sum_{m=M+1}^{\infty} \frac{y^m}{m} .$$

The lemma follows from the fact that

$$\sum_{m=M+1}^{\infty} \frac{y^m}{m} \le \frac{y^{M+1}}{M+1}\left(\frac{1}{1-y}\right)$$

and

$$\sum_{m=M+1}^{\infty} \frac{y^m}{m} \ge \sum_{m=M+1}^{\infty} \frac{y^m}{(M+1)2^{(M+1)-m}} \ge \frac{y^{M+1}}{M+1}\left(\frac{1}{1-(y/2)}\right).$$

Lemma 1.18. If $0 \le \mu \le 200$, then

$$\exp\left(\sum_{n=1}^{36} \underline{b}(n,\mu) \int_{V^+}^{A} \frac{d\theta(u)}{u^{2n}\log u}\right) \le \prod_{V<p<A} G(p,\mu)$$

$$\le \exp\left(\sum_{n=1}^{36} \bar{b}(n,\mu) \int_{V^+}^{A} \frac{d\theta(u)}{u^{2n}\log u}\right).$$

Proof. From (16), we obtain

$$1+\sum_{k=1}^{5} u_k+u_6(1-\operatorname{sgn}(u_6)T_5) \le G(p,\mu) \le 1+\sum_{k=1}^{5} u_k+u_6(1+\operatorname{sgn}(u_6)T_5).$$

Since $T_5 \le 2.1 \cdot 10^{-10}$, we have

$$1+S_L(p,\mu) \le G(p,\mu) \le 1+S_H(p,\mu).$$

Since $0 \le -S_H(p,\mu) \le -S_L(p,\mu) \le 7.485 \cdot 10^{-9}$, we can use the above lemma to obtain

$$B_L(p,\mu) \leq \log G(p,\mu) \leq B_H(p,\mu).$$

To complete the proof of Lemma 1.18, we note that

$$\sum_{V<p<A} B_L(p,\mu) = \sum_{n=1}^{36} \underline{b}(n,\mu) \sum_{V<p<A} p^{-2n} = \sum_{n=1}^{36} \underline{b}(n,\mu) \int_{V^+}^{A} \frac{d\theta(u)}{u^{2n}\log u}$$

and

$$\sum_{V<p<A} B_H(p,\mu) = \sum_{n=1}^{36} \bar{b}(n,\mu) \int_{V^+}^{A} \frac{d\theta(u)}{u^{2n}\log u}.$$

To bound the integrals in the above lemma, we have

Lemma 1.19. If $n \geq 1$, then

$$I_L(n) \leq \int_{V^+}^{A} \frac{d\theta(u)}{u^{2n}\log u} \text{ and } \int_{V^+}^{A} \frac{d\theta(u)}{u^{2n}\log u} \leq I_H(n).$$

Proof. Schoenfeld [24, p360] has shown that $\theta(u) < (1.000081)u$ for all $u > 0$ and $\theta(u) < u$ for $0 < u \leq 10^{11}$. Using this result, we obtain

$$\int_{V^+}^{A} \frac{d\theta(u)}{u^{2n}\log u} = \int_{V^+}^{10^{11}} \frac{d\theta(u)}{u^{2n}\log u} + \int_{10^{11}}^{A} \frac{d\theta(u)}{u^{2n}\log u}$$

$$= \frac{\theta(10^{11})}{(10^{11})^{2n}\log 10^{11}} - \frac{\theta(V)}{V^{2n}\log V} + \int_{V^+}^{10^{11}} \frac{2n\,\theta(u)du}{u^{2n+1}\log^2 u}$$

$$+ \int_{V^+}^{10^{11}} \frac{\theta(u)\,du}{u^{2n+1}\log^2 u}$$

$$+ \frac{\theta(A)}{A^{2n}\log 10^{11}} - \frac{\theta(10^{11})}{(10^{11})^{2n}\log 10^{11}} + \int_{10^{11}}^{A} \frac{2n\theta(u)du}{u^{2n+1}\log u}$$

$$+ \int_{10^{11}}^{A} \frac{\theta(u)du}{u^{2n+1}\log^2 u}$$

$$\leq I_H(n).$$

The lower bound is obtained similarly.

Using Lemma 1.18 and Lemma 1.19, we have

$$\exp(Q_L(\mu)) = \exp \sum_{n=1}^{36} \underline{b}(n,\mu)J_L(n,\mu)) \leq \prod_{V<p<A} G(p,\mu)$$

$$\leq \exp \sum_{n=1}^{36} \bar{b}(n,\mu)J_H(n,\mu)) = \exp(Q_H(\mu)),$$

which completes the proof of Theorem 1.16.

In computing F(x, A), we use the fact that $\theta(1155901) = 1150824.716$.

In addition, we use the following result from Abramowitz and Stegun [1, p. 231]:

<u>Lemma 1.20.</u> Let

$$a_1 = 8.5733287401,$$

$$a_2 = 18.0590169730,$$

$$a_3 = 8.6347608925,$$

$$a_4 = 0.2677737343,$$

$$b_1 = 9.5733223454,$$

$$b_2 = 25.6329561486,$$

$$b_3 = 21.0996530827 \quad \text{and}$$

$$b_4 = 3.9584969228.$$

Then

$$E_1(v) = \int_v^\infty \frac{e^{-y}dy}{y} = \frac{1}{ve^v}\left(\frac{v^4+a_1v^3+a_2v^2+a_3v+a_4}{v^4+b_1v^3+b_2v^2+b_3v+b_4} + \epsilon(v)\right),$$

where $|\epsilon(v)| \le 2\cdot 10^{-8}$.

Since we compute $G(p,\mu)$ with an error of $\le \frac{4}{3}\cdot 10^{-16}$, the relative error in computing $\prod_{2\le p\le V} G(p,\mu)$ is less than 10^{-10}. Thus, we can compute the bounds in Theorem 1.16 with an absolute error $< 1.9\cdot 10^{-9}$. Taking $\delta = e^{-40}$ in Lemma 1.7, we see that we can compute $D(x)$ with an absolute error $< 2\cdot 10^{-9}$.

<u>5. Calculation of $L(1,\chi)$.</u> It is well known (see Davenport [6, pp. 67-69]) that

$$\chi(n) = \frac{1}{\tau(\bar{\chi})}\sum_{m=1}^{q} \bar{\chi}(m)\, e(mn/q),$$

where q is a prime and

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m)\, e(m/q).$$

Hence

$$L(1,\chi) = \frac{1}{\tau(\bar{\chi})} \sum_{m=1}^{q} \bar{\chi}(m) \sum_{n=1}^{\infty} e(mn/q)/n$$

$$= \frac{-1}{\tau(\bar{\chi})} \sum_{m=1}^{q-1} \bar{\chi}(m) \log(1-e(m/q)).$$

Suppose that $\chi(-1) = -1$. We have

$$\bar{\chi}(m)\log(1-e(m/q))+\bar{\chi}(q-m)\log(1-e((-m)/q)) = \bar{\chi}(m)\log\left(\frac{1-e(m/q)}{1-e(-m/q)}\right).$$

But $(1-e(m/q))/(1-e(-m/q)) = e^{\pi i(-1+\frac{2m}{q})}$, so the above is

$$\bar{\chi}(m)\pi i(-1+\frac{2m}{q}).$$

Thus,

$$L(1,\chi) = \frac{-1}{2\tau(\bar{\chi})} \sum_{m=1}^{q-1} \bar{\chi}(m)\, \pi i\left(-1+\frac{2m}{q}\right)$$

$$= -\frac{\pi i}{\tau(\bar{\chi})q} \sum_{m=1}^{q-1} m\bar{\chi}(m).$$

Suppose that $\chi(-1) = 1$. We have

$$\bar{\chi}(m) \log(1-e(m/q)) + \bar{\chi}(q-m) \log(1-e((-m/q)) = 2\bar{\chi}(m) \log|1-e(m/q)|.$$

But $|1-e(m/q)| = 2 \sin \frac{\pi m}{q}$, so the above is

$$2\bar{\chi}(m)\left(\log 2 + \log \sin \frac{\pi m}{q}\right).$$

Hence

$$L(1,\chi) = \begin{cases} \dfrac{-1}{\tau(\bar{\chi})} \displaystyle\sum_{m=1}^{q-1} \bar{\chi}(m) \log \sin \frac{\pi m}{q} & \text{if } \bar{\chi}(-1) = 1, \\ \dfrac{-\pi i}{\tau(\bar{\chi})q} \displaystyle\sum_{m=1}^{q-1} m\bar{\chi}(m) & \text{if } \bar{\chi}(-1) = -1. \end{cases}$$

Given a primitive root g mod q, we compute

$$(19)\quad \tau(\bar{\chi}_k) = \sum_{r=1}^{q-1} e\left(\frac{-kr}{q-1}\right) e((g^r \bmod q)/q),$$

$$(20)\quad L(1,\chi_k) = \begin{cases} \dfrac{-1}{\tau(\bar{\chi}_k)} \displaystyle\sum_{r=1}^{q-1} e\left(\frac{-kr}{q-1}\right) \log \sin \frac{\pi (g^r \bmod q)}{q} & \text{if } \bar{\chi}(-1) = 1, \\ \dfrac{-\pi i}{\tau(\bar{\chi}_k)q} \displaystyle\sum_{r=1}^{q-1} e\left(\frac{-kr}{q-1}\right)(g^r \bmod q) & \text{if } \bar{\chi}(-1) = -1, \end{cases}$$

for $1 \le k \le \frac{q-1}{2}$.

CHAPTER II

THE DISTRIBUTION OF THE ERROR TERM IN THE PRIME NUMBER THEOREM

1. Statement of results. When we examine the distribution of prime numbers, $\psi(u)$ is often a more natural function to use than $\pi(u)$. If we define

$$\psi_0(u) = \begin{cases} \psi(u) & \text{if } u \neq p^m, \\ \psi(u) - \frac{1}{2}\Lambda(u) & \text{otherwise,} \end{cases}$$

then we have the von Mangoldt formula

$$\psi_0(u) = u - \sum_{\rho} \frac{u^{\rho}}{\rho} - \log 2\pi - \tfrac{1}{2}\log(1-u^{-2})$$

for $u > 1$, where the sum is taken over all non-trivial zeros of $\zeta(s)$. We will examine

$$H(u) = \frac{\psi_0(u) - u + \log 2\pi + \frac{1}{2}\log(1-u^{-2})}{u^{\frac{1}{2}}} = -\sum_{\rho} \frac{u^{\rho-\frac{1}{2}}}{\rho}$$

assuming the Riemann hypothesis and the rational linear independence of the positive imaginary parts of the non-trivial zeros of $\zeta(s)$.

Given the Riemann hypothesis, we write the non-trivial zeros of $\zeta(s)$ in the form $\rho = \frac{1}{2} + i\gamma$, and obtain

$$H(u) = -\sum_{\gamma} \frac{u^{i\gamma}}{\frac{1}{2}+i\gamma}.$$

Setting $u = e^{v}$, we write

(1)
$$H(u) = h(v) = -\sum_{\gamma>0} \left(\frac{\exp(i\gamma v)}{\frac{1}{2}+i\gamma} + \frac{\exp(-i\gamma v)}{\frac{1}{2}-i\gamma} \right)$$

$$= -\sum_{\gamma>0} \frac{\exp(i(\gamma v - \arg \rho))}{|\rho|} - \sum_{\gamma>0} \frac{\exp(-i(\gamma v - \arg \rho))}{|\rho|} = h_1(v) + h_2(v).$$

This function is related to the probability distribution of a random variable in

Theorem 2.1. Let $\Psi = \sum_{\gamma>0} \frac{2 \sin 2\pi\, \theta_\gamma}{|\rho|}$, where the θ_γ are independent and uniformly distributed in $[0, 1)$ (almost sure convergence follows by arguments analogous to those used in Theorem 1.1). Let $G(x)$ be the distribution function of Ψ. If the $\gamma > 0$ are linearly independent over the rationals, then for every real x,

$$\lim_{V\to\infty} \frac{1}{V} \operatorname{meas}\{v : 0 \le v \le V,\ h(v) \le x\} = G(x).$$

We need several preliminary results. From Wintner and Jessen [33, Theorem 29] we have

Lemma 2.2. Let Φ_j be the circular equidistribution on $S_j = \{z : |z| = r_j\}$, where for any Borel set E, $\Phi_j(E) = \mu(S_j \cap E)/\mu(S_j)$ (μ is one-dimensional measure). Let $r_1, r_2, \ldots, r_n$ be positive, $\lambda_1, \lambda_2, \ldots, \lambda_n$ be linearly independent and $\delta_1, \delta_2, \ldots, \delta_n$ be real. Then the asymptotic distribution of

$$s_n(t) = r_1\exp(i(\lambda_1 t + \delta_1)) + r_2\exp(i(\lambda_2 t + \delta_2)) + \cdots + r_n\exp(i(\lambda_n t + \delta_n))$$

is the distribution $\psi_n = \Phi_1 * \Phi_2 * \cdots * \Phi_n$.

From Wintner and Jessen [33, Theorem 7] we have

Lemma 2.3. The convergence of the series $r_1^2 + r_2^2 + \cdots$ is necessary and sufficient both for the convergence and the absolute convergence of the infinite convolution $\Phi_1 * \Phi_2 * \cdots$.

Since $N(T) << T \log T$, we have

$$\sum_{\gamma>0} \frac{1}{|\rho|^2} \leq \sum_{\gamma>0} \frac{1}{\gamma^2} << \sum_{n=1}^{\infty} \frac{\log n}{n^2} < \infty .$$

Using the previous two lemmas we see that the asymptotic distribution of $h_1(v)$ is $\Phi_1 * \Phi_2 * \cdots$ and the asymptotic distribution of $h_2(v)$ is $\overline{\Phi}_1 * \overline{\Phi}_2 * \cdots$, where Φ_j is the distribution of $\frac{e(\theta\gamma_j)}{|\rho_j|}$. Hence, $h(v)$ has asymptotic distribution $\Phi_1 * \overline{\Phi}_1 * \Phi_2 * \overline{\Phi}_2 * \cdots$. But the distribution $\Phi_j * \overline{\Phi}_j$ is the distribution of $\frac{2\sin 2\pi\theta\gamma_j}{|\rho_j|}$. Thus, the asymptotic distribution of $h(v)$ is the distribution of Υ, which completes the proof of the theorem.

The following table contains typical values for $G(x)$:

x	G(x)	Maximum possible error
-1.0	0.000000263	0.000000194
-0.8	0.000047207	0.0000002
-0.6	0.00213582	0.0000002
-0.4	0.0309492	0.0000002
-0.2	0.178533	0.000001
0.0	0.5	0.0
0.2	0.821467	0.000001
0.4	0.9690508	0.0000002
0.6	0.99786418	0.0000002
0.8	0.999952793	0.0000002
1.0	0.999999737	0.000000194

We obtain these results using computer programs (Appendix B) based on

Theorem 2.12. A graph of G(x) for $0 \le x \le 0.8$ appears in Figure 2. Montgomery [18] has shown that G(x) is continuous,

$$G(-x) + G(x) = 1 \text{ and for } x \ge 1$$

$$\exp\left(-c_1\sqrt{x}\, e^{\sqrt{2\pi x}}\right) \le 1 - G(x) \le \exp\left(-c_2\sqrt{x}\, e^{\sqrt{2\pi x}}\right).$$

Here c_1 and c_2 are positive absolute constants.

For large values of u, we show in section 4 that

$$G'(u) = \frac{\exp(f(T))}{\sqrt{8\pi f''(T)}} \left(1 + O\left(\frac{1}{\sqrt{T \log T}}\right)\right),$$

where

$$f(z) = \sum_{\gamma > 0} \log I_0\left(\frac{z}{|\rho|}\right) - zu/2$$

and T is chosen so that $f'(T) = 0$.

Using this formula, we can find constants a_1, a_2, a_3 and a_4 such that

$$G'(u) = \exp\left(-\left(\sqrt{a_1 u + a_2} - a_3\right) e^{\sqrt{2\pi u + a_4}} + O(\sqrt{u})\right).$$

2. Approximation of G(x) by an infinite sum. In this section, we first establish

Theorem 2.4. Let G(x, B) be the distribution function of $\Psi_B = \sum_{0 < \gamma \le B} \frac{2\sin 2\pi\theta_\gamma}{|\rho|}$. If $0 \le x \le 1$, $B \ge e^{58.223085}$ and $R = N(B)$, then

$$|G(x) - G(x, B)| < 10^{-12.24}.$$

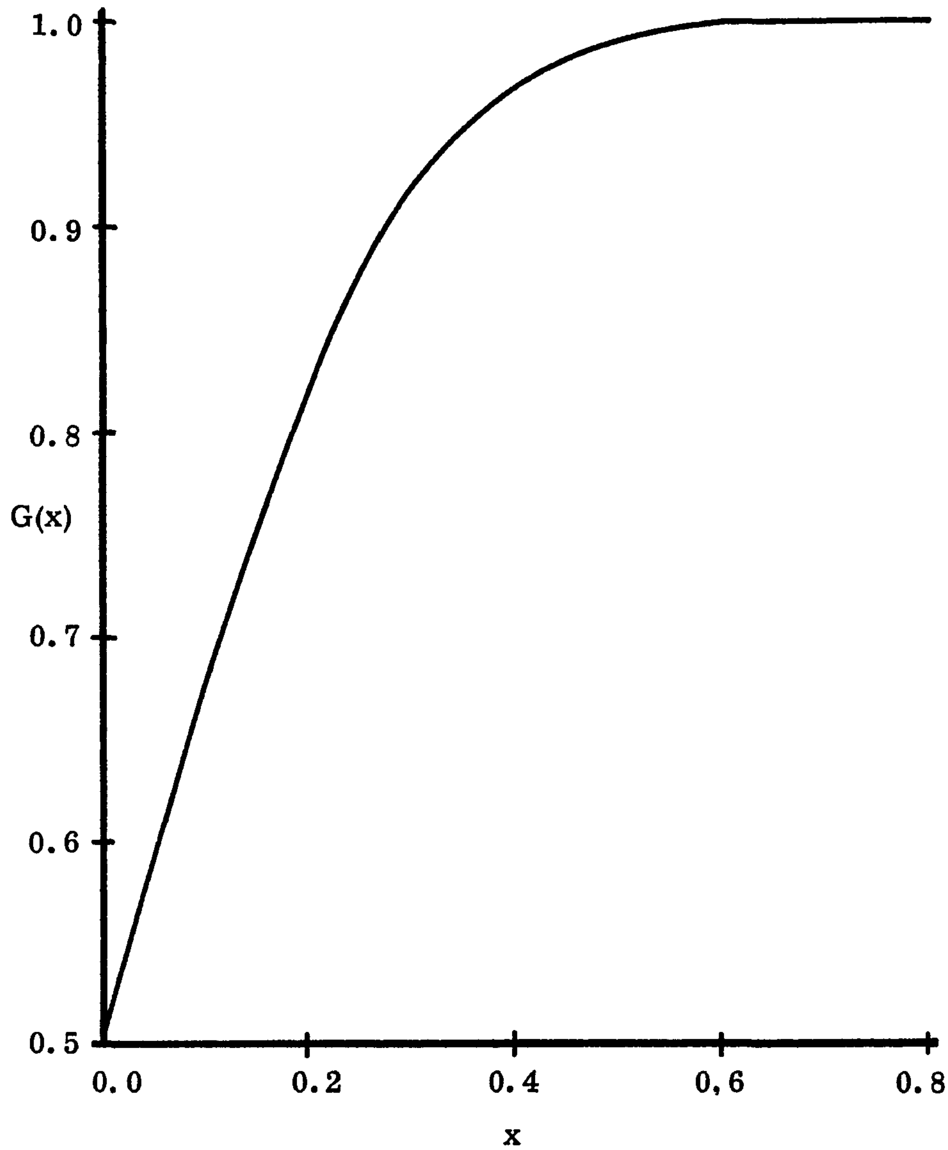

Fig. 2 - A Graph of G(x) for $0 \leq x \leq 0.8$

We need several preliminary results. From Edwards [7, p. 45] we have

Lemma 2.5. Let $f(z)$ be an analytic function on the disk $|z| \leq r$, let $f(0) = 0$, and let M be the maximum value of $\mathrm{Re} f(z)$ on the circle $|z| = r$. If $r_1 < r$, then $|f(z)| \leq 2r_1 M/(r - r_1)$ on the disk $|z| \leq r_1$.

We use this result in proving

Lemma 2.6. Let $S_1(T) = \int_0^T S(t)\,dt = \frac{1}{\pi}\int_0^T \arg \zeta(\frac{1}{2}+it)\,dt$. Let

$$A_L = -1.7412588,$$
$$B_L = -2.864789,$$
$$C_L = -0.3936986,$$
$$D_L = -3.8661494,$$
$$A_H = 0.0198943,$$
$$B_H = 0.4774648 \text{ and}$$
$$D_H = -0.666871.$$

If the Riemann hypothesis is true and $T \geq 1005$, then

$$A_L \log T + B_L \log\log T + C_L \log\log(T + \sqrt{\tfrac{7}{4}}) + D_L$$
$$\leq S_1(t) \leq A_H \log T + B_H \log\log T + D_H.$$

Proof. From Titchmarsh [26, p. 188] we have

$$S_1(T) = \frac{1}{\pi}\int_{\frac{1}{2}}^{\infty} \log|\zeta(\sigma + iT)|\,d\sigma - \frac{1}{\pi}\int_{\frac{1}{2}}^{\infty} \log|\zeta(\sigma)|\,d\sigma.$$

From Edwards [7, p. 192] we obtain

$$\left| \int_2^{\infty} \log|\zeta(\sigma+iT)|\,d\sigma \right| \leq \int_2^{\infty} 3\cdot 2^{-\sigma}\,d\sigma = \frac{3}{4\log 2}$$

and

$$0 \leq \int_2^\infty \log|\zeta(\sigma)|\,d\sigma \leq \frac{3}{4\log 2}\ .$$

Hence,

(1)

$$S_1(T) = \frac{1}{\pi}\int_{\frac{1}{2}}^{2} \log|\zeta(\sigma+iT)|\,d\sigma - \frac{1}{\pi}\int_{\frac{1}{2}}^{2}\log|\zeta(\sigma)|\,d\sigma + R(T) = K_1 - K_2 + R(T),$$

where

$$\frac{-3}{\pi\, 2\log 2} \leq R(T) \leq 0.$$

We first bound K_1. For a fixed T we let $z_0 = 2+iT$ and let C be the circle centered at z_0 with radius $R = 2 + \epsilon$ such that no zeros of $\zeta(s)$ lie on C. Since ϵ can be made arbitrarily small, we can assume that $R = 2$ in our computations. Assuming the Riemann hypothesis, we let $z_j = \frac{1}{2}+it_j$, for $1 \leq j \leq n$, be the n zeros of $\zeta(s)$ inside C. Finally we let $F(z) = \zeta(z) \prod_{j=1}^{n} a_j(z)$, where $a_j(z) = \dfrac{R^2 - (\bar{z}_j - \bar{z}_t)(z - z_0)}{R(z-z_j)}$.

We apply Lemma 2.5 to the disk $|z - z_0| \leq R$ with $f(z) = \mathrm{Re}\log(F(z)/F(z_0))$. Since f(z) is analytic in the disk and $f(z_0) = 0$, f(z) satisfies the conditions of the lemma. Thus to use the lemma we need only to bound

(2) $$\mathrm{Re}\log(F(z)/F(z_0)) = \mathrm{Re}\log F(z) - \mathrm{Re}\log F(z_0).$$

On $|z - z_0| = R$, we have

$$\mathrm{Re}\log F(z) = \log|F(z)| = \log|\zeta(z)| + \sum_{j=1}^{n}\log|a_j(z)|.$$

From Backland [2, pp. 362-367], we obtain, for $z = \sigma + it$ and $50 \leq t$,

$$|\zeta(z)| \le \begin{cases} \log t - .048 & \text{if } 1 < \sigma \\ \left(\frac{t}{2\pi}\right)^{\frac{1-\sigma}{2}} \log t \left(1 + \frac{4}{t^2}\right)^{-1} \left(1 + \frac{5}{t^2}\right)^{-1} & \text{if } \frac{1}{2} \le \sigma \le 1 \\ \left(\frac{t}{2\pi}\right)^{\frac{1-\sigma}{2}} \log t \left(\frac{t^2}{t^2 - 4}\right) & \text{if } 0 \le \sigma \le \frac{1}{2}. \end{cases}$$

Since $|a_j(z)| = 1$ on $|z - z_0| = R$ we have

(3) $\operatorname{Re} \log F(z) \le \operatorname{Re} \log \zeta(z)$

$$\le \begin{cases} \log(\log t - .048) & \text{if } 1 < \sigma \\ \frac{1-\sigma}{2} \log \frac{t}{2\pi} + \log\log t - \log\left(1 + \frac{4}{t^2}\right)\left(1 + \frac{5}{t^2}\right) & \text{if } \frac{1}{2} \le \sigma \le 1 \\ \frac{1-\sigma}{2} \log \frac{t}{2\pi} + \log\log t + \log\left(\frac{t^2}{t^2 - 4}\right) & \text{if } 0 \le \sigma \le \frac{1}{2} \end{cases}$$

for $|z - z_0| = R$.

We next obtain a lower bound for

(4) $$\operatorname{Re} \log F(z_0) = \log|\zeta(z_0)| + \sum_{j=1}^{n} \log|a_j(z_0)|.$$

From Edwards [7, p. 190] we have

(5) $$\log|\zeta(z_0)| \ge -\log \zeta(2) = -\log \frac{\pi^2}{6}.$$

In addition,

$$|a_j(z_0)| = \left| \frac{R}{(z_0 - z_j)} \right| \geq 1$$

and thus

(6) $$\sum_{j=1}^{n} \log|a_j(z_0)| \geq 0.$$

Combining (2), (3), (4), (5), and (6) we obtain

$$f(z) = \text{Re} \log(F(z)/F(z_0)) \leq \tfrac{1}{2} \log T + \log\log T - 0.4212345 = M$$

for $1005 \leq t$ and $|z - z_0| = R$. Applying Lemma 2.5 to f(z) we have

(7) $$|f(z)| \leq 6M$$

in $|z - z_0| \leq \frac{3}{2}$. From this it follows directly that

$$|\text{Re} \log F(z) - \text{Re} \log F(z_0)| \leq 6M.$$

and combining this with (4), (5) and (6) we obtain

(8) $$\text{Re} \log F(z) \geq -6M - \log \frac{\pi^2}{6}.$$

We can now bound K_1. Using (3), we obtain

(9) $$K_1 = \frac{1}{\pi} \int_{\frac{1}{2}}^{1} \text{Re} \log \zeta(\sigma + iT)\, d\sigma + \frac{1}{\pi} \int_{1}^{2} \text{Re} \log \zeta(\sigma + iT)\, d\sigma$$

$$\leq \frac{1}{\pi} \int_{\frac{1}{2}}^{1} \left(\frac{1-\sigma}{2} \log \frac{T}{2\pi} + \log\log T\right) d\sigma + \frac{1}{\pi} \int_{1}^{2} \log\log T\, d\sigma$$

$$\leq 0.0198943 \log T + 0.4774648 \log\log T - 0.0365633.$$

Using (8), we obtain

(10) $$K_1 = \frac{1}{\pi} \int_{\frac{1}{2}}^{2} \text{Re} \log \zeta(\sigma + i T)\, d\sigma$$

$$= \frac{1}{\pi} \int_{\frac{1}{2}}^{2} \text{Re} \log F(\sigma + i T)\, d\sigma - \frac{1}{\pi} \sum_{j=1}^{n} \int_{\frac{1}{2}}^{2} \text{Re} \log a_j(\sigma + i T)\, d\sigma$$

$$> \frac{1}{\pi} \left(-6M - \log \frac{\pi^2}{6} \right) - \frac{1}{\pi} \sum_{j=1}^{n} \int_{\frac{1}{2}}^{2} \text{Re} \log a_j(\sigma + i T)\, d\sigma .$$

Thus we need to find an upper bound for

(11) $$\sum_{j=1}^{n} \int_{\frac{1}{2}}^{2} \text{Re} \log a_j(\sigma + i T)\, d\sigma \leq n \max_j \int_{\frac{1}{2}}^{2} \log \left| a_j(\sigma + i T) \right| .$$

From Backland [2, p. 355], we have

(12)
$$\left| N(T) - \left(\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} \right) \right| < 0.137 \log T + 0.443 \log\log T + 4.350$$

for $T > 2$. Since we are assuming the Riemann hypothesis, the zeros z_j inside $|z - z_0| = R$ have real part equal to $\frac{1}{2}$ and for $T > 1005$

(13) $$n \leq N\left(T + \sqrt{\tfrac{7}{4}}\right) - N\left(T - \sqrt{\tfrac{7}{4}}\right)$$

$$\leq \frac{T}{2\pi} \left(\log \frac{T + \sqrt{\frac{7}{4}}}{2\pi} - \log \frac{T - \sqrt{\frac{7}{4}}}{2\pi} \right) + \frac{1}{2\pi} \sqrt{\frac{7}{4}} \left(\log \frac{T + \sqrt{\frac{7}{4}}}{2\pi} + \log \frac{T - \sqrt{\frac{7}{4}}}{2\pi} \right) - \frac{1}{\pi} \sqrt{\frac{7}{4}}$$

$$+ 0.137 \left(\log \left(T + \sqrt{\tfrac{7}{4}} \right) + \log \left(T - \sqrt{\tfrac{7}{4}} \right) \right) + 0.443 \left(\log\log \left(T + \sqrt{\tfrac{7}{4}} \right) + \log\log \left(T - \sqrt{\tfrac{7}{4}} \right) \right) + 8.7$$

$$\leq \frac{T}{2\pi}\left(\frac{2\sqrt{\frac{7}{4}}}{T-\sqrt{\frac{7}{4}}}\right)+\frac{1}{2\pi}\sqrt{\frac{7}{4}}\left(2\log\frac{T}{2\pi}\right)-\frac{1}{\pi}\sqrt{\frac{7}{4}}$$

$$+0.137\ (2\log T)+0.886\log\log\left(T+\sqrt{\frac{7}{4}}\right)+8.7$$

$$\leq 0.6950843\log T+0.886\log\log\left(T+\sqrt{\frac{7}{4}}\right)+7.9266538.$$

From the definition of $a_j(\sigma+iT)$, we write

$$\int_{\frac{1}{2}}^{2}\log\left|a_j(\sigma+iT)\right|d\sigma=\int_{\frac{1}{2}}^{2}\log\left|\frac{R^2-(\bar{z}_j-\bar{z}_0)(z-z_0)}{R(z-z_j)}\right|d\sigma$$

$$=\int_{\frac{1}{2}}^{2}\log\left|\frac{4-(((\frac{1}{2}-it_j)(2-iT))((\sigma+iT)-(2+iT)))}{2((\sigma+iT)-(\frac{1}{2}+it_j))}\right|d\sigma$$

$$=\int_{\frac{1}{2}}^{2}\log\left|\frac{1+\frac{3}{2}\sigma-ib_j(\sigma-2)}{2\sigma-1+i2b_j}\right|d\sigma,$$

where $b_j=T-t_j$. Since we are assuming the Riemann hypothesis, $|b_j|\leq\sqrt{\frac{7}{4}}$, and in this range

$$0\leq\log\left|\frac{1+\frac{3}{2}\sigma-ib_j(\sigma-2)}{2\sigma-1+i2b_j}\right|=\frac{1}{2}\log\left(\frac{(1+\frac{3}{2}\sigma)^2+(\sigma-2)^2b_j^2}{(2\sigma-1)^2+4b_j^2}\right)$$

$$<\frac{1}{2}\log\left(\frac{(1+\frac{3}{2}\sigma)^2}{(2\sigma-1)^2}\right).$$

Combining the above two results we obtain

(14)

$$0\leq\int_{\frac{1}{2}}^{2}\log\left|a_j(\sigma+iT)\right|d\sigma\leq\frac{1}{2}\int_{\frac{1}{2}}^{2}\log\left(\frac{(1+\frac{3}{2}\sigma)^2}{(2\sigma-1)^2}\right)d\sigma\leq 1.3959830.$$

Combining (10), (11), (13) and (14) we obtain

(15)

$$\frac{1}{\pi}\int_{\frac{1}{2}}^{2} \log|\zeta(\sigma+iT)|\,d\sigma \geq \frac{1}{\pi}\left(-6M - \log\frac{\pi^2}{6}\right) - \frac{1}{\pi}(1.3959830)\,(0.6950843 \log T$$

$$+ 0.886 \log\log\left(T + \sqrt{\frac{7}{4}}\right) + 7.9266538),$$

which gives us a lower bound for K_1.

To bound K_2 we use the formula (see Edwards [7, p.114-115])

$$\zeta(\sigma) = \frac{1}{\sigma-1} + \frac{1}{2} - \sigma\int_1^{\infty} \frac{\hat{B}_1(u)\,du}{u^{\sigma+1}} = \frac{1}{\sigma-1} + \frac{1}{2} + R_0,$$

where $0 \leq R_0 \leq \frac{1}{12}$. This gives us

$$\frac{1}{2} + \frac{1}{\sigma-1} \leq \zeta(\sigma) \leq \frac{1}{2} + \frac{1}{\sigma-1} + \frac{\sigma}{12}.$$

Thus,

(16) $$\log\left|\frac{1}{\sigma-1} + \frac{7}{12}\right| \leq \log|\zeta(\sigma)| \leq \log\left|\frac{1}{\sigma-1} + \frac{1}{2}\right| \quad \left(\tfrac{1}{2} \leq \sigma < 1\right)$$

and

(17) $$\log\left|\frac{1}{\sigma-1} + \frac{1}{2}\right| \leq \log|\zeta(\sigma)| \leq \log\left|\frac{1}{\sigma-1} + \frac{2}{3}\right| \quad (1 < \sigma \leq 2).$$

Since

$$\int_{\frac{1}{2}}^{1} \log\left|\frac{1}{\sigma-1} + c\right| = \frac{1}{2}\log|2-c| - \frac{1}{c}\log\left|1 - \frac{c}{2}\right|$$

for $\frac{1}{2} \leq c \leq \frac{7}{12}$, we use (16) to obtain

(18) $$\tfrac{1}{2}\log\frac{17}{12} - \frac{12}{7}\log\frac{17}{24} \le \int_{\frac{1}{2}}^{1} \log|\zeta(\sigma)|\, d\sigma \le \tfrac{1}{2}\log\frac{3}{2} - 2\log\frac{3}{4}.$$

Since

$$\int_1^2 \log\left|\frac{1}{\sigma-1} + c\right| = (1+\frac{1}{c})\log|1+c|$$

for $\frac{1}{2} \le c \le \frac{2}{3}$, we use (17) to obtain

(19) $$3\log\frac{3}{2} \le \int_1^2 \log|\zeta(\sigma)|\, d\sigma \le \frac{5}{2}\log\frac{5}{3}.$$

Combining (18) and (19), we have

(20) $$0.6303077 \le K_2 \le 0.6541774.$$

Combining (1), (9), (15) and (20) we complete the proof of the lemma.

We next prove

Lemma 2.7. If $m \ge 1$, then

(i) $$\int_a^b \frac{\log y\, dy}{y^{m+2}} = \frac{1}{(m+1)\,a^{m+1}}\left(\log a + \frac{1}{m+1}\right) - \frac{1}{(m+1)\,b^{m+1}}\left(\log b + \frac{1}{m+1}\right)$$

(ii) $$\int_a^b \frac{\log\log y\, dy}{y^{m+2}} \le \frac{1}{(m+1)\,a^{m+1}}\left(\log\log a + \frac{1}{(m+1)\log a}\right) - \frac{1}{(m+1)\,b^{m+1}}\left(\log\log b + \frac{1}{(m+1)\log a}\right)$$

(iii) $$\int_a^b \frac{\log\log\left(y+\sqrt{\frac{7}{4}}\right)dy}{y^{m+2}} \le \frac{1}{(m+1)a^{m+1}}\left(\log\log\left(a+\sqrt{\frac{7}{4}}\right) + \frac{1}{(m+1)\log\left(a+\sqrt{\frac{7}{4}}\right)}\right) - \frac{1}{(m+1)b^{m+1}}\left(\log\log\left(b+\sqrt{\frac{7}{4}}\right) + \frac{1}{(m+1)\log\left(a+\sqrt{\frac{7}{4}}\right)}\right)$$

and

(iv) $$\int_a^b \frac{dy}{y^{m+2}} = \frac{1}{m+1}\left(\frac{1}{a^{m+1}} + \frac{1}{b^{m+1}}\right).$$

Proof. We have

$$\int_a^b \frac{\log y\, dy}{y^{m+2}} = \frac{\log a}{(m+1)a^{m+1}} - \frac{\log b}{(m+1)b^{m+1}} + \frac{1}{m+1}\int_a^b \frac{dy}{y^{m+2}}$$

$$= \frac{1}{(m+1)a^{m+1}}\left(\log a + \frac{1}{(m+1)}\right) - \frac{1}{(m+1)b^{m+1}}\left(\log b + \frac{1}{m+1}\right),$$

$$\int_a^b \frac{\log\log y\, dy}{y^{m+2}} = \frac{\log\log a}{(m+1)a^{m+1}} - \frac{\log\log b}{(m+1)b^{m+1}} + \frac{1}{(m+1)}\int_a^b \frac{dy}{y^{m+2}\log y}$$

$$\leq \frac{\log\log a}{(m+1)a^{m+1}} - \frac{\log\log b}{(m+1)b^{m+1}} + \frac{1}{(m+1)\log a}\int_a^b \frac{dy}{y^{m+2}}$$

$$= \frac{1}{(m+1)a^{m+1}}\left(\log\log a + \frac{1}{(m+1)\log a}\right) - \frac{1}{(m+1)b^{m+1}}\left(\log\log b + \frac{1}{(m+1)\log a}\right),$$

$$\int_a^b \frac{\log\log(y + \sqrt{\tfrac{7}{4}})dy}{y^{m+2}} \leq \frac{1}{(m+1)a^{m+1}}\left(\log\log\left(a + \sqrt{\tfrac{7}{4}}\right) + \frac{1}{(m+1)\log\left(a + \sqrt{\tfrac{7}{4}}\right)}\right)$$

$$= \frac{1}{(m+1)b^{m+1}}\left(\log\log\left(b + \sqrt{\tfrac{7}{4}}\right) + \frac{1}{(m+1)\log\left(a + \sqrt{\tfrac{7}{4}}\right)}\right)$$

and

$$\int_a^b \frac{1}{y^{m+2}} = \frac{1}{m+1}\left(\frac{1}{a^{m+1}} - \frac{1}{b^{m+1}}\right).$$

We use this lemma in proving

<u>Lemma 2.8.</u> Let $n \in \mathbb{Z}^+$. Let A_L, B_L, C_L, D_L, A_H, B_H and D_H be as in Lemma 2.6. Let $N(y) = \frac{y}{2\pi}\log\frac{y}{2\pi} - \frac{y}{2\pi} + \frac{7}{8} + S(y) + G(y)$, where $S(y) = \frac{1}{\pi}\arg\zeta(\frac{1}{2}+iy)$. If $a \geq 1005$, then

$$\int_a^b \frac{dN(y)}{|\frac{1}{2}+iy|} \leq \frac{1}{4\pi}\left(\left(\log\frac{b}{2\pi}\right)^2 - \left(\log\frac{a}{2\pi}\right)^2\right) + \frac{S(b)+G(b)}{b} - \frac{S(a)+G(a)}{a}$$

$$+ \frac{1}{a^2}\left(\frac{25}{96\pi} + (A_H - A_L)\log a + (B_H - B_L)\log\log a - C_L\log\log\left(a+\sqrt{\frac{7}{4}}\right) + (D_H - D_L)\right.$$

$$\left. + \frac{A_H}{2} + \frac{B_H}{2\log a}\right) - \frac{1}{2b^2}\left(\frac{25}{48\pi} + A_H + \frac{B_H}{\log a}\right),$$

$$\int_a^b \frac{dN(y)}{|\frac{1}{2}+iy|^{2n}} \leq \frac{1}{2\pi(2n-1)a^{2n-1}}\left(\log\frac{a}{2\pi} + \frac{1}{2n-1}\right) - \frac{1}{2\pi(2n-1)b^{2n-1}}\left(\log\frac{b}{2\pi} + \frac{1}{2n-1}\right)$$

$$+ \frac{S(b)+G(b)}{b^{2n}} - \frac{S(a)+G(a)}{a^{2n}} + \frac{2n}{a^{2n+1}}\left(\frac{25}{48\pi(2n+1)} + (A_H - A_L)\log a + (B_H - B_L)\log\log a\right.$$

$$\left. - C_L\log\log\left(a+\sqrt{\frac{7}{4}}\right) + (D_H - D_L) + \frac{A_H}{2n+1} + \frac{B_H}{(2n+1)\log a}\right)$$

$$- \frac{2n}{(2n+1)b^{2n+1}}\left(\frac{25}{48\pi} + A_H + \frac{B_H}{\log a}\right)$$

and

$$\int_a^b \frac{dN(y)}{|\frac{1}{2}+iy|^{2n}} \geq \frac{1}{2\pi(2n-1)a^{2n-1}}\left(\log\frac{a}{2\pi} + \frac{1}{2n-1}\right) - \frac{1}{2\pi(2n-1)b^{2n-1}}\left(\log\frac{b}{2\pi} + \frac{1}{2n-1}\right)$$

$$+ \frac{S(b)+G(b)}{b^{2n}} - \frac{S(a)+G(a)}{a^{2n}} + \frac{2n}{a^{2n+1}}\left(\frac{-25}{48\pi(2n+1)} + (A_L - A_H)\log a + (B_L - B_H)\log\log a\right.$$

$$+ C_L \log\log\left(a+\sqrt{\tfrac{7}{4}}\right) + (D_L - D_H) + \frac{A_L}{2n+1} + \frac{B_L}{(2n+1)\log a} + \frac{C_L}{(2n+1)\log\left(a+\sqrt{\tfrac{7}{4}}\right)}\Bigg)$$

$$- \frac{2n}{(2n+1)b^{2n+1}}\left(\frac{-25}{48\pi} + A_L + \frac{B_L}{\log a} + \frac{C_L}{\log\left(a+\sqrt{\tfrac{7}{4}}\right)}\right)$$

$$- \frac{n}{4}\left(\frac{1}{2\pi(2n+1)a^{2n+1}}\left(\log\frac{a}{2\pi} + \frac{1}{2n+1}\right) - \frac{1}{2\pi(2n+1)b^{2n+1}}\left(\log\frac{b}{2\pi} + \frac{1}{2n+1}\right)\right.$$

$$+ \frac{S(b)+G(b)}{b^{2n+2}} - \frac{S(a)+G(a)}{a^{2n+2}} + \frac{2n+2}{a^{2n+3}}\Bigg(\frac{-25}{48\pi(2n+3)} + (A_L - A_H)\log a + (B_L - B_H)\log\log a$$

$$+ C_L \log\log\left(a+\sqrt{\tfrac{7}{4}}\right) + (D_L - D_H) + \frac{A_L}{2n+3} + \frac{B_L}{(2n+3)\log a} + \frac{C_L}{(2n+3)\log\left(a+\sqrt{\tfrac{7}{4}}\right)}\Bigg)$$

$$\left.- \frac{2n+2}{(2n+3)b^{2n+3}}\left(\frac{-25}{48\pi} + A_L + \frac{B_L}{\log a} + \frac{C_L}{\log\left(a+\sqrt{\tfrac{7}{4}}\right)}\right)\right) .$$

Proof. We have

(21)
$$\int_a^b \frac{dN(y)}{\left|\tfrac{1}{2}+iy\right|} \le \int_a^b \frac{dN(y)}{y}$$

$$= \int_a^b \frac{d\left(\frac{y}{2\pi}\log\frac{y}{2\pi} - \frac{y}{2\pi} + \frac{7}{8}\right)}{y} + \int_a^b \frac{d(S(y)+G(y))}{y}$$

$$= \frac{1}{4\pi}\left(\left(\log\frac{b}{2\pi}\right)^2 - \left(\log\frac{a}{2\pi}\right)^2\right) + \int_a^b \frac{d(S(y)+G(y))}{y}$$

and similarly

(22)
$$\int_a^b \frac{dN(y)}{\left|\tfrac{1}{2}+iy\right|^{2n}} \le \int_a^b \frac{dN(y)}{y^{2n}}$$

$$= \frac{1}{2\pi(2n-1)a^{2n-1}}\left(\log\frac{a}{2\pi}+\frac{1}{2n-1}\right)-\frac{1}{2\pi(2n-1)b^{2n-1}}\left(\log\frac{b}{2\pi}+\frac{1}{2n-1}\right)$$

$$+\int_a^b \frac{d(S(y)+G(y))}{y^{2n}}.$$

Since

$$(y^2+\tfrac{1}{4})^n - y^{2n} \leq \frac{n}{4}\left(y^{2n-2}+\binom{n-1}{1}y^{2n-4}\left(\tfrac{1}{4}\right)+\binom{n-1}{2}y^{2n-6}\left(\tfrac{1}{4}\right)^6+\cdots+\left(\tfrac{1}{4}\right)^{n-1}\right)$$

$$= \frac{n}{4}(y^2+\tfrac{1}{4})^{n-1}$$

we have

$$\int_a^b \frac{dN(y)}{\left|\frac{1}{2}+iy\right|^{2n}} \geq \int_a^b \frac{dN(y)}{y^{2n}} - \frac{n}{4}\int_a^b \frac{dN(y)}{y^{2n+2}}$$

$$= \frac{1}{2\pi(2n-1)a^{2n-1}}\left(\log\frac{a}{2\pi}+\frac{1}{2n-1}\right)-\frac{1}{2\pi(2n-1)b^{2n-1}}\left(\log\frac{b}{2\pi}+\frac{1}{2n-1}\right)$$

$$-\frac{n}{4}\left(\frac{1}{2\pi(2n+1)a^{2n+1}}\left(\log\frac{a}{2\pi}+\frac{1}{2n+1}\right)-\frac{1}{2\pi(2n+1)b^{2n+1}}\left(\log\frac{b}{2\pi}+\frac{1}{2n-1}\right)\right)$$

$$+\int_a^b \frac{d(S(y)+G(y))}{y^{2n}} - \frac{n}{4}\int_a^b \frac{d(S(y)+G(y))}{y^{2n}}.$$

For $m \geq 1$, we integrate by parts to see that

$$\text{(24)}\quad \int_a^b \frac{d(S(y)+G(y))}{y^m} = \frac{S(b)+G(b)}{b^m} - \frac{S(a)+G(a)}{a^m} + m\int_a^b \frac{S(y)dy}{y^{m+1}} + m\int_a^b \frac{G(y)dy}{y^{m+1}}$$

$$= \frac{S(b)+G(b)}{b^m} - \frac{S(a)+G(a)}{a^m} + m\left(\frac{S_1(b)}{b^{m+1}} - \frac{S_1(a)}{a^{m+1}} + (m+1)\int_a^b \frac{S_1(y)dy}{y^{m+2}} + \int_a^b \frac{G(y)dy}{y^{m+1}}\right).$$

Using Lemma 2.6 and Lemma 2.7, we obtain

(26) $$\int_a^b \frac{S_1(y)dy}{y^{m+2}} \leq \frac{A_H}{m+1}\left(\frac{1}{a^{m+1}}\left(\log a + \frac{1}{m+1}\right) - \frac{1}{b^{m+1}}\left(\log b + \frac{1}{m+1}\right)\right)$$

$$+ \frac{B_H}{m+1}\left(\frac{1}{a^{m+1}}\left(\log\log a + \frac{1}{(m+1)\log a}\right) - \frac{1}{b^{m+1}}\left(\log\log b + \frac{1}{(m+1)\log a}\right)\right)$$

$$+ \frac{D_H}{m+1}\left(\frac{1}{a^{m+1}} - \frac{1}{b^{m+1}}\right)$$

and

(27) $$\int_a^b \frac{S_1(y)dy}{y^{m+2}} \geq \frac{A_L}{m+1}\left(\frac{1}{a^{m+1}}\left(\log a + \frac{1}{m+1}\right) - \frac{1}{b^{m+1}}\left(\log b + \frac{1}{m+1}\right)\right)$$

$$+ \frac{B_L}{m+1}\left(\frac{1}{a^{m+1}}\left(\log\log a + \frac{1}{(m+1)\log a}\right) - \frac{1}{b^{m+1}}\left(\log\log b + \frac{1}{(m+1)\log a}\right)\right)$$

$$+ \frac{C_L}{m+1}\left(\frac{1}{a^{m+1}}\left(\log\log\left(a+\sqrt{\frac{7}{4}}\right) + \frac{1}{(m+1)\log\left(a+\sqrt{\frac{7}{4}}\right)}\right) - \frac{1}{b^{m+1}}\left(\log\log\left(b+\sqrt{\frac{7}{4}}\right)\right.\right.$$

$$\left.\left. + \frac{1}{(m+1)\log\left(a+\sqrt{\frac{7}{4}}\right)}\right)\right)$$

$$+ \frac{D_L}{m+1}\left(\frac{1}{a^{m+1}} - \frac{1}{b^{m+1}}\right).$$

Backland [2, p. 374] showed that

$$|G(y)| \leq \frac{25}{48\pi y}$$

for $y \geq 50$. Hence for $m \geq 1$

$$\frac{-25}{48\pi(m+1)}\left(\frac{1}{a^{m+1}}-\frac{1}{b^{m+1}}\right)\leq \int_a^b \frac{G(y)dy}{y^{m+1}} \leq \frac{25}{48\pi(m+1)}\left(\frac{1}{a^{m+1}}-\frac{1}{b^{m+1}}\right).$$

The upper bound for

$$\int_a^b \frac{dN(y)}{|\frac{1}{2}+iy|}$$

follows from (21), (24), (26), (28) and Lemma 2.6. The upper bound for

$$\int_a^b \frac{dN(y)}{|\frac{1}{2}+iy|^{2n}}$$

follows from (22), (24), (26), (28) and Lemma 2.6. The lower bound for

$$\int_a^b \frac{dN(y)}{|\frac{1}{2}+iy|^{2n}}$$

follows from (23), (24), (27), (28) and Lemma 2.6.

We now have the tools which we need to prove the theorem. To calculate $G(x, B)$, we write

$$G(x,B) = \tfrac{1}{2} + \int_{-\infty}^{\infty} \frac{e(\lambda x)-1}{2\pi i\lambda}\, \hat{G}(\lambda, B)d\lambda,$$

where

$$\hat{G}(\lambda, B) = \int_{-\infty}^{\infty} e(-\lambda x)\, dG(x, B)$$

as in Theorem 1.9. Here

$$\hat{G}(\lambda, B) = \int_{I^R} e(-\lambda \mathbf{Y}_B)\, d\underline{\theta} = \prod_{0<\gamma\leq B} \int_0^1 e\left(\frac{-2\lambda}{|\rho|}\sin 2\pi\theta\right)d\theta.$$

$$= \prod_{0<\gamma\leq B} \frac{1}{\pi}\int_0^\pi e^{i\frac{4\pi\lambda}{|\rho|}\cos\theta}\, d\theta$$

$$= \prod_{0<\gamma\leq B} J_0\left(\frac{4\pi\lambda}{|\rho|}\right),$$

since $J_0(z) = \frac{1}{\pi}\int_0^\pi e^{iz\cos\theta}\, d\theta$ (see Abromowitz and Stegun [1, p. 360]). We note here that $|J_0(z)| \leq \exp(|\mathrm{Im}(z)|)$. Hence

$$G(x, B) = \tfrac{1}{2} + \int_{-\infty}^{\infty} \frac{e(\lambda x)-1}{2\pi i\lambda} \prod_{0<\gamma\leq B} J_0\left(\frac{4\pi\lambda}{|\rho|}\right)$$

and similarly

$$G(x) = \tfrac{1}{2} + \int_{-\infty}^{\infty} \frac{e(\lambda x)-1}{2\pi i\lambda} \prod_{\gamma>0} J_0\left(\frac{4\pi\lambda}{|\rho|}\right).$$

Thus for any $c \leq \frac{B}{4\pi}$ we have

(29)

$$|G(x) - G(x, B)| \leq \left| \int_0^\infty \frac{2\sin 2\pi\lambda x}{2\pi\lambda} \prod_{0<\gamma\leq B} J_0\left(\frac{4\pi\lambda}{|\rho|}\right)\left(1 - \prod_{\gamma>B} J_0\left(\frac{4\pi\lambda}{|\rho|}\right)\right) d\lambda \right|$$

$$\leq 2\int_0^c \left| 1 - \prod_{\gamma>B} J_0\left(\frac{4\pi\lambda}{|\rho|}\right)\right| d\lambda + \int_c^\infty \frac{1}{\lambda} \prod_{0<\gamma\leq B} \left| J_0\left(\frac{4\pi\lambda}{|\rho|}\right)\right| d\lambda = L_1 + L_2.$$

To bound the first integral, we observe that for $|z| \leq 6$,

$$J_0(z) \geq 1 - \frac{z^2}{4}.$$

Thus

$$L_1 \leq 2\int_0^c \left| 1 - \exp\left(\sum_{\gamma>B} \log\left(1 - \frac{(4\pi\lambda)^2}{4|\rho|^2}\right)\right)\right| d\lambda.$$

Applying Lemma 1.17, we see that the above integral

$$\leq 2\int_0^c \left| 1 - \exp\left(- \sum_{\gamma > B} \left(\frac{(4\pi\lambda)^2}{4|\rho|^2} + \frac{(4\pi\lambda)^4}{24|\rho|^4} \right) \right) \right| d\lambda.$$

$$= 2\int_0^c \left| 1 - \exp\left(- \frac{(4\pi\lambda)^2}{4} \int_{B^+}^{\infty} \frac{dN(y)}{|\frac{1}{2}+iy|^2} - \frac{(4\pi\lambda)^4}{24} \int_{B}^{\infty} \frac{dN(y)}{|\frac{1}{2}+iy|^4} \right) \right| d\lambda.$$

Next we apply Lemma 2.8 and obtain

$$\text{(30)} \qquad L_1 \leq 2\int_0^c \left| 1 - \exp\left(\frac{-(4\pi\lambda)^2 \log B}{8\pi B} - \frac{(4\pi\lambda)^4 \log B}{144\pi B^3} \right) \right| d\lambda$$

$$\leq 2\left(\frac{(4\pi)^2 c^3 \log B}{24\pi B} + \frac{(4\pi)^4 c^5 \log B}{720\pi B^3} \right).$$

To bound the second integral, we use the well-known result (see Watson [30, pp.206-208]) that

$$J_0(t) = \left(\frac{2}{\pi t}\right)^{\frac{1}{2}} \left(\cos(t - \tfrac{\pi}{4})\, P(t,0) - \sin(t - \tfrac{\pi}{4})\, Q(t,0)\right)$$

where $0 \leq t$, $0 \leq P(t,0) \leq 1$ and $\frac{-1}{8t} \leq Q(t,0) \leq 0$. Thus,

$$\text{(31)} \qquad |J_0(t)| \leq \min\left(\left(\frac{2}{\pi t}\right)^{\frac{1}{2}} \left(1 + \frac{1}{8t}\right),\ 1 \right).$$

Using the above result, we obtain

$$\text{(32)} \qquad L_2 \leq \frac{2}{653} \prod_{0<\gamma<1005} \left(\left(\frac{|\rho|}{2\pi^2 c} \right)^{\frac{1}{2}} \left(1 + \frac{|\rho|}{32\pi c} \right) \right).$$

We choose $c = e^{10}/4\pi$. Using (30) we obtain $L_1 \leq 10^{-12.25}$. Using (32) we obtain $L_2 \leq 10^{-78}$. Combining these results with (29), we complete the proof of the theorem.

Next we prove

Theorem 2.9. If $0 \leq x \leq 1$, $h \leq \frac{1}{512}$ and $B = e^{58.2230805}$, then

$$\left| G(x) - \left(\tfrac{1}{2} + h \sum_{n=-\infty}^{\infty} \left(\frac{e(nhx)-1}{2\pi i n h} \prod_{0<\gamma\leq B} J_0\left(\frac{4\pi n h}{|\rho|}\right)\right)\right) \right| \leq 10^{-12}.$$

We need several preliminary results.

Lemma 2.10. Let $f(z) = \frac{e(zx)-1}{2\pi i z} \hat{G}(z, B)$. If $0 \leq x \leq 1$ and

$$0 < h \leq \frac{1}{2\left(\sum_{0<\gamma\leq B} \frac{1}{|\rho|} + \frac{1}{4\pi} + \frac{x}{2} \right)},$$

then $\int_{-\infty}^{\infty} f(\lambda)\, d\lambda = h \sum_{-\infty}^{\infty} f(nh)$.

Proof. By Corollary 1.11 and (31), it is sufficient to show that

$$\hat{f}(t) = 0 \quad \text{for } |t| \geq 2\left(\sum_{0<\gamma\leq B} \frac{1}{|\rho|} + \frac{1}{4\pi} + \frac{x}{2} \right).$$

We have

$$\hat{f}(t) = \int_{-\infty}^{\infty} e(-tz) f(z)\, dz$$

$$= \lim_{r\to\infty} \int_{\Gamma} e(-tz) f(z)\, dz.$$

From Lemma 1.12, we know that the above limit equals zero if $e(-tz - \frac{z}{2\pi}) f(z) \to 0$ uniformly with respect to arg z as $|z| \to \infty$ for $0 \leq \arg z \leq \pi$.

For $t \leq 0$, $z = r\cos\theta + i r \sin\theta$ and $0 \leq \theta \leq \pi$, we have

$$\left| e\left(-tz - \frac{z}{2\pi}\right) f(z) \right| \leq \frac{2\exp(r\sin\theta(1+2\pi x+2\pi t))}{r} \prod_{0<\gamma\leq B} J_0\left(\frac{4\pi z}{|\rho|}\right).$$

In addition

$$\prod_{0<\gamma\leq B}\left|J_o\left(\frac{4\pi z}{|\rho|}\right)\right| \leq \exp\left(r\sin\theta\sum_{0<\gamma\leq B}\frac{4\pi z}{|\rho|}\right)$$

so that

$$\left|e\left(-tz-\frac{z}{2\pi}\right)f(z)\right| \leq \frac{2}{r}\exp\left(4\pi\; r\sin\theta\left(\sum_{0<\gamma\leq B}\frac{1}{|\rho|}+\frac{1}{4\pi}+\frac{x}{2}+\frac{t}{2}\right)\right).$$

Thus $\left|e\left(-tz-\frac{z}{2\pi}f(z)\right)\right|\to 0$ uniformly with respect to arg z as $|z|\to\infty$ for $0\leq \arg z\leq \pi$ if $t\leq -2\left(\sum_{0<\gamma\leq B}\frac{1}{|\rho|}+\frac{1}{4\pi}+\frac{x}{2}\right)$, which shows that $\hat{f}(t)=0$ for these t. For $t\geq 0$, we have

$$\int_{-\infty}^{\infty} e(-tz)\, f(z)dz = -\int_{-\infty}^{\infty} e(tz)\,\frac{e(-zx)-1}{2\pi iz}\prod_{0<\lambda\leq B} J_0\left(\frac{4\pi z}{|\rho|}\right),$$

and proceeding as before, we obtain $\hat{f}(t)=0$ for

$t\geq 2\left(\sum_{0<\gamma\leq B}\frac{1}{|\rho|}+\frac{1}{4\pi}+\frac{x}{2}\right)$. This completes the proof of the lemma.

We next prove

Lemma 2.11. If $0 \le x \le 1$ and $B \ge e^{58.223085}$, then

$$\frac{1}{512} \le \frac{1}{2\left(\sum_{0<\gamma\le B} \frac{1}{|\rho|} + \frac{1}{4\pi} + \frac{x}{2}\right)}.$$

Proof. Put

$$\sum_{0<\gamma\le B} \frac{1}{|\rho|} = \sum_{0<\gamma<1005} \frac{1}{|\rho|} + \sum_{1005<\gamma\le B} \frac{1}{|\rho|} = T_1 + T_2.$$

We obtain $T_1 = 2.0325543$ by direct calculation. From Lemma 2.8, we have

$$T_2 \le \int_{1005}^{B+} \frac{dN(\gamma)}{|\frac{1}{2}+iy|} \le \frac{1}{4\pi}\left(\left(\log\frac{B}{2\pi}\right)^2 - \left(\log\frac{1005}{2\pi}\right)^2\right)$$

$$+ \frac{S(B^+)+G(B^+)}{B} - \frac{S(1005)+G(1005)}{1005} + \frac{1}{(1005)^2}\left(\frac{25}{96\pi} + (A_H-A_L)\log 1005\right.$$

$$\left. + (B_H-B_L)\log\log 1005 - C_L\log\log\left(1005+\sqrt{\frac{7}{4}}\right) + D_H - D_L + \frac{A_H}{2} + \frac{B_H}{2\log 1005}\right)$$

$$- \frac{1}{B^2}\left(\frac{25}{96\pi} + \frac{A_H}{2} + \frac{B_H}{2\log 1005}\right).$$

Since $|S(B^+)+G(B^+)| \le 0.137\log B + 0.443\log\log B + 4.35$ and $S(1005) + G(1005) \ge 0.34741$ we have

$$T_2 \le 251.$$

This completes the proof of the lemma.

The theorem follows directly form Theorem 2.9, Lemma 2.10 and Lemma 2.11.

3. The calculation of G(x). In Theorem 2.9, we reduced the problem of evaluating G(x) for $0 \leq x \leq 1$ to that of evaluating

$$\frac{1}{2} + h \sum_{n=-\infty}^{\infty} \left(\frac{e(nhx)-1}{2\pi i n h} \prod_{0 \leq \gamma \leq B} J_0\left(\frac{4\pi n h}{|\rho|}\right)\right)$$

when $0 < h \leq \frac{1}{512}$ and $B = e^{58.2230805}$.

For n = 0, we have

$$\lim_{n \to 0} h \frac{e(nhx)-1}{2\pi i n h} \prod_{0<\gamma\leq B} J_0\left(\frac{4\pi n h}{|\rho|}\right) = hx.$$

For n > 0, we have

$$\frac{e(nhx)-1}{2\pi i n h} \prod_{0<\gamma\leq B} J_0\left(\frac{4\pi nh}{|\rho|}\right) + \frac{e(-nhx)-1}{-2\pi i n h} \prod_{0<\gamma\leq B} J_0\left(\frac{-4\pi nh}{|\rho|}\right)$$

$$= \frac{2\sin 2\pi nhx}{2\pi nh} \prod_{0<\gamma\leq B} J_0\left(\frac{4\pi nh}{|\rho|}\right).$$

Using the above equalities, we obtain

$$G(x) = \frac{1}{2} + hx + 2h \sum_{n=1}^{\infty} \frac{\sin 2\pi nhx}{2\pi nh} \prod_{0<\gamma\leq B} J_0\left(\frac{4\pi nh}{|\rho|}\right) + \epsilon$$

where $|\epsilon| \leq 10^{-12.24}$.

We now take $h = \frac{1}{512}$. We first truncate the above sum.

Using (31), we obtain

$$\left|\frac{1}{256\pi}\sum_{n=8065}^{\infty}\frac{\sin \pi nx/256}{n/256}\prod_{0<\gamma\leq B} J_0\left(\frac{\pi n/128}{|\rho|}\right)\right|$$

$$\leq \frac{1}{256\,\pi}\sum_{n=8065}^{\infty}\frac{1}{n/256}\prod_{0<\gamma\leq B}\min\left(\left(\frac{256\,|\rho|}{\pi^2 n}\right)^{\frac{1}{2}}\left(1+\frac{16\,|\rho|}{\pi n}\right),\ 1\right)$$

$$\leq \frac{1}{\pi}\int_{31.5}^{\infty}\frac{1}{t}\prod_{0<\gamma\leq 235}\left(\frac{|\rho|}{\pi^2 t}\right)^{\frac{1}{2}}\left(1+\frac{|\rho|}{16\pi t}\right)dt$$

$$\leq 0.5\cdot 10^{-18}.$$

Using this, we obtain

$$G(x) = \tfrac{1}{2} + \frac{x}{512} + \frac{1}{256\pi}\sum_{n=1}^{8064}\frac{\sin \pi\, nx/256}{n/256}\prod_{0<\nu\leq B} J_0\left(\frac{\pi n/128}{|\rho|}\right) + \epsilon,$$

with $|\epsilon| \leq 10.^{-12.23}$.

Through calculation we show that

$$\left|\frac{1}{256\pi}\sum_{n=5377}^{8064}\frac{\sin \pi\, nx/256}{n/256}\prod_{0<\gamma\leq B} J_0\left(\frac{\pi\, n/128}{|\rho|}\right)\right| \leq 3.36\cdot 10^{-24}.$$

Hence

$$G(x) = \tfrac{1}{2} + \frac{x}{512} + \frac{1}{256\pi} \sum_{n=1}^{5376} \frac{\sin \pi nx/256}{n/256} \prod_{0<\gamma\leq B} J_0\left(\frac{\pi n/128}{|\rho|}\right) + \epsilon$$

with $|\epsilon| \leq 10^{-12.22}$.

Since we cannot compute $J_0\left(\frac{t}{|\rho|}\right)$ for all $0 < \gamma \leq B$, we need

<u>Theorem 2.12.</u> Let $J_0\left(\frac{t}{|\rho|}\right) = \sum_{k=0}^{\infty} u_k$,

$$T_L\left(\frac{t}{|\rho|}\right) = \sum_{k=1}^{5} u_k + 0.9999121\, u_6$$

and

$$T_H\left(\frac{t}{|\rho|}\right) = \sum_{k=1}^{6} u_k.$$

Let

$$C_L\left(|\rho|, t\right) = \sum_{m=1}^{36} \underline{c}\,(m, t)\,|\rho|^{-2m}$$

$$= T_L\left(\frac{t}{|\rho|}\right) - \frac{T_L\left(\frac{t}{|\rho|}\right)^2}{2} + \frac{T_L\left(\frac{t}{|\rho|}\right)^3}{3} - \frac{T_L\left(\frac{t}{|\rho|}\right)^4}{4} + \frac{T_L\left(\frac{t}{|\rho|}\right)^5}{5}$$

$$- \frac{1.0043233\; T_L\left(\frac{t}{|\rho|}\right)^6}{6}$$

and

$$C_H(|\rho|, t) = \sum_{m=1}^{36} \overline{c}\,(m, t)\,|\rho|^{-2m}$$

$$= T_H\left(\frac{t}{|\rho|}\right) - \frac{T_H\left(\frac{t}{|\rho|}\right)^2}{2} + \frac{T_H\left(\frac{t}{|\rho|}\right)^3}{3} - \frac{T_H\left(\frac{t}{|\rho|}\right)^4}{4} + \frac{T_H\left(\frac{t}{|\rho|}\right)^5}{5} - \frac{T_H\left(\frac{t}{|\rho|}\right)^6}{6} .$$

Put a = 1005. Put b = B. Let

$$K_L(n) = \frac{1}{2\pi(2n-1)a^{2n-1}}\left(\log\frac{a}{2\pi} + \frac{1}{2n-1}\right) - \frac{1}{2\pi(2n-1)b^{2n-1}}\left(\log\frac{b}{2\pi} + \frac{1}{2n-1}\right)$$

$$+ \frac{S(b)+G(b)}{b^{2n}} - \frac{0.34741}{a^{2n}} + \frac{2n}{a^{2n+1}}\left(\frac{-25}{48\pi(2n+1)} + (A_L - A_H)\log a\right.$$

$$+ (B_L - B_H)\log\log a + C_L \log\log\left(a+\sqrt{\frac{7}{4}}\right) + (D_L - D_H) + \frac{A_L}{2n+1} + \frac{B_L}{(2n+1)\log a}$$

$$\left. + \frac{C_L}{(2n+1)\log\left(a+\sqrt{\frac{7}{4}}\right)}\right) - \frac{2n}{(2n+1)b^{2n+1}}\left(\frac{-25}{48\pi} + A_L + \frac{B_L}{\log a} + \frac{C_L}{\log\left(a+\sqrt{\frac{7}{4}}\right)}\right)$$

$$- \frac{n}{4}\left(\frac{1}{2\pi(2n+1)a^{2n+1}}\left(\log\frac{a}{2\pi} + \frac{1}{2n+1}\right) - \frac{1}{2\pi(2n+1)b^{2n+1}}\left(\log\frac{b}{2\pi}\right.\right.$$

$$\left. + \frac{1}{2n+1}\right) + \frac{S(b)+G(b)}{b^{2n+2}} - \frac{0.34741}{a^{2n+2}} + \frac{2n+2}{a^{2n+3}}\left(\frac{-25}{48\pi(2n+3)} + (A_L - A_H)\log a\right.$$

$$+ (B_L - B_H)\log\log a + C_L \log\log\left(a+\sqrt{\frac{7}{4}}\right) + (D_L - D_H) + \frac{A_L}{2n+3} + \frac{B_L}{(2n+3)\log a}$$

$$\left.\left. + \frac{C_L}{(2n+3)\log a + \sqrt{\frac{7}{4}}}\right) - \frac{2n+2}{(2n+3)b^{2n+3}}\left(\frac{-25}{48\pi} + A_L + \frac{B_L}{\log a} + \frac{C_L}{\log a + \sqrt{\frac{7}{4}}}\right)\right)$$

and

$$K_H(n) = \frac{1}{2\pi(2n-1)a^{2n-1}}\left(\log\frac{a}{2\pi} + \frac{1}{2n-1}\right) - \frac{1}{2\pi(2n-1)b^{2n-1}}\left(\log\frac{b}{2\pi}\right.$$

$$\left. + \frac{1}{2n-1}\right) + \frac{S(b) + G(b)}{b^{2n}} - \frac{0.34741}{a^{2n}} + \frac{2n}{a^{2n+1}}\left(\frac{25}{48\pi(2n+1)} + (A_H - A_L)\log a\right.$$

$$+ (B_H - B_L)\log\log a - C_L \log\log\left(a + \sqrt{\frac{7}{4}}\right) + (D_H - D_L) \quad + \frac{A_H}{2n+1}$$

$$\left. + \frac{B_H}{(2n+1)\log a}\right) \quad - \frac{2n}{(2n+1)b^{2n+1}}\left(\frac{25}{48\pi} + A_H + \frac{B_H}{\log a}\right).$$

Let

$$L_L(n, t) = \begin{cases} K_L(n) & \text{if } \underline{c}\,(n, t) \geq 0, \\ K_H(n) & \text{if } \underline{c}\,(n, t) < 0, \end{cases}$$

$$L_H(n, t) = \begin{cases} K_H(n) & \text{if } \overline{c}\,(n, t) \geq 0, \\ K_L(n) & \text{if } \overline{c}\,(n, t) < 0, \end{cases}$$

$$M_L(t) = \sum_{n=1}^{36} \underline{c}\,(n, t)\, L_L(n, t)$$

and

$$M_H(t) = \sum_{n=1}^{36} \overline{c}\,(n, t)\, L_H(n, t).$$

Let

$$N_L(t) = \begin{cases} \exp(M_L(t)) & \text{if } \dfrac{\sin xt/2}{t} \prod_{0<\gamma<1005} J_0\left(\dfrac{t}{|\rho|}\right) \geq 0, \\ \exp(M_H(t)) & \text{otherwise} \end{cases}$$

and

$$N_H(t) = \begin{cases} \exp(M_H(t)) & \text{if } \dfrac{\sin xt/2}{t} \prod_{0<\gamma<1005} J_0\left(\dfrac{t}{|\rho|}\right) \geq 0, \\ \exp(M_L(t)) & \text{otherwise.} \end{cases}$$

If $0 \leq x \leq 1$, then

$$\tfrac{1}{2} + \frac{x}{512} + \frac{1}{256\pi} \sum_{n=1}^{5376} \frac{\sin \pi nx/256}{n/256} \prod_{0<\gamma<1005} J_0\left(\frac{\pi n/128}{|\rho|}\right) N_L\left(\frac{\pi n}{128}\right)$$

$$- \epsilon \leq G(x) \leq \tfrac{1}{2} + \frac{x}{512}$$

$$+ \frac{1}{256\pi} \sum_{n=1}^{5376} \frac{\sin \pi nx/256}{n/256} \prod_{0<\gamma<1005} J_0\left(\frac{\pi n/128}{|\rho|}\right) N_H\left(\frac{\pi n}{128}\right) + \epsilon,$$

where $|\epsilon| \leq 10^{-12}$.

To prove Theorem 2.12, we need the following auxilliary results.

Lemma 2.13. Let $J_0(z) = \sum_{n=0}^{N} \dfrac{(-1)^m (z/2)^{2m}}{(m!)^2} + R_N$. If N is odd and $z \leq 2(N+3)$, then

$$\frac{\left(\frac{z^2}{4}\right)^{N+1}}{((N+1)!)^2}\left(1 - \frac{\left(\frac{z^2}{4}\right)}{(N+2)^2} + \frac{\left(\frac{z^2}{4}\right)^2}{(N+2)^2(N+3)^2} - \frac{\left(\frac{z^2}{4}\right)^3}{(N+2)^2(N+3)^2(N+4)^2}\right)$$

$$\leq R_N \leq \frac{\left(\frac{z^2}{4}\right)^{N+1}}{((N+1)!)^2} .$$

<u>Proof.</u> We have

$$R_N = \frac{\left(\frac{z^2}{4}\right)^{N+1}}{((N+1)!)^2} - \frac{\left(\frac{z^2}{4}\right)^{N+2}}{((N+2)!)^2} + \ldots .$$

If $M \geq N$ and M is odd, then

$$- \frac{\left(\frac{z^2}{4}\right)^{M+2}}{((M+2)!)^2} + \frac{\left(\frac{z^2}{4}\right)^{M+3}}{((M+3)!)^2}$$

$$= \frac{\left(\frac{z^2}{4}\right)^{M+2}}{((M+2)!)^2}\left(\frac{\frac{z^2}{4}}{(M+3)^2} - 1\right) \leq 0.$$

The lemma follows directly.

<u>Lemma 2.14.</u> If $0 \leq t \leq 42\pi$, then

$$\exp\left(\sum_{n=1}^{36} \underline{c}(n,t)\int_{1005}^{B^+} \frac{dN(y)}{|\frac{1}{2}+iy|^{2n}}\right) \leq \prod_{1005<\gamma\leq B} J_0\left(\frac{t}{|\rho|}\right) \leq$$

$$\exp\left(\sum_{n=1}^{36} \overline{c}(n,t)\int_{1005}^{B^+} \frac{dN(y)}{|\frac{1}{2}+iy|^{2n}}\right).$$

Proof. From the previous lemma we have

$$1+\sum_{k=1}^{5} u_k + u_6\left(1-\frac{\left(\frac{t}{2|\rho|}\right)^2}{7^2 8^2}+\frac{\left(\frac{t}{2|\rho|}\right)^4}{7^2 8^2}-\frac{\left(\frac{t}{2|\rho|}\right)^6}{7^2 8^2 9^2}\right)\leq J_0\left(\frac{t}{|\rho|}\right)$$

$$\leq 1+\sum_{k=1}^{6} u_k.$$

Since $\frac{t}{|\rho|} \leq 0.1312904$, we have

$$1+T_L\left(\frac{t}{|\rho|}\right)\leq J_0\left(\frac{t}{|\rho|}\right)\leq 1+T_H\left(\frac{t}{|\rho|}\right).$$

Since $0 \leq -T_H\left(\frac{t}{|\rho|}\right) \leq -T_L\left(\frac{t}{|\rho|}\right) \leq 0.0043047$, we can use Lemma 1.17 to obtain

$$C_L(|\rho|, t) \leq \log J_0\left(\frac{t}{|\rho|}\right) \leq C_H(|\rho|, t).$$

To complete the proof of Lemma 2.14, we note that

$$\sum_{1005<\gamma\le B} c_L\left(|\rho|, t\right) = \sum_{1005<\gamma\le B} \sum_{n=1}^{36} \underline{c}\,(n,t)\,|\rho|^{-2n}$$

$$= \sum_{n=1}^{36} \underline{c}\,(n,t) \int_{1005}^{B^+} \frac{dN(y)}{\left|\frac{1}{2}+iy\right|^{2n}}$$

and

$$\sum_{1005<\gamma\le B} c_H\left(|\rho|, t\right) \doteq \sum_{n=1}^{36} \bar{c}\,(n,t) \int_{1005}^{B^+} \frac{dN(y)}{\left|\frac{1}{2}+iy\right|^{2n}} .$$

Using Lemma 28, we obtain

$$K_L(n) \le \int_{1005}^{B^+} \frac{dN(y)}{\left|\frac{1}{2}+iy\right|^{2n}} \le K_H(n).$$

Thus

$$\exp(M_L(t)) = \exp\left(\sum_{n=1}^{36} \underline{c}\left(|\rho|, t\right) L_N(n,t)\right) \le \prod_{1005<\gamma\le B} J_0\left(\frac{t}{|\rho|}\right)$$

$$\le \exp\left(\sum_{n=1}^{36} \bar{c}\left(|\rho|, t\right) L_H(n,t)\right) = \exp(M_H(t)),$$

which completes the proof of the theorem.

In computing G(x), we use the tables in Haselgrove [12, pp. 58-64] to approximate the value of γ with an error of $\leq 2.10^{-6}$. Denoting the values from the table by $\underline{\gamma}$, we have

$$2.032554186 \leq \sum_{0<\underline{\gamma}<1005} \frac{1}{|\frac{1}{2}+i\underline{\gamma}|} \leq 2.032554275.$$

Thus, if H(x) is the number we calculate

$$H(x - 10^{-7}) \leq G(x) \leq H(x + 10^{-7}).$$

Our calculations show that this causes a relative error in G(x) of at most 10^{-7}.

The relative error in calculating $J_0\left(\frac{t}{|\underline{\rho}|}\right)$, where $\underline{\rho} = \frac{1}{2} + i\underline{\gamma}$, is

$$\leq \begin{cases} 0.30 \cdot 10^{-15} & \text{if } 0 \leq \frac{t}{|\rho|} \leq 4, \\ 0.60 \cdot 10^{-15} & \text{if } 4 \leq \frac{t}{|\rho|} \leq 8, \\ 0.20 \cdot 10^{-11} & \text{if } 8 \leq \frac{t}{|\rho|} \leq 10. \end{cases}$$

Thus for $0 \leq t \leq 21\pi$ we have a relative error in calculating $\prod_{0<\underline{\gamma}<1005} J_0\left(\frac{t}{|\rho|}\right)$ of $\leq 2.0 \cdot 10^{-13}$. For $21\pi \leq t \leq 42\pi$ we have a relative error in calculating $\prod_{0<\underline{\gamma}<1005} J_0\left(\frac{t}{|\rho|}\right)$ of $\leq 0.30 \cdot 10^{-11}$. Therefore, our absolute error in calculating H(x) is $\leq 5 \cdot 10^{-10}$.

4. Asymptotic results for $G'(u)$. In this section we prove

Theorem 2.15. Let $f(z) = \sum_{\gamma>0} \log I_0\left(\frac{z}{|\rho|}\right) - zu/2$. If $T > 50$ is chosen so that $f'(T) = 0$, then

$$G'(u) = \frac{\exp(f(T))}{\sqrt{8\pi f''(T)}}\left(1 + O\left(\frac{1}{\sqrt{T \log T}}\right)\right).$$

We need several preliminary results.

Lemma 2.16. Let $G'(u)$ be the density function of the random variable $\mathbf{Y} = \sum_{\gamma>0} \frac{2\sin 2\pi\theta_\gamma}{|\rho|}$. Then

$$G'(u) = \frac{1}{4\pi i}\int_{T-i\infty}^{T+i\infty} \exp(f(z))\,dz.$$

Proof. As in Theorem 2.4 we have

$$G'(u) = \int_{-\infty}^{\infty} e(\lambda u)\hat{G}(\lambda)\,d\lambda$$

where

$$G(\lambda) = \int_{-\infty}^{\infty} e(-\lambda x)\,dG(x) = \sum_{\gamma>0} J_0\left(\frac{4\pi\lambda}{|\rho|}\right).$$

Since $I_0(z) = J_0(-iz)$ we have

$$G'(u) = -i\int_{-i\infty}^{i\infty} \exp(-2\pi z u) \prod_{\gamma>0} I_0\left(\frac{4\pi z}{|\rho|}\right) dz$$

$$= \frac{1}{4\pi i}\int_{T-i\infty}^{T+i\infty} \exp(-zu/2) \prod_{\gamma>0} I_0\left(\frac{z}{|\rho|}\right) dz,$$

which completes the proof of the lemma.

Lemma 2.17. Let

$$U_1 = \frac{1}{4\pi i}\int_{T-iT}^{T-iT^{2/3}/\left(\log\frac{T}{2\pi}\right)^{\frac{1}{2}}} \exp(f(z))dz + \frac{1}{4\pi i}\int_{T+iT^{2/3}/\left(\log\frac{T}{2\pi}\right)^{\frac{1}{2}}}^{T+iT} \exp(f(z))dz$$

and

$$U_2 = \frac{1}{4\pi i}\int_{T-i\infty}^{T-iT} \exp(f(z))dz + \frac{1}{4\pi i}\int_{T+iT}^{T+i\infty} \exp(f(z))dz.$$

If $T > 50$, then

$$G'(u) = \frac{1}{4\pi i}\int_{T-iT^{2/3}/\left(\log\frac{T}{2\pi}\right)^{\frac{1}{2}}}^{T+iT^{2/3}/\left(\log\frac{T}{2\pi}\right)^{\frac{1}{2}}} \exp(f(z))dz + U_1 + U_2,$$

where

$$|U_1| \leq \exp(f(T))\left(\frac{\exp\left(-0.0207121\,T^{1/3}\right)T^{1/3}}{0.0414242\,\pi\sqrt{\log\frac{T}{2\pi}}\left(1+\sqrt{1+\frac{4}{0.0207121\,\pi\,T^{1/3}}}\right)}\right)$$

and

$$|U_2| < \exp(f(T))\,\frac{\exp(-0.0429599T)}{0.0859198\pi}.$$

<u>Proof</u>. From the previous lemma we have

$$G'(u) = \frac{1}{4\pi i}\int_{T-iT^{2/3}/\left(\log\frac{T}{2\pi}\right)^{\frac{1}{2}}}^{T+iT^{2/3}/\left(\log\frac{T}{2\pi}\right)^{\frac{1}{2}}} \exp(f(z))dz + U_1 + U_2.$$

In order to estimate U_1 and U_2 we first examine, for $z = T + it$,

$$\operatorname{Re} f(z) - f(T) = \operatorname{Re} f(T+it) - f(T)$$

$$= \log \prod_{\gamma>0}\left|I_0\left(\frac{z}{|\rho|}\right)\right| - \log \prod_{0<\gamma} I_0\left(\frac{T}{|\rho|}\right)$$

$$= \log \prod_{\gamma > 0} \left| \frac{I_0\left(\frac{T+it}{|\rho|}\right)}{I_0\left(\frac{T}{|\rho|}\right)} \right| .$$

We have

$$\left| I_0\left(\frac{T+it}{|\rho|}\right) \right| \le \frac{1}{\pi} \int_0^{\pi} \left| e^{\frac{T}{|\rho|}\cos\theta} \right| \left| e^{\frac{it}{|\rho|}\cos\theta} \right| d\theta$$

$$\le \frac{1}{\pi} \int_0^{\pi} e^{\frac{T}{|\rho|}\cos\theta} \, d\theta = I_0\left(\frac{T}{|\rho|}\right).$$

Combining the two above equations we obtain

$$\text{(33)} \quad \operatorname{Re} f(z) - f(t) \le \sum_{\substack{\gamma > 0 \\ \frac{T}{|\rho|} \le 1, \ \frac{|t|}{|\rho|} \le 1}} \log \left| \frac{I_0\left(\frac{T+it}{|\rho|}\right)}{I_0\left(\frac{T}{|\rho|}\right)} \right| = \frac{1}{2} \sum_{\substack{\gamma > 0 \\ T, |t| \le |\rho|}} \log \left| \frac{I_0\left(\frac{T+it}{|\rho|}\right)}{I_0\left(\frac{T}{|\rho|}\right)} \right|^2$$

$$= \frac{1}{2} \sum_{\substack{\gamma > 0 \\ T, |t| < |\rho|}} \log\left(1 + I_0\left(\frac{T}{|\rho|}\right)^{-2} g(z, T, |\rho|)\right)$$

where

$$g(z, T, |\rho|) = 2 \operatorname{Re}\left(I_0\left(\frac{z}{|\rho|}\right) - I_0\left(\frac{T}{|\rho|}\right)\right) I_0\left(\frac{T}{|\rho|}\right) + \operatorname{Re}^2\left(I_0\left(\frac{z}{|\rho|}\right) - I_0\left(\frac{T}{|\rho|}\right)\right) + \operatorname{Im}^2\left(I_0\left(\frac{z}{|\rho|}\right) - I_0\left(\frac{T}{|\rho|}\right)\right)$$

We have, for $|x|$, $|y| \le 1$, $\operatorname{Re}(I_0(x+iy)) - I_0(x) \le 0$, and hence

(34)

$$2 \operatorname{Re}(I_0(x+iy) - I_0(x)) I_0(x) + \operatorname{Re}^2(I_0(x+iy) - I_0(x)) + \operatorname{Im}^2(I_0(x+iy) - I_0(x))$$

$$\le 2 \operatorname{Re}(I_0(x+iy) - I_0(x)) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^6} + \frac{x^6}{2^8 3^2} + \cdots + \frac{x^{18}}{2^{32} 3^8 5^2 7^2}\right)$$

$$+ \mathrm{Re}^2(I_0(x+iy) - I_0(x)) + \mathrm{Im}^2 I_0(x+iy).$$

For $s = x+iy$ we write

$$I_0(s) = 1 + \frac{s^2}{2^2} + \frac{s^4}{2^6} + \frac{s^6}{2^8 3^2} + \frac{s^8}{2^{14} 3^2} + \frac{s^{10}}{2^{16} 3^2 5^2} + \frac{s^{12}}{2^{20} 3^4 5^2} E(s),$$

where $E(s) = 1 + \frac{s^2}{4 \cdot 7^2} + \frac{s^4}{4^2 7^2 8^2} + \cdots$. Using this in (34), we obtain

(35)

$$2\,\mathrm{Re}(I_0(x+iy) - I_0(x))\,I_0(x) + \mathrm{Re}^2(I_0(x+iy) - I_0(x)) + \mathrm{Im}^2(I_0(x+iy) - I_0(x))$$

$$\leq f_0 y^2 + f_2 y^2 x^2 + f_4 y^2 x^4 + f_6 y^2 x^6 + f_8 y^2 x^8 + f_{10} x^{10} + g(x,y),$$

where

(36) $$g(x,y) = f_{12} x^{12} + f_{14} x^{14} + f_{16} x^{16} + f_{18} x^{18} + f_{20} x^{20} + f_{22} x^{22}$$

$$+ f_{24} x^{24} + f_{26} x^{26} + f_{28} x^{28} + f_{30} x^{30}$$

and

(38)

$$f_0 = \frac{-1}{2} + \frac{3y^2}{2^5} - \frac{5y^4}{2^6 3^2} + \frac{35y^6}{2^{13} 3^2} - \frac{7y^8}{2^{14} 5^2} + \frac{461 y^{10}}{2^{19} 3^4 5^2} - \frac{17 y^{12}}{2^{20} 3^4 5^2} + \frac{19 y^{14}}{2^{28} 3^3 5^2}$$

$$- \frac{y^{16}}{2^{29} 3^4 5^2} + \frac{y^{18}}{2^{32} 3^4 5^4} + \left(\frac{y^{10}}{2^{14} 3^4 5^2} - \frac{y^{12}}{2^{21} 3^4 5^2} + \frac{y^{14}}{2^{25} 3^4 5^2} - \frac{y^{16}}{2^{27} 3^6 5^2} \right.$$

$$\left. + \frac{y^{18}}{2^{33} 3^6 5^2} - \frac{y^{20}}{2^{35} 3^6 5^4} \right) \mathrm{Re}\, E(s) + \frac{|E(s)|^2 y^{22}}{2^{40} 3^8 5^4} + x\left(\frac{y^9}{2^{17} 3^3 5^2} - \frac{y^{11}}{2^{20} 3^4 5} + \frac{y^{13}}{2^{22} 3^4 5^2} \right.$$

$$-\frac{y^{15}}{2^{26}3^{5}5^{2}}+\frac{y^{17}}{2^{31}3^{6}5^{2}}-\frac{y^{19}}{2^{34}3^{6}5^{4}}\Bigg)\operatorname{Im}E(s),$$

$$(38)\quad f_2=\frac{-1}{2^{4}}+\frac{y^{2}}{2^{6}3}-\frac{7y^{4}}{2^{12}3^{2}}+\frac{7y^{6}}{2^{14}3^{2}5}+\frac{19y^{8}}{2^{18}3^{3}5^{2}}-\frac{13y^{10}}{2^{20}3^{4}5^{2}}+\frac{y^{12}}{2^{25}3^{4}}+\frac{53y^{14}}{2^{29}3^{4}5^{2}}$$

$$+\frac{y^{16}}{2^{31}3^{4}5^{3}}+\Bigg(\frac{-11y^{8}}{2^{18}3^{3}5^{2}}+\frac{43y^{10}}{2^{21}3^{4}5^{2}}-\frac{y^{12}}{2^{22}3^{3}5^{2}}+\frac{y^{14}}{2^{27}3^{4}5^{2}}+\frac{31y^{16}}{2^{32}3^{7}5^{2}}-\frac{y^{18}}{2^{32}3^{5}5^{3}}\Bigg)$$

$$\cdot\operatorname{Re}E(s)-\frac{41|E(s)|^{2}y^{20}}{2^{40}3^{7}5^{4}}+x\Bigg(\frac{-11y^{7}}{2^{17}3^{4}5}+\frac{y^{9}}{2^{20}3^{4}5}-\frac{y^{11}}{2^{22}3^{3}5^{2}}-\frac{y^{13}}{2^{23}3^{6}5^{2}}+\frac{7y^{15}}{2^{31}3^{4}5^{2}}$$

$$+\frac{53y^{17}}{2^{33}3^{6}5^{4}}\Bigg)\operatorname{Im}E(s),$$

$$(39)\quad f_4=\frac{-5}{2^{7}3}+\frac{y^{2}}{2^{12}3}-\frac{11y^{4}}{2^{13}3\cdot 5}-\frac{19y^{6}}{2^{19}3^{3}5}+\frac{7y^{8}}{2^{20}3^{2}5^{2}}+\frac{y^{10}}{2^{26}3^{3}5}-\frac{29y^{12}}{2^{26}3^{4}5}+\frac{2569y^{14}}{2^{32}3^{4}5^{3}}$$

$$+\Bigg(\frac{11y^{6}}{2^{19}3^{2}5}-\frac{121y^{8}}{2^{21}3^{4}5^{2}}-\frac{y^{10}}{2^{23}3^{2}5^{2}}+\frac{y^{12}}{2^{25}3^{5}5}-\frac{139y^{14}}{2^{33}3^{4}5^{2}}-\frac{893y^{16}}{2^{35}3^{5}5^{4}}\Bigg)\operatorname{Re}E(s)$$

$$+\frac{209|E(s)|^{2}y^{18}}{2^{37}3^{7}5^{4}}+x\Bigg(\frac{11y^{5}}{2^{16}3^{2}5^{2}}-\frac{11y^{7}}{2^{20}3^{4}5^{2}}-\frac{y^{9}}{2^{22}3^{5}}+\frac{y^{11}}{2^{26}3^{5}5^{2}}-\frac{151y^{13}}{2^{29}3^{7}5}$$

$$-\frac{15719y^{15}}{2^{34}3^{6}5^{4}}\Bigg)\operatorname{Im}E(s),$$

$$(40)\quad f_6=\frac{-7}{2^{12}3^{2}}+\frac{29y^{2}}{2^{12}3^{2}5}-\frac{3599y^{4}}{2^{17}3^{4}5^{2}}-\frac{41y^{6}}{2^{19}3^{3}5^{2}}+\frac{41y^{8}}{2^{25}3^{2}5^{2}}+\frac{217y^{10}}{2^{24}3^{4}5^{2}}-\frac{71y^{12}}{2^{29}3\cdot 5^{3}}$$

$$+\Bigg(\frac{-77y^{4}}{2^{17}3^{3}5^{2}}-\frac{11y^{6}}{2^{21}3^{3}5}+\frac{11y^{8}}{2^{22}3^{4}5^{2}}+\frac{29y^{10}}{2^{25}3^{6}5^{2}}+\frac{571y^{12}}{2^{29}3^{6}5^{2}}+\frac{8729y^{14}}{2^{35}3^{6}5^{3}}\Bigg)\operatorname{Re}E(s)$$

$$-\frac{12595|E(s)|^2y^{16}}{2^{40}3^85^3}+x\left(\frac{-11y^3}{2^{16}3^25^2}-\frac{11y^5}{2^{18}3^35^2}-\frac{11y^7}{2^{22}3^45^2}+\frac{y^9}{2^{23}3^45^2}+\frac{931y^{11}}{2^{29}3^65^2}\right.$$

$$\left.+\frac{11803y^{13}}{2^{32}3^65^4}\right)\operatorname{Im}E(s),$$

(41) $$f_8=\frac{-67}{2^{14}3^25}+\frac{701y^2}{2^{19}3^35}+\frac{41y^4}{2^{19}3^35^2}+\frac{1099y^6}{2^{27}3^45^2}-\frac{8209y^8}{2^{28}3^45^2}+\frac{6407y^{10}}{2^{31}3^35^3}$$

$$+\left(\frac{11y^2}{2^{14}3^25}+\frac{11y^4}{2^{21}3^35}+\frac{11y^6}{2^{24}3^25^2}+\frac{11y^8}{2^{26}3^55^2}-\frac{34811y^{10}}{2^{33}3^65^2}-\frac{36086y^{12}}{2^{34}3^65^4}\right)\operatorname{Re}E(s)$$

$$+\frac{7667|E(s)|^2y^{14}}{2^{40}3^65^3}+x\left(\frac{11y}{2^{17}3^45}-\frac{11y^3}{2^{20}3^45^2}+\frac{11y^5}{2^{22}3^45^2}+\frac{11y^7}{2^{25}3^65}-\frac{1841y^9}{2^{30}3^65^2}\right.$$

$$\left.-\frac{47757y^{11}}{2^{33}3^65^4}\right)\operatorname{Im}E(s),$$

(42) $$f_{10}=\frac{-31y^2}{2^{18}3^35^2}-\frac{7y^4}{2^{20}3^25^2}-\frac{41y^6}{2^{25}3^25^2}+\frac{6169y^8}{2^{28}3^45^2}-\frac{63937y^{10}}{2^{30}3^45^4}+\left(\frac{-11y^2}{2^{18}3^35^2}\right.$$

$$\left.-\frac{121y^4}{2^{21}3^45^2}+\frac{11y^6}{2^{22}3^45^2}-\frac{11y^8}{2^{26}3^55^2}+\frac{2233y^{10}}{2^{31}3^65^2}+\frac{97571y^{12}}{2^{35}3^65^4}\right)\operatorname{Re}E(s)$$

$$-\frac{17017|E(s)|^2y^{14}}{2^{39}3^55^4}+x\left(\frac{-y}{2^{17}3^35^2}+\frac{y^3}{2^{18}3^4}+\frac{y^5}{2^{22}3^35}+\frac{y^7}{2^{23}3^45^2}+\frac{1519y^9}{2^{30}3^75^2}\right.$$

$$\left.+\frac{3812y^{11}}{2^{31}3^55^4}\right)\operatorname{Im}E(s),$$

$$(43)\quad f_{12} = \frac{y^2}{2^{21}3^4} + \frac{17y^4}{2^{26}3^4 5^2} - \frac{3143y^6}{2^{27}3^6 5} + \frac{691961y^8}{2^{33}3^6 5^3}$$

$$+ \frac{(\mathrm{Re}\,E(s) - \mathrm{Re}\,E(s)\,E(x) + 634032\,|E(s)|^2)y^{12}}{2^{34}3^8 5^4} + \left(\frac{1}{2^{19}3^4 5^2} - \frac{43y^2}{2^{21}3^4 5^2} - \frac{y^4}{2^{23}3^2 5^2}\right.$$

$$\left. - \frac{29y^6}{2^{25}3^6 5^2} - \frac{49y^8}{2^{32}3^5 5} - \frac{24011y^{10}}{2^{34}3^5 5^4}\right)\mathrm{Re}\,E(s) + \left(\frac{-1}{2^{19}3^4 5^2} + \frac{y^2}{2^{24}3^3 5^2} - \frac{y^4}{2^{25}3^4 5^2}\right.$$

$$\left. + \frac{y^6}{2^{27}3^6 5^2} + \frac{y^8}{2^{33}3^6 5^2} + \frac{y^{10}}{2^{35}3^6 5^4}\right)E(x) + x\left(\frac{-y}{2^{19}3^3 5^2} + \frac{y^3}{2^{22}3^3 5^2} + \frac{y^5}{2^{24}3^5 5^2}\right.$$

$$\left. - \frac{7y^7}{2^{24}3^5 5^2} - \frac{30817y^9}{2^{33}3^6 5^4} + \frac{(1 - E(x))y^{11}}{2^{37}3^7 5^4}\right)\mathrm{Im}\,E(s),$$

$$(44)\quad f_{14} = \frac{103y^2}{2^{25}3^2 5^2 7^2} + \frac{21439y^4}{2^{26}3^5 5^2 7^2} - \frac{101293y^6}{2^{29}3^6 5 \cdot 7^2} + \frac{7y^8}{2^{34}3^6 5^3}$$

$$+ \frac{(-22\,\mathrm{Re}\,E(s) + 22\,\mathrm{Re}\,E(s)E(x) - 153153\,|E(s)|^2)y^{10}}{2^{39}3^7 5^4} + \left(\frac{1}{2^{21}3^4 5^2} - \frac{y^2}{2^{22}3^3 5^2}\right.$$

$$\left. - \frac{y^4}{2^{25}3^5 5} - \frac{y^6}{2^{25}3^7 5^2} + \frac{1577y^8}{2^{34}3^6 5^2} + \frac{y^{12}}{2^{41}3^8 5^4 7^2}\right)\mathrm{Re}\,E(s) + \left(\frac{-1}{2^{21}3^4 5^2} + \frac{y^2}{2^{24}3^3 5^2}\right.$$

$$\left. - \frac{y^4}{2^{27}3^5 \cdot 5} + \frac{7y^6}{2^{31}3^7 5^2} + \frac{y^8}{2^{35}3^4 5^3}\right)E(x) + x\left(\frac{-y}{2^{22}3^4 5^2} - \frac{y^3}{2^{33}3^6 5^2} - \frac{y^5}{2^{29}3^5 5^2}\right.$$

$$\left. + \frac{5069y^7}{2^{32}3^6 5^4} + \frac{(-11 + 11\,E(x))y^9}{2^{37}3^8 5^3} + \frac{y^{11}}{2^{39}3^7 5^4 7^2}\right)\mathrm{Im}\,E(s),$$

$$(45)\quad f_{16} = \frac{-7747y^2}{2^{31}3^5 5 \cdot 7^2} + \frac{9238237y^4}{2^{35}3^6 5^3 7^2} - \frac{8797y^6}{2^{37}3^6 5^3 7^2} + \frac{83y^8}{2^{43}3^6 5^3 7}$$

$$+ \frac{(22\,\mathrm{Re}\,E(s) - 22\,\mathrm{Re}\,E(s)\,E(x) + 7667\,|E(s)|^2)y^8}{2^{40}3^6 5^3} + \left(\frac{1}{2^{25}3^4 5^2} - \frac{y^2}{2^{27}3^4 5^2}\right.$$

$$- \frac{19y^4}{2^{33}3^6 5^2} - \frac{3157y^6}{2^{35}3^6 5^3} - \frac{11y^{10}}{2^{40}3^7 5^4 7^2} + \frac{y^{12}}{2^{41}3^8 5^4 7^2}\Bigg) \operatorname{Re} E(s) + \Bigg(\frac{-1}{2^{25}3^4 5^2}$$

$$+ \frac{y^2}{2^{27}3^5 5} + \frac{7y^4}{2^{32}3^6 5} + \frac{7y^6}{2^{34}3^6 5^3}\Bigg) \operatorname{Re} E(x) + x\Bigg(\frac{-y}{2^{26}3^5 5^2} - \frac{7y^3}{2^{31}3^6 5^2} - \frac{2237y^5}{2^{34}3^6 5^4}$$

$$+ \frac{11(1-E(x))y^7}{2^{36}3^6 5^4} - \frac{11y^9}{2^{39}3^8 5^3 7^2} + \frac{y^{11}}{2^{47}3^7 5^4 7^2}\Bigg) \operatorname{Im} E(s),$$

(46) $$f_{18} = \frac{181799y^2}{2^{34}3^8 5^3 7^2} + \frac{87368y^4}{2^{37}3^8 5^3 7^2} - \frac{2729y^6}{2^{41}3^{10}5^3 7^2} + \frac{367y^8}{2^{44}3^{10}5^3 7^2}$$

$$+ \frac{(1848(1-E(x))\operatorname{Re} E(s) - 62975|E(s)|^2)y^6}{2^{40}3^8 5^4} + \Bigg(\frac{1}{2^{27}3^6 5^2} + \frac{y^2}{2^{32}3^6 5^2} + \frac{803y^4}{2^{35}3^6 5^4}$$

$$+ \frac{11y^8}{2^{41}3^6 5^3 7^2} - \frac{11y^{10}}{2^{48}3^7 5^4 7^2} + \frac{y^{12}}{2^{51}3^{12}5^4 7^4}\Bigg) \operatorname{Re} E(s) + \Bigg(\frac{-1}{2^{27}3^6 5^2} + \frac{7y^2}{2^{31}3^6 5^2}$$

$$- \frac{7y^4}{2^{34}3^5 5^3}\Bigg) E(x) + x\Bigg(\frac{-y}{2^{31}3^6 5^2} - \frac{y^3}{2^{33}3^4 5^3} + \frac{11(E(x)-1)y^5}{2^{36}3^6 5^4} + \frac{11y^7}{2^{38}3^6 5^4 7^2}$$

$$- \frac{11y^9}{2^{47}3^8 5^3 7^2} + \frac{y^{11}}{2^{49}3^{11}5^4 7^4}\Bigg) \operatorname{Im} E(s),$$

(47) $$f_{20} = \frac{-7207y^2}{2^{37}3^7 5^3 7^2} + \frac{3919y^4}{2^{42}3^6 5^3 7^2} - \frac{1771y^6}{2^{44}3^{10}5^3 7^2} + \frac{y^8}{2^{47}3^8 5^3 7^2}$$

$$+ \frac{(165(1-E(x))\operatorname{Re} E(s) + 836|E(s)|^2)y^4}{2^{39}3^7 5^4} + \Bigg(\frac{1}{2^{33}3^6 5^2} + \frac{y^2}{2^{35}3^4 5^4} - \frac{11y^6}{2^{39}3^7 5^4 7}$$

$$+ \frac{11y^8}{2^{49}3^6 5^3 7^2} - \frac{11y^{10}}{2^{50}3^{11}5^4 7^4}\Bigg) \operatorname{Re} E(s) + \Bigg(\frac{-1}{2^{33}3^6 5^4} + \frac{y^2}{2^{35}3^4 5^3}\Bigg) E(x) + x\Bigg(\frac{-y}{2^{34}3^6 5^4}$$

$$+ \frac{11(1-E(x))y^3}{2^{37}3^8 5^4} - \frac{11y^5}{2^{38}3^6 5^4 7^2} + \frac{11y^7}{2^{46}3^6 5^4 7^2} - \frac{11y^9}{2^{49}3^{12}5^3 7^4}\Bigg) \operatorname{Im} E(s),$$

$$(48)\quad f_{22} = \frac{-12709y^2}{2^{41}3^9 5^3 7^2} + \frac{377y^4}{2^{43}3^{10}5^3 7} - \frac{y^6}{2^{46}3^9 5^3 7}$$

$$+ \frac{(44(E(x)-1)\,\mathrm{Re}\,E(s) - 41|E(s)|^2)y^2}{2^{40}3^7 5^4} + \frac{\mathrm{Re}\,E(s) - E(x)}{2^{35}3^6 5^4} + \left(\frac{11y^4}{2^{41}3^6 5^3 7^2}\right.$$

$$\left. - \frac{11y^6}{2^{47}3^7 5^4 7} + \frac{11y^8}{2^{51}3^{10}5^3 7^4}\right)\mathrm{Re}\,E(s) + x\left(\frac{(E(x)-1)y}{2^{37}3^7 5^4} + \frac{11y^3}{2^{39}3^8 5^3 7^2} - \frac{11y^5}{2^{46}3^6 5^4 7^2}\right.$$

$$\left. + \frac{11y^7}{2^{48}3^{10}5^4 7^4}\right)\mathrm{Im}\,E(s),$$

$$(49)\quad f_{24} = \frac{-869y^2}{2^{45}3^{10}5^3 7^2} + \frac{y^4}{2^{46}3^9 5^3 7} + \frac{2(\mathrm{Re}\,E(s) - \mathrm{Re}\,E(s)E(x) - E(x)) + E(x)^2 + |E(s)|^2}{2^{40}3^8 5^4}$$

$$+\left(\frac{-11y^2}{2^{40}3^7 5^4 7^2} + \frac{11y^4}{2^{49}3^6 5^3 7^2} - \frac{11y^6}{2^{44}3^{11}5^4 7^3}\right)\mathrm{Re}\,E(s) + x\left(\frac{-y}{2^{39}3^7 5^4 7^2} + \frac{11y^3}{2^{47}3^8 5^3 7^2}\right.$$

$$\left. - \frac{11y^5}{2^{48}3^{10}5^4 7^4}\right)\mathrm{Im}\,E(s),$$

$$(50)\quad f_{26} = \frac{-1}{2^{47}3^8 5^3 7^2} + \frac{\mathrm{Re}\,E(s) - E(x)}{2^{41}3^8 5^4 7^2} + \left(\frac{-11y^2}{2^{48}3^7 5^4 7^2} + \frac{11y^4}{2^{51}3^{10}5^3 7^4}\right)\mathrm{Re}\,E(s)$$

$$+ x\left(\frac{-y}{2^{47}3^7 5^4 7^2} + \frac{11y^3}{2^{49}3^{12}5^3 7^4}\right)\mathrm{Im}\,E(s),$$

$$(51)\quad f_{28} = \frac{\mathrm{Re}\,E(s) - E(x)}{2^{49}3^8 5^4 7^2} - \frac{11y^2\,\mathrm{Re}\,E(s)}{2^{50}3^{11}5^4 7^4} + \frac{-xy\,\mathrm{Im}\,E(s)}{2^{49}3^{11}5^4 7^4}$$

and

$$(52)\quad f_{30} = \frac{\mathrm{Re}\,E(s) - E(x)}{2^{51}3^{12}5^4 7^4}\,.$$

We now assume that $0 \leq x \leq 1$ and $-1 \leq y \leq 0$. In order to bound the terms in (35), we need to examine $E(x+iy)$ more closely. We have, for $s = x+iy$

$$E(s) = 1 + \frac{s^2}{4\cdot 7^2} + \frac{s^4}{4^2\cdot 7^2\cdot 8^2} + \frac{s^6}{4^3\cdot 7^2\cdot 8^2\cdot 9^2} + \cdots$$

$$= 1 + \frac{x^2+\binom{2}{1}x(iy)+(iy)^2}{4\cdot 7^2} + \frac{x^4+\binom{4}{1}x^3(iy)+\binom{4}{2}x^2(iy)^2+\binom{4}{3}x(iy)^3+(iy)^4}{4^2\cdot 7^2\cdot 8^2} + \cdots$$

Thus

(53) $$\operatorname{Re} E(s) = 1 + \frac{x^2-y^2}{4\cdot 7^2} + \frac{x^4-\binom{4}{2}x^2y^2+y^4}{4^2\cdot 7^2\cdot 8^2} + \cdots$$

and

(54) $$\operatorname{Im} E(s) = \frac{\binom{2}{1}xy}{4\cdot 7^2} + \frac{\binom{4}{1}x^3y-\binom{4}{3}xy^3}{4^2\cdot 7^2\cdot 8^2} + \cdots .$$

In addition, we write

$$E(x) = 1 + \frac{x^2}{4\cdot 7^2} + \frac{x^4}{4^2\cdot 7^2\cdot 8^2} + \cdots .$$

Since $\sum_{i=0}^{n}\binom{n}{i} = 2^n$, the sum of the coefficients of the degree n polynomials in (53) and (54) is $< 2^n$. Thus

(55) $$\left|\operatorname{Re} E(s)-1\right| \leq \frac{1}{4\cdot 7^2} + \frac{1}{4^2\cdot 7^2\cdot 8} + \frac{1}{7^2\cdot 8^2\cdot 9^2} + \frac{1}{7^2\cdot 8^2\cdot 9^2\cdot 10^2} + \cdots$$

$$\leq 0.0052653,$$

$$\frac{xy}{2\cdot 7^2} + 0.0001633xy \leq \operatorname{Im} E(s) \leq \frac{xy}{2\cdot 7^2} - 0.0001633xy$$

and

$$\frac{-y^2}{4\cdot 7^2} - 0.0001633y^2 \leq \operatorname{Re} E(s) - E(x) \leq \frac{-y^2}{4\cdot 7^2} + 0.0001633y^2.$$

From the above two equations we obtain

(56) $$0.0103673xy \leq \operatorname{Im} E(s) \leq 0.0100407xy$$

and

(57) $$-0.0052653y^2 \leq \operatorname{Re} E(s) - E(x) \leq -0.0049387y^2.$$

Using (55), (56), and (57) in (36) we obtain

$$g(x,y) < \frac{y^2x^{12}}{2^{20}3^4} + \frac{2(\operatorname{Re} E(s) - \operatorname{Re} E(s)E(x) - E(x)) + E(x)^2 + |E(s)|^2}{2^{40}3^8 5^4} - \frac{(0.0103673)^2 y^2}{2^{40}3^8 5^4}.$$

Since

$$2(\operatorname{Re} E(s) - \operatorname{Re} E(s)E(x) - E(x)) + E(x)^2 + |E(s)|^2$$

$$= (\operatorname{Re} E(s) - E(x))(\operatorname{Re} E(s) - E(x) + 2) + \operatorname{Im} E(s)^2$$

$$\leq (0.0103673)^2 y^2,$$

we have

(58) $$g(x,y) < \frac{x^{12}y^2}{2^{20}3^4}.$$

We next examine the leading terms in (35). We have, from (37),

$$
\begin{aligned}
(59)\quad f_0 = {} & \frac{-1}{2} + \frac{3y^2}{2^5} - \frac{5y^4}{2^6 3^2} + \frac{35y^6}{2^{13}3^2} - \frac{y^8}{2^{14}5} + \frac{y^8}{2^{14}5^2}\Bigg(-2 + \frac{xy\,\mathrm{Im}\,E(s)}{2^3 3^3} \\
& + \frac{y^2(461 + \mathrm{Re}\,E(s))}{2^5 3^4} - \frac{5xy^3\,\mathrm{Im}\,E(s)}{2^6 3^4}\Bigg) + \frac{y^{12}}{2^{21}3^4 5^2}\Bigg(-34 - \mathrm{Re}\,E(s) + 2xy\,\mathrm{Im}\,E(s) \\
& + \frac{y^2(57 + 128\,\mathrm{Re}\,E(s))}{2^7} - \frac{xy^3\,\mathrm{Im}\,E(s)}{2^3 3}\Bigg) + \frac{y^{16}}{2^{29}3^6 5^2}\Bigg(-9 - 4\mathrm{Re}\,E(s) + \frac{xy\,\mathrm{Im}\,E(s)}{2^2} \\
& + \frac{y^2(18 + 25\,\mathrm{Re}\,E(s))}{2^5 5^2} - \frac{xy^3\,\mathrm{Im}\,E(s)}{2^5 5^2}\Bigg) + \frac{y^{20}}{2^{35}3^6 5^4}\Bigg(-\mathrm{Re}\,E(s) + \frac{y^2|E(s)|^2}{2^5 3^2}\Bigg) \\
\le {} & \frac{-1}{2} + \frac{3y^2}{2^5} - \frac{5y^4}{2^6 3^2} + \frac{35y^6}{2^{13}3^2} - \frac{y^8}{2^{14}5} \le -0.3994805y^2,
\end{aligned}
$$

since each of the terms in parentheses is negative and $-\frac{1}{2} + \frac{3y^2}{2^5} - \frac{5y^4}{2^6 3^2} + \frac{35y^6}{2^{13}3^2} - \frac{y^8}{2^{14}5}$ is an increasing function of y^2 for $y^2 \le 1$. From (38) we obtain

$$
\begin{aligned}
(60)\quad f_2 = {} & \frac{-1}{2^4} + \frac{y^2}{2^6 3} - \frac{7y^4}{2^{13}3^2} + \frac{y^4}{2^{13}3^2}\Bigg(-7 + \frac{7y^2}{2\cdot 5} - \frac{11xy^3\,\mathrm{Im}\,E(s)}{2^4 3^2 5} + \frac{19y^4}{2^5\cdot 3\cdot 5^2} \\
& + \frac{y^5\,\mathrm{Im}\,E(s)}{2^7 3^2 5}\Bigg) + \frac{y^8}{2^{18}3^3 5^2}\Bigg(-11\mathrm{Re}\,E(s) - \frac{13y^2}{2^2\cdot 3} + \frac{43y^2\,\mathrm{Re}\,E(s)}{2^3\cdot 3} - \frac{y^4\,\mathrm{Re}\,E(s)}{2^4} \\
& + \frac{y^4}{2^7\cdot 3\cdot 5^2} + \frac{y^6(53 + 4\,\mathrm{Re}\,E(s))}{2^{11}3} + \frac{y^8(54 + 155\,\mathrm{Re}\,E(s))}{2^{14}3^4 5} - \frac{y^2\,\mathrm{Re}\,E(s)}{2^{14}3^2 5} \\
& - \frac{41y^2|E(s)|^2}{2^{22}3^4 5^2} - \frac{xy^3\,\mathrm{Im}\,E(s)}{2^4} - \frac{xy^5\,\mathrm{Im}\,E(s)}{2^5 3^3}\Bigg) \le \frac{-1}{2^4} + \frac{y^2}{2^6 3} - \frac{7y^4}{2^{13}3^2} \le -0.0573866,
\end{aligned}
$$

since each of the terms in parentheses is negative and $\frac{-1}{2^4} + \frac{y^2}{2^6 3} - \frac{7y^4}{2^{13}3^2}$ is an increasing function of y^2 for $y^2 \le 1$. From (39) we have

(61) $$f_4 = \frac{-5}{2^7 3} + \frac{y^2}{2^{12} 3} - \frac{10y^4}{2^{13} \cdot 3 \cdot 5} + \frac{y^4}{2^{13} 3 \cdot 5}\Big(-1 + \frac{11xy \operatorname{Im} E(s)}{2^3 3 \cdot 5}$$

$$+ \frac{y^2(-19 + 33 \operatorname{Re} E(s))}{2^6 3^2} - \frac{11xy^3 \operatorname{Im} E(s)}{2^7 3^3 5} + \frac{y^4(126 - 121 \operatorname{Re} E(s))}{2^8 3^3 5} - \frac{xy^5 \operatorname{Im} E(s)}{2^9 3^2}$$

$$+ \frac{y^6(5-24 \operatorname{Re} E(s))}{2^{13} 3^2 5} + \frac{xy^7 \operatorname{Im} E(s)}{2^{13} 3^4 5}\Big) + \frac{y^{12}}{2^{26} 3^4 5}\Big(-29 + \frac{2 \operatorname{Re} E(s)}{3} - \frac{151xy \operatorname{Im} E(s)}{2^3 3^3}$$

$$+ \frac{2569y^2}{2^6 5^2} - \frac{139y^2 \operatorname{Re} E(s)}{2^7 5} - \frac{15719xy^3 \operatorname{Im} E(s)}{2^8 3^2 5^3} - \frac{893y^4 \operatorname{Re} E(s)}{2^9 3 \cdot 5^3} + \frac{209y^6 |E(s)|^2}{2^{11} 3^3 5^3}\Big)$$

$$\leq \frac{-5}{2^7 3} + \frac{y^2}{2^{12} 3} - \frac{10y^4}{2^{13} 3 \cdot 5} \leq -.01298,$$

since each of the terms in parentheses is negative and $\frac{-5}{2^7 3} + \frac{y^2}{2^{12} 3} - \frac{10y^4}{2^{13} 3 \cdot 5}$ has its maximum at $y^2 = \frac{1}{2}$. From (40) we obtain

(62) $$f_6 = \frac{-7}{2^{12} 3^2} + \frac{29y^2}{2^{12} 3^2 5} - \frac{11xy^3 \operatorname{Im} E(s)}{2^{16} 3^2 5^2} - \frac{3599y^4}{2^{17} 3^4 5^2} + \frac{y^4}{2^{17} 3^4 5^2}\Big(-231 \operatorname{Re} E(s)$$

$$- \frac{33xy \operatorname{Im} E(s)}{2} - \frac{3y^2(164 + 55 \operatorname{Re} E(s))}{2^4} - \frac{11xy^3 \operatorname{Im} E(s)}{2^5} + \frac{y^4(369 + 88 \operatorname{Re} E(s))}{2^8}$$

$$+ \frac{xy^5 \operatorname{Im} E(s)}{2^6} + \frac{y^6(3906 + 29 \operatorname{Re} E(s))}{2^8 3^2} + \frac{931xy^7 \operatorname{Im} E(s)}{2^{12} 3^2} + \frac{y^8(-17253 + 2855 \operatorname{Re} E(s))}{2^{12} 3^2 5}$$

$$+ \frac{11803xy^9 \operatorname{Im} E(s)}{2^{15} 3^2 5^2} + \frac{8729y^{10} \operatorname{Re} E(s)}{2^{18} 3^2 5^2} - \frac{12595y^{12} |E(s)|^2}{2^{23} 3^4 5}\Big) \leq \frac{1}{2^{12} 3^2}\Big(-7 + \frac{29y^2}{5}$$

$$- \frac{11xy^3 \operatorname{Im} E(s)}{2^4 5^2} - \frac{3599y^4}{2^{17} 3^4 5^2}\Big) \leq \frac{-3.4627824}{2^{12} 3^2},$$

since each of the terms in parentheses is negative and $-7 + \frac{29y^2}{5} - \frac{11xy^3 \operatorname{Im} E(s)}{2^4 5^2} - \frac{3599y^4}{2^{17} 3^4 5^2}$ is a decreasing function of y for $-1 \leq y \leq 0$.

From (41) we have

$$(63)\quad f_8 = \frac{-67}{2^{14}3^2 5} + \frac{11xy \operatorname{Im} E(s)}{2^{17}3^4 5} + \frac{y^2(701 + 33 \operatorname{Re} E(s))}{2^{19}3^3 5} - \frac{11xy^3 \operatorname{Im} E(s)}{2^{20}3^4 5^2}$$

$$+ \frac{y^4(164 + 55 \operatorname{Re} E(s))}{2^{21}3^3 5^2} + \frac{11xy^5 \operatorname{Im} E(s)}{2^{22}3^4 5^2} + \frac{y^6(1099 + 792 \operatorname{Re} E(s))}{2^{27}3^4 5^2} + \frac{11xy^7 \operatorname{Im} E(s)}{2^{25}3^6 5}$$

$$+ \frac{y^8}{2^{28}3^4 5^2}\Big(-8209 + \frac{44 \operatorname{Re} E(s)}{3} - \frac{1841xy \operatorname{Im} E(s)}{2^2 3^2} + \frac{y^2(-691956 - 174055 \operatorname{Re} E(s))}{2^5 3^2 5}$$

$$- \frac{47757xy^3 \operatorname{Im} E(s)}{2^5 3^2 5^2} - \frac{36086y^4 \operatorname{Re} E(s)}{2^6 3^2 5^2} + \frac{7667y^6 |E(s)|^2}{2^{12}3^2 5}\Big)$$

$$\leq \frac{1}{2^{14}3^2 5}\Big(-67 + \frac{11xy \operatorname{Im} E(s)}{2^3 3^2} + \frac{y^2(701 + 33 \operatorname{Re} E(s))}{2^5 3} - \frac{11xy^3 \operatorname{Im} E(s)}{2^6 3^2 5}$$

$$+ \frac{y^4(164 + 55 \operatorname{Re} E(s))}{2^7 3 \cdot 5} + \frac{11xy^5 \operatorname{Im} E(s)}{2^8 3^2 5} + \frac{y^6(1099 + 792 \operatorname{Re} E(s))}{2^{13}3^2 5} + \frac{11xy^7 \operatorname{Im} E(s)}{2^{11}3^4}\Big)$$

$$\leq \frac{-59.23165}{2^{14}3^2 5},$$

since each of the terms in parentheses is negative and

$$-67 + \frac{11xy \operatorname{Im} E(s)}{2^3 3^2} + \frac{y^2(701 + 33 \operatorname{Re} E(s))}{2^5 3} - \frac{11xy^3 \operatorname{Im} E(s)}{2^6 3^2 5} + \frac{y^4(164 + 55 \operatorname{Re} E(s))}{2^7 3 \cdot 5}$$

$$+ \frac{11xy^5 \operatorname{Im} E(s)}{2^8 3^2 5} + \frac{y^6(1009 + 792) \operatorname{Re} E(s))}{2^{13}3^2 5} + \frac{11xy^7 \operatorname{Im} E(s)}{2^{11}3^4}$$ is a decreasing function

of y for $-1 \leq y \leq 0$. Finally, it is clear from (42) that

(64) $$f_{10} \leq 0 \text{ for } -1 \leq y \leq 0.$$

Combining (35), (58), (59), (60), (61), (62), (63) and (64) we obtain

(65) $$2\,\mathrm{Re}(I_0(x+iy) - I_0(x))I_0(x) + \mathrm{Re}^2(I_0(x+iy) - I_0(x)) + \mathrm{Im}^2(I_0(x+iy) - I_0(x)) \leq -0.3994805y^2.$$

Using the fact that $I_0(x) \leq 0.46576e$ and (65) in (33), we have, for $t \leq 0$,

(66) $$\mathrm{Re}\, f(T+it) - f(T) \leq \tfrac{1}{2} \sum_{\substack{\gamma > 0 \\ T \leq |\rho| \\ -|\rho| \leq t \leq 0}} \log\left(1 - \frac{0.3994805 \frac{t^2}{|\rho|^2}}{(0.46576)^2 e^2}\right)$$

$$\leq \tfrac{1}{2} \sum_{\substack{\gamma > 0 \\ T \leq |\rho| \\ -|\rho| \leq t \leq 0}} \log \exp\left(-0.260276 \frac{t^2}{|\rho|^2}\right)$$

$$= -0.130138 \sum_{\substack{\gamma > 0 \\ T \leq |\rho| \\ -|\rho| \leq t \leq 0}} \frac{t^2}{|\rho|^2}$$

$$\leq \begin{cases} -0.130138t^2 \int_{t-}^{\infty} \frac{dN(y)}{|\frac{1}{2}+iy|^2} & \text{if } T \leq |t|, \\ -0.130138t^2 \int_{T-}^{\infty} \frac{dN(y)}{|\frac{1}{2}+iy|^2} & \text{if } |t| \leq T. \end{cases}$$

Using Lemma 2.8 we obtain

$$\int_c^\infty \frac{dN(\gamma)}{|w|^2} \geq \frac{\log \frac{c}{2\pi}}{2\pi c} - \frac{1}{c^2}$$

for c $\geq$ 50. Applying this to (66), we obtain

$$\text{(67)} \qquad \operatorname{Re} f(T+it) - f(T) \leq \begin{cases} -0.0207121\, t \log \frac{t}{2\pi} & \text{if } T \leq |t|, \\ -0.0207121 \frac{t^2}{T} \log \frac{T}{2\pi} & \text{if } |t| \leq T. \end{cases}$$

We can now bound U_1 and U_2. Since $f(T-it) = \overline{f(T+it)}$, we have

$$U_1 = \frac{1}{2\pi} \int_{-T}^{-T^{2/3}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}} \operatorname{Re} \exp(f(T+it))dt.$$

Using (67) we obtain

$$|U_1| < \frac{1}{2\pi} \int_{-T}^{-T^{2/3}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}} \exp(\operatorname{Re} f(T+it))\, dt$$

$$\leq \frac{1}{2\pi} \exp(f(T)) \int_{-T}^{-T^{2/3}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}} \exp\left(-0.0207121 \frac{t^2}{T} \log \frac{T}{2\pi}\right) dt$$

$$= \frac{1}{2\pi} \exp(f(T)) \sqrt{\frac{T}{0.0207121 \log \frac{T}{2\pi}}} \int_{\sqrt{0.0207121\, T^{1/3}}}^{\sqrt{0.0207121\, T \log \frac{T}{2\pi}}} \exp(-u^2)\, du.$$

Using a result from Abromowitz and Stegun [1, p.298], that

$$\frac{\exp(-x^2)}{x+\sqrt{x^2+2}} < \int_x^\infty \exp(-t^2)\,dt \le \frac{\exp(-x^2)}{x+\sqrt{x^2+\frac{4}{\pi}}} \quad (x \ge 0),$$

we have

(69) $$|U_1| \le \exp(f(T))\exp\left(-0.0207121\, T^{1/3}\right)$$

$$\cdot\left(\frac{T^{1/3}}{0.0414242\pi\left(1+\sqrt{1+\frac{4}{0.0207121\pi T^{1/3}}}\right)\sqrt{\log\frac{T}{2\pi}}}\right)$$

Similarly

$$U_2 = \frac{1}{2\pi}\int_{-\infty}^{-T} \operatorname{Re}\exp(f(T+it))\,dt$$

and for $50 \le T$,

(70) $$|U_2| \le \exp(f(T))\,\frac{\exp(-0.0429599T)}{0.0859198\pi}.$$

This completes the proof of Lemma 2.17.

Finally, we prove

Lemma 2.18. If $f'(T) = 0$, then

$$\frac{1}{4\pi i}\int_{T-iT^{2/3}/(\log\frac{T}{2\pi})^{1/2}}^{T+iT^{2/3}/(\log\frac{T}{2\pi})^{1/2}} \exp(f(z))dz = \frac{\exp(f(T))}{\sqrt{8\pi f''(T)}}\left(1+O\left(\frac{1}{\sqrt{T\log T}}\right)\right).$$

Proof. We write

(71)
$$\frac{1}{4\pi i}\int_{T-iT^{2/3}/(\log\frac{T}{2\pi})^{1/2}}^{T+iT^{2/3}/(\log\frac{T}{2\pi})^{1/2}} \exp(f(z))dz = \frac{1}{4\pi}\int_{-T^{2/3}/(\log\frac{T}{2\pi})^{1/2}}^{T^{2/3}/(\log\frac{T}{2\pi})^{1/2}} \exp\Big(f(T)$$

$$+ \frac{df(T+iv)}{dv}\bigg|_{v=0} t + \frac{d^2f(T+iv)}{dv^2}\bigg|_{v=0} \frac{t^2}{2} + R_2(t)\Big)dt,$$

where $|R_2(t)| \le \max_{0\le v\le t} \left|\frac{d^3(f(T+iv))}{dv^3}\right| \frac{|t|^3}{6}$. We compute the derivatives using the formula $\frac{d^nf(T+iv)}{dv^n} = (i)^n \frac{d^nf(z)}{dz^n}\Big|_{T+iv}$. From Abramowitz and Stegun [1, p.376], we have

$$\frac{dI_j(z)}{dz} = I_{j-1}(z) - \frac{j}{z}I_j(z) = I_{j+1}(z) + \frac{j}{z}I_j(z).$$

Since $f(z) = \sum_{\gamma>0} \log I_0\left(\frac{z}{|\rho|}\right) - zu/2$, we obtain

$$f'(z) = \sum_{\gamma>0} \frac{1}{|\rho|}\frac{I_1\left(\frac{z}{|\rho|}\right)}{I_0\left(\frac{z}{|\rho|}\right)} - u/2,$$

$$f''(z) = \sum_{\gamma>0} \frac{1}{|\rho|^2} \frac{I_0\left(\frac{z}{|\rho|}\right)^2 - I_1\left(\frac{z}{|\rho|}\right)^2 - \frac{|\rho|}{z} I_0\left(\frac{z}{|\rho|}\right) I_1\left(\frac{z}{|\rho|}\right)}{I_0\left(\frac{z}{|\rho|}\right)^2}$$

and

$$f'''(z) = \sum_{\gamma>0} \frac{h(z, |\rho|)}{|\rho|^3 I_0\left(\frac{z}{|\rho|}\right)^3},$$

where

$$h(z, |\rho|) = \frac{|\rho|}{z} I_0\left(\frac{z}{|\rho|}\right)\left(I_1\left(\frac{z}{|\rho|}\right)^2 - I_0\left(\frac{z}{|\rho|}\right)^2\right) + 2I_1\left(\frac{z}{|\rho|}\right)\left(I_1\left(\frac{z}{|\rho|}\right)^2 - I_0\left(\frac{z}{|\rho|}\right)^2\right.$$

$$\left. + \frac{|\rho|}{z} I_0\left(\frac{z}{|\rho|}\right) I_1\left(\frac{z}{|\rho|}\right)\right) + 2 \frac{|\rho|^2}{z^2} I_0\left(\frac{z}{|\rho|}\right) I_1\left(\frac{z}{|\rho|}\right)^2 .$$

From Olver [19, p 269] we have, for Re $s \geq 1$

$$I_0(s) = \frac{\exp(s)}{\sqrt{2\pi s}}(A_0(s) - i \exp(-2s)B_0(s))$$

and

$$I_1(s) = \frac{\exp(s)}{\sqrt{2\pi s}} (A_1(s) + i \exp(-2s)B_1(s)),$$

where

$$A_0(s) = 1 + \frac{1}{8s} + \frac{9}{128s^2} + \frac{75}{1024s^3} + C_0(s),$$

$$B_0(s) = 1 - \frac{1}{8s} + \frac{9}{128s^2} - \frac{75}{1024s^3} + D_0(s),$$

$$A_1(s) = 1 - \frac{3}{8s} - \frac{15}{128s^2} - \frac{105}{1024s^3} + C_1(s),$$

$$B_1(s) = 1 + \frac{3}{8s} - \frac{15}{128s^2} + \frac{105}{1024s^3} + D_1(s),$$

$$|C_0(s)|, |D_0(s)| \le \frac{2.637}{(\mathrm{Re}\ s)^4}$$

and

$$|C_1(s)|, |D_1(s)| \le \frac{16.31}{(\mathrm{Re}\ s)^4}.$$

Hence,

(73)

$$\frac{1}{s} I_0(s)(I_1(s)^2 - I_0(s)^2) + 2I_1(s)(I_1(s)^2 - I_0(s)^2 + \frac{1}{s} I_0(s)I_1(s)) + \frac{2}{s^2} I_0(s)I_1(s)^2$$

$$= \frac{\exp(3s)}{(2\pi s)^{3/2}} \left(\frac{A_0(s)(A_1(s)^2 - A_0(s)^2)}{s} + 2A_1(s)\left(A_1(s)^2 - A_0(s)^2 + \frac{A_0(s)A_1(s)}{s}\right)\right.$$

$$+ \frac{2A_0(s)A_1(s)^2}{s^2} + \exp(-4s) \frac{A_0(s)(B_1(s)^2 - B_0(s)^2)}{s} + \frac{2B_0(s)(A_1(s)B_1(s) + A_0(s)B_0(s))}{s}$$

$$+ 2A_1(s)\left(B_1(s)^2 - B_0(s)^2 - \frac{B_0(s)B_1(s)}{s}\right) + 2B_1(s)\left(2A_1(s)A_0(s) + 2A_0(s)A_1(s)\right.$$

$$\left.+ \frac{A_0(s)B_1(s) - B_0(s)A_1(s)}{s}\right) + \frac{2A_0(s)B_1(s)^2}{s^2} + \left.\frac{4A_1(s)B_1(s)B_0(s)}{s^2}\right) + i\left(\exp(-2s)\right.$$

$$\left(\frac{A_0(s)(A_0(s)B_0(s)+A_1(s)B_1(s))}{s} - \frac{B_0(s)(A_1(s)^2-B_0(s)^2)}{s}\right.$$

$$+ 2B_1(s)\left(A_1(s)^2-A_0(s)^2 + \frac{A_0(s)A_1(s)}{s}\right)$$

$$+ 2A_1(s)\left(2A_1(s)B_1(s)+2A_0(s)B_0(s) + \frac{A_0(s)B_1(s)-B_0(s)A_1(s)}{s}\right)$$

$$\left. + \frac{4A_0(s)A_1(s)B_1(s)}{s^2} - \frac{2B_0(s)A_1(s)^2}{s^2}\right)$$

$$+ \exp(-6s)\left(\frac{B_0(s)(B_1(s)^2-A_0(s)^2)}{s} - 2B_1(s)\left(B_1(s)^2-B_0(s)^2 - \frac{B_0(s)B_1(s)}{s}\right)\right.$$

$$\left.\left. + \frac{2B_0(s)B_1(s)^2}{s^2}\right)\right) \quad .$$

Using our definitions of $A_0(s)$ and $A_1(s)$, we obtain

$$(74) \quad \frac{A_0(s)(A_1(s)^2-A_0(s)^2)}{s} + A_1(s)\left(A_1(s)^2-A_0(s)^2+\frac{A_0(s)A_1(s)}{s}\right)+\frac{2A_0(s)A_1(s)}{s^2}$$

$$= \frac{-1}{s^2} - \frac{3}{8s^3} - \frac{49}{128s^4} - \left(\frac{9183}{16384}+\frac{9}{8\cdot 32}+\frac{9}{4\cdot 128}+\frac{75}{8\cdot 128}\right)\frac{1}{s^4} - \left(\frac{9183}{8\cdot 16384}+\frac{81}{32\cdot 128}\right.$$

$$\left.+\frac{75}{32\cdot 128} - \frac{225}{16384}\right)\frac{1}{s^6} - \left(\frac{9\cdot 9183}{128\cdot 16384}+\frac{9\cdot 75}{8\cdot 128\cdot 32} - \frac{225}{8\cdot 16384}\right)\frac{1}{s^7}$$

$$- \left(\frac{75\cdot 9183}{8\cdot 128\cdot 16384} - \frac{9\cdot 225}{128\cdot 16384}\right)\frac{1}{s^8} + \frac{75\cdot 225}{8\cdot 128\cdot 16384}\,\frac{1}{s^9} + C_0(s)\left(\frac{-1}{s^2} - \frac{1}{4s^3} - \frac{9}{32s^4}\right.$$

$$\left.- \frac{9183}{16384s^5} + \frac{225}{16384s^6}\right) + \left(2+\frac{1}{4s}+\frac{9}{64s^2}+\frac{75}{4\cdot 128s^3}+C_0(s)\right)\left(\frac{-C_0(s)}{s} - \frac{1}{s} - \frac{1}{8s^2}\right.$$

$$\left.- \frac{9}{128s^3} - \frac{75}{8\cdot 128s^4}\right)C_0(s) + \left(2+\frac{3}{4s}+\frac{15}{64s^2}+\frac{105}{4\cdot 128s^3} - C_1(s)\right)\left(\frac{-C_0(s)}{s} - \frac{1}{s} - \frac{1}{8s^2}\right.$$

$$-\frac{9}{128s^3} - \frac{75}{8\cdot 128s^4}\Big) C_1(s) + \frac{2}{s^2} - \frac{5}{4s^3} - \frac{15}{64s^4} - \Big(\frac{15}{64} + \frac{3}{8\cdot 16} + \frac{27}{2\cdot 128} - \frac{150}{8\cdot 128}\Big)$$

$$\cdot\frac{1}{s^5} - \Big(\frac{15}{8\cdot 64} + \frac{27}{16\cdot 128} + \frac{225}{16\cdot 128} - \frac{1485}{16\cdot 128}\Big)\frac{1}{s^6} - \Big(\frac{15\cdot 9}{64\cdot 128} + \frac{225}{8\cdot 16\cdot 128}$$

$$-\frac{1575}{16\cdot 128} - \frac{1485}{8\cdot 64\cdot 128}\Big)\frac{1}{s^7} + \Big(\frac{(105)^2}{32(128)^2} + \frac{1575}{16(128)^2} + \frac{9\cdot 1485}{64(128)^2} - \frac{15\cdot 75}{8\cdot 64\cdot 128}\Big)\frac{1}{s^8}$$

$$+\Big(\frac{(105)^2}{4\cdot 8^2(128)^2} + \frac{9\cdot 1575}{2(128)^3} + \frac{75\cdot 1485}{8\cdot 64(128)^2}\Big)\frac{1}{s^9} + \Big(\frac{9(105)^2}{32(128)^3} + \frac{75\cdot 1575}{16(128)^3}\Big)\frac{1}{s^{10}}$$

$$+\frac{75(105)^2}{8\cdot 32(128)^3}\,\frac{1}{s^{11}} - C_1(s)\Big(\frac{2}{s^2} + \frac{3}{4s^3} + \frac{15}{64s^4} + \frac{105}{4\cdot 128s^5} - \frac{C_1(s)}{s^2}\Big)(2)\Big(1 + \frac{1}{8s}$$

$$+\frac{9}{128s^2} + \frac{75}{8\cdot 128s^3} + C_0(s)\Big) + \frac{2C_0(s)}{s^2}\Big(1 - \frac{3}{4s} - \frac{3}{32s^2} - \frac{15}{128s^3} + \frac{1485}{(128)^2s^4}$$

$$+\frac{1575}{4(128)^2s^5} + \frac{(105)^2}{8^2(128)^2s^6}\Big) - \frac{1}{s^2} - \frac{3}{8s^3} + \Big(\frac{-2\cdot 10335}{16384} + \frac{9}{32} + \frac{15}{128}\Big)\frac{1}{s^4} + \Big(\frac{-2\cdot 36480}{8^2(128)^2}$$

$$+\frac{6\cdot 10335}{8\cdot 16384} + \frac{45}{4\cdot 128} + \frac{105}{8\cdot 128}\Big)\frac{1}{s^5} + \Big(\frac{-2\cdot 2070}{8(128)^2} + \frac{6\cdot 36480}{8^3(128)^2} + \frac{30\cdot 10335}{128\cdot 16384} + \frac{315}{8\cdot 4\cdot 128}\Big)$$

$$\cdot\frac{1}{s^6} + \Big(\frac{-150\cdot 105}{(8\cdot 128)^2} + \frac{6\cdot 2070}{(8\cdot 128)^2} + \frac{30\cdot 36480}{8^2(128)^3} + \frac{2\cdot 105\cdot 10335}{8\cdot 128\cdot 16384}\Big)\frac{1}{s^7} + \Big(\frac{450\cdot 105}{8^3(128)^2}$$

$$+\frac{30\cdot 2070}{8(128)^3} + \frac{210\cdot 36480}{8^3(128)^3}\Big)\frac{1}{s^8} + \Big(\frac{15\cdot 150\cdot 105}{8^2(128)^3} + \frac{210\cdot 2070}{8^2(128)^3}\Big)\frac{1}{s^9} + \frac{(105)^2 150}{(8\cdot 128)^3}\,\frac{1}{s^{10}}$$

$$+2C_1(s)\Big(\frac{-1}{2s^2} - \frac{3}{8s^3} - \frac{10335}{16384s^4} - \frac{36480}{8^2(128)^2s^5} - \frac{2070}{8(128)^2s^6} - \frac{75\cdot 105}{(8\cdot 128)^2s^7}\Big)$$

$$+2\Big(1 - \frac{3}{8s} - \frac{15}{128s^2} - \frac{105}{8\cdot 128s^3} + C_1(s)\Big)\Big(C_1(s)\Big(\frac{1}{s} + \frac{1}{8s} + \frac{9}{128s^3} + \frac{75}{8\cdot 128s^4}\Big)\Big)$$

$$+ C_0(s)\left(\frac{1}{s} - \frac{3}{8s^2} - \frac{15}{128s^3} - \frac{105}{8\cdot 128s^4} + \frac{C_1(s)}{s}\right) - \frac{C_1(s)}{s}\left(2 + \frac{3}{4s} + \frac{15}{64s^2} + \frac{105}{4\cdot 128s^3}\right.$$

$$\left. - C_1(s)\right) - \frac{C_0(s)}{s}\left(2 + \frac{1}{4s} + \frac{9}{64s^2} + \frac{75}{4\cdot 128s^3} + C_0(s)\right)\Bigg)$$

$$= \frac{-2}{s^3} + O\left(\frac{1}{(\operatorname{Re} s)^4}\right) = O\left(\frac{1}{(\operatorname{Re} s)^3}\right) .$$

Combining (73) and (74), we obtain, for $\operatorname{Re} s \geq 1$.

(75) $$\frac{1}{s} I_0(s)(I_1(s)^2 - I_0(s)^2) + 2\, I_1(s)\left(I_1(s)^2 - I_0(s)^2 + \frac{I_0(s)I_1(s)}{s}\right)$$

$$+ \frac{2\, I_0(s)I_1(s)^2}{s^2} = O\left(\frac{\exp(3s)}{s^{3/2}(\operatorname{Re} s)^3}\right).$$

We have

$$I_0(s) = \sum_{k=0}^{\infty} \frac{\left(\frac{s^2}{4}\right)^k}{(k!)^2}$$

and

$$I_1(s) = \frac{s}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{s^2}{4}\right)^k}{k!\,(k+1)!} .$$

Thus

$$I_1(s) = I_0(s)\, \frac{I_1(s)}{I_0(s)} = \frac{s}{2}\, I_0(s)\, Q(s),$$

where

$$Q(s) = \frac{\sum_{k=0}^{\infty} \frac{\left(\frac{s^2}{4}\right)^k}{(k!)(k+1)!}}{\sum_{k=0}^{\infty} \frac{\left(\frac{s^2}{4}\right)^k}{(k!)^2}} = \frac{1 + \frac{s^2}{8} + \frac{s^4}{192} + \cdots}{1 + \frac{s^2}{4} + \frac{s^4}{64} + \cdots} .$$

For $\operatorname{Re} s \leq 1$ and $|\operatorname{Im} s| \leq \operatorname{Re} s$, we have

$$|Q(s)| \leq 1 + \frac{2}{8} + \frac{2^2}{4^2 \cdot 2 \cdot 6} + \frac{2^3}{4^3 \cdot 6 \cdot 24} + \cdots \leq 1.28.$$

Hence for $\operatorname{Re} s \leq 1$ and $|\operatorname{Im} s| \leq \operatorname{Re} s$ we have

(76)

$$\frac{1}{s} I_0(s)(I_1(s)^2 - I_0(s)^2) + 2 I_1(s)\left(I_1(s)^2 - I_0(s)^2 + \frac{I_0(s)I_1(s)}{s}\right) + \frac{2\, I_0(s)I_1(s)^2}{s^2}$$

$$= \frac{1}{s} I_0(s)\left(\frac{s^2}{4} I_0(s)^2 Q(s)^2 - I_0(s)^2\right) + s\, I_0(s)Q(s)\left(\frac{s^2}{4} I_0(s)^2 Q(s)^2 - I_0(s)^2 + \frac{I_0(s)^2 Q(s)}{2}\right) + \frac{I_0(s)^3 Q(s)^2}{2}$$

$$= I_0(s)^3\left(\frac{s^3 Q(s)^3}{4} + s\left(\frac{3Q(s)^2}{4} - Q(s)\right) + \frac{Q(s)^2}{2} - \frac{1}{s}\right).$$

Using (72), (75), (76) and Lemma 2.8, we obtain

(77)

$$|f'''(z)| << \sum_{\substack{0<\gamma \\ |\rho| \leq T}} \frac{1}{|\rho|^3 \operatorname{Re}\left(\frac{z}{|\rho|}\right)^3} + \sum_{\substack{0<\gamma \\ |\rho| \geq T}} \frac{1}{|\rho|^3}\left(\frac{|\rho|}{z} + 1 + \frac{z}{|\rho|} + \frac{z^3}{|\rho|^3}\right)$$

$$<< \frac{\log T}{T^2} .$$

We next estimate

$$f''(T) = \sum_{\gamma > 0} \frac{I_0\left(\frac{T}{|\rho|}\right)^2 - I_1\left(\frac{T}{|\rho|}\right)^2 - \frac{I_0\left(\frac{T}{|\rho|}\right) I_1\left(\frac{T}{|\rho|}\right)}{(T/|\rho|)}}{|\rho|^2 I_0\left(\frac{T}{|\rho|}\right)^2} .$$

From Abromowitz and Stegun [1, 9.1.14, 9.63], we have, for real x

$$I_0(x)^2 = J_0(ix)^2 = \sum_{k=0}^{\infty} \frac{(2k)!\left(\frac{x^2}{4}\right)^k}{(k!)^4} ,$$

$$I_1(x)^2 = -J_1(ix)^2 = \left(\frac{x}{2}\right)^2 \sum_{k=0}^{\infty} \frac{(2k+2)!\left(\frac{x^2}{4}\right)^k}{k!\,((k+1)!)^2 (k+2)!}$$

and

$$I_0(x)I_1(x) = -i\, J_0(ix)J_1(ix) = \frac{x}{2} \sum_{k=0}^{\infty} \frac{(2k+1)!\left(\frac{x^2}{4}\right)^k}{(k!)^2((k+1)!)^2} .$$

Hence

$$I_0(x)^2 - I_1(x)^2 - \frac{I_0(x)I_1(x)}{x} = \sum_{k=0}^{\infty} \frac{(2k)!\left(\frac{x^2}{4}\right)^k}{(k!)^4} - \sum_{k=0}^{\infty} \frac{(2k+2)!\left(\frac{x^2}{4}\right)^{k+1}}{k!\,((k+1)!)^2 (k+2)!}$$

$$- \tfrac{1}{2} \sum_{k=0}^{\infty} \frac{(2k+1)!\left(\frac{x^2}{4}\right)^k}{(k!)^2((k+1)!)^2}$$

$$= \tfrac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(2k)!\left(\frac{x^2}{4}\right)^k}{(k!)^2((k+1)!)^2} \right) \le \tfrac{1}{2} I_0(x)^2 .$$

Thus

(78)
$$0 \le \frac{I_0(x)^2 - I_1(x)^2 - \frac{I_0(x)I_1(x)}{x}}{I_0(x)^2} \le \tfrac{1}{2}.$$

As before, we write $I_0(x)$ and $I_1(x)$ in terms of $A_0(x)$, $B_0(x)$, $A_1(x)$ and $B_1(x)$ for $x \ge 1$. We have

$$I_0(x)^2 - I_1(x)^2 - \frac{I_0(x)I_1(x)}{x} = \frac{\exp(2x)}{2\pi x}\Bigg(A_0(x)^2 - A_1(x)^2 - \frac{A_0(x)A_1(x)}{x}$$

$$+ \exp(-4x)\left(-B_0(x)^2 + B_1(x)^2 - \frac{B_0(x)B_1(x)}{x}\right) + i\,\exp(-2x)(-2A_0(x)B_0(x)$$

$$+ 2A_1(x)B_1(x) + A_0(x)B_1(x) - A_1(x)B_0(x)\Bigg)$$

and

$$A_0(x)^2 - A_1(x)^2 - \frac{A_0(x)A_1(x)}{x} = \frac{1}{2x^2} + \frac{3}{8x^3} + \frac{3}{1024x^4} + \frac{285}{8192x^5}$$

$$+ \frac{1395}{131072x^6} + \frac{7875}{(1024)^2x^7}$$

$$+ C_0(x)\left(2 - \frac{3}{4x} + \frac{66}{128x^2} + \frac{270}{1024x^3} + \frac{105}{1024x^4} + C_0(x) - C_1(x)\right)$$

$$- C_1(x)\left(2 + \frac{1}{4x} - \frac{14}{128x^2} - \frac{138}{1024x^3} + \frac{75}{1024x^4} + C_1(x)\right).$$

Since $|C_0(x)| \le \frac{2.637}{x^4}$ and $|C_1(x)| \le \frac{16.31}{x^4}$, we have, for $x \ge 8$

$$\frac{0.0324768}{x^2} \le A_0(x)^2 - A_1(x)^2 - \frac{A_0(x)A_1(x)}{x} \le \frac{1.1422185}{x^2}.$$

Thus there are constants $d_1,\ d_2,\ d_3 > 0$ such that if $x \geq d_1$,

(79) $$\frac{d_2}{x^2} \leq \frac{I_0(x)^2 - I_1(x)^2 - \frac{I_0(x)I_1(x)}{x}}{I_0(x)^2} \leq \frac{d_3}{x^2}.$$

Combining (78) and (79), we have

$$\sum_{\substack{\gamma > 0 \\ |\rho| \leq T/d_1}} \frac{d_2}{T^2} \leq f''(T) \leq \sum_{\substack{\gamma > 0 \\ |\rho| \leq T/d_1}} \frac{d_3}{T^2} + \tfrac{1}{2} \sum_{\substack{\gamma > 0 \\ |\rho| > T/d_1}} \frac{1}{|\rho|^2}.$$

Applying Lemma 2.8 to the above equation, we obtain

(80) $$\frac{\log T}{T} << f''(T) << \frac{\log T}{T}.$$

Combining (71) and (77), we obtain

$$G'(u) - U_1 - U_2 = \frac{1}{4\pi} \int_{-T^{2/3}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}}^{T^{2/3}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}} \exp(f(T) - \frac{f''(T)t^2}{2} - R_2(t))\,dt,$$

where $|R_2(t)| << \frac{\log T |t|^3}{T^2}$. Letting $v = \sqrt{\frac{f''(T)}{2}}\, t$, we write

(81)
$$G'(u) - U_1 - U_2 = \frac{\exp(f(T))}{4\pi\sqrt{\frac{f''(T)}{2}}} \int_{-T^{2/3}\sqrt{\frac{f''(T)}{2}}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}}^{T^{2/3}\sqrt{\frac{f''(T)}{2}}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}} \exp\left(-v^2 + R_2\left(\frac{v}{\sqrt{\frac{f''(T)}{2}}}\right)\right) dv$$

$$= \frac{\exp(f(T))}{\pi\sqrt{8f''(T)}} \int_{-T^{2/3}\sqrt{\frac{f''(T)}{2}}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}}^{T^{2/3}\sqrt{\frac{f''(T)}{2}}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}} \exp(-v^2)\left(1 + O\left(\frac{\log T |v|^3}{T^2 (f''(T))^{3/2}}\right)\right) dv$$

$$= \frac{\exp(f(T))}{\sqrt{8\pi f''(T)}}\left(1 + O\left(\frac{1}{\sqrt{T \log T}}\right)\right),$$

which completes the proof of Lemma 2.18.

We obtain Theorem 2.15 by combining Lemma 2.16, Lemma 2.17, and Lemma 2.18.

We now examine f(x) more closely. We have

$$f(x) = \sum_{\gamma > 0} \log I_0\left(\frac{x}{|\rho|}\right) \tag{82}$$

$$= \int_1^\infty \log I_0\left(\frac{x}{|\frac{1}{2}+iy|}\right) dN(y).$$

Olver [19, p. 93] has shown that

$$I_0(v) = e^v\left\{\sum_{s=0}^{n-1} \frac{\Gamma(s+\frac{1}{2})\, a_{2s}}{v^{s+(\frac{1}{2})}} - \epsilon_{2n,1}(v) + \epsilon_{2n,2}(v)\right\} + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} e^{v\cos t}\, dt,$$

where for $v > \frac{1}{2}$,

$$a_0 = \frac{1}{\pi\, 2^{\frac{1}{2}}},$$

$$|\epsilon_{2,1}(v)| < \frac{e^{-v}}{\pi v}$$

and

$$|\epsilon_{2,2}(v)| < \frac{\Gamma(3/2)}{(v-\frac{1}{2})^{3/2}\, \pi\, 2^{5/2}}$$

Thus for $v \geq 1$, we have

(83) $$I_0(v) = \frac{\exp(v)}{\sqrt{2\pi v}} E(v),$$

where

$$E(v) = 1 + O\left(\frac{1}{v}\right).$$

Since

$$I_0(v) = \sum_{k=0}^{\infty} \frac{\left(\frac{v}{2}\right)^{2k}}{(k!)^2} = 1 + \frac{v^2}{4} + \cdots ,$$

we have, for $0 < v < 1$,

(84) $$I_0(v) = 1 + \frac{v^2}{4} + O(v^4).$$

Since

$$0 \leq \frac{1}{y} - \frac{1}{(\frac{1}{2}+y^2)^{\frac{1}{2}}} = \frac{\left(1 + \frac{1}{2y^2}\right)^{\frac{1}{2}} - 1}{(\frac{1}{2}+y^2)^{\frac{1}{2}}} \leq \frac{1}{2y^3} ,$$

we can write

$$\frac{1}{(\frac{1}{2}+y^2)^{\frac{1}{2}}} = \frac{1}{y}(1 + F(y))$$

where

$$\frac{-1}{2y^2} \leq F(y) < 0.$$

We write $N(y) = \frac{y}{2\pi} \log \frac{y}{2\pi} - \frac{y}{2\pi} + \frac{7}{8} + S(y) + G(y)$. Then

$$f(x) = \frac{1}{2\pi} \int_1^x \log I_0\left(\frac{x}{y}(1+F(y))\right) \log \frac{y}{2\pi} dy + \int_x^\infty \log I_0\left(\frac{x}{y}(1+F(y))\right) \log \frac{y}{2\pi} dy$$

$$+\int_1^x \log I_0\left(\frac{x}{y}(1+F(y))\right) d(S(y)+G(y)) + \int_x^\infty \log I_0\left(\frac{x}{y}(1+F(y))\right) d(S(y)+G(y))$$

$$= K_1 + K_2 + K_3 + K_4 \, .$$

Using (83) and letting $t = \frac{x}{y}$, we obtain

$$K_1 = \frac{x}{2\pi} \int_1^x \left(t - t F\left(\frac{x}{t}\right) - \tfrac{1}{2} \log 2\pi t - \tfrac{1}{2} \log\left(1 + F\left(\frac{x}{t}\right)\right)\right.$$

$$\left. + \log E\left(t\left(1 + F\left(\frac{x}{t}\right)\right)\right)\right) \log \frac{x}{2\pi t} \frac{dt}{t^2}$$

$$= \frac{x}{2\pi} \int_1^x \left(t - \tfrac{1}{2} \log 2\pi t + \log E(t)\right) \log \frac{x}{2\pi t} \frac{dt}{t^2}$$

$$+ \frac{x}{2\pi} \int_1^u \left(t F\left(\frac{x}{t}\right) - \tfrac{1}{2} \log\left(1 + F\left(\frac{x}{t}\right)\right) + \left(\log E\left(t\left(1 + F\left(\frac{x}{t}\right)\right)\right) - \log E(t)\right) \log \frac{x}{2\pi t} \frac{dt}{t^2}$$

$$= L_1 + L_2.$$

Since

$$\frac{x}{2\pi} \int_1^x \log E(t) \log \frac{x}{2\pi t} \frac{dt}{t^2} = \frac{x}{2\pi} \log \frac{x}{2\pi} \left(\int_1^\infty \frac{\log E(t)\, dt}{t^2} + O\left(\frac{1}{x^2}\right)\right)$$

$$- \frac{x}{2\pi} \left(\int_1^\infty \frac{\log E(t) \log t \, dt}{t^2} + O\left(\frac{\log x}{x^2}\right)\right),$$

we have

(85)

$$L_1 = \frac{x}{4\pi} \log^2 x + C_{1,1}\, x \log x + C_{1,2}\, x + C_{1,3} \log x + C_{1,4} + O\left(\frac{\log x}{x}\right).$$

To evaluate L_2 we let $y = \frac{x}{t}$, so that

(86)

$$L_2 = \frac{1}{2\pi} \int_1^x \left(\frac{x}{y} F(y) - \tfrac{1}{2}\log(1+F(y)) + \log E\left(\frac{x}{y}(1+F(y))\right) - \log E\left(\frac{x}{y}\right)\right) \log \frac{y}{2\pi}\, dy$$

$$= \frac{1}{2\pi} \int_1^\infty \left(\frac{x}{y} F(y) - \tfrac{1}{2}\log(1+F(y))\right) \log \frac{y}{2\pi}\, dy + O\left(\frac{\log x}{x}\right)$$

$$= C_{1,2}' x + C_{1,4}' + O\left(\frac{\log x}{x}\right) .$$

Using (84) and letting $t = \frac{x}{y}$, we obtain

$$K_2 = \frac{x}{2\pi} \int_0^1 \log I_0\left(t\left(1 + F\left(\frac{x}{t}\right)\right)\right) \log \frac{x}{2\pi t} \frac{dt}{t^2}$$

$$= \frac{x}{2\pi} \int_0^1 \log\left(1 + \frac{t^2\left(1 + F\left(\frac{x}{t}\right)\right)^2}{4} + \cdots\right) \log \frac{x}{2\pi t} \frac{dt}{t^2}$$

$$= C_{2,1}\, x \log x + C_{2,2}\, x + O\left(x \int_0^1 \frac{t^4}{x^2} \log \frac{x}{t} \frac{dt}{t^2}\right)$$

$$= C_{2,1}\, x \log x + C_{2,2}\, x + O\left(\frac{\log x}{x}\right) .$$

Recall that $S(y) + G(y) = O(\log y)$. Using (83), we have

(88)

$$K_3 = \int_1^x \left(\frac{x}{y}(1+F(y)) - \tfrac{1}{2}\log \frac{2\pi x(1+F(y))}{y} + \log E\left(\frac{x}{y}(1+F(y))\right)\right) d\,(S(y)+G(y))$$

$$= (S(x)+G(x))\,(1+F(x) - \tfrac{1}{2}\log 2\pi(1+F(x)) + \log E(1+F(x))$$

$$- (S(1)+G(1))\,(x(1+F(1)) - \tfrac{1}{2}(\log 2\pi x)(1+F(1)) + \log E(x(1+F(1)))$$

$$+ \int_1^x (S(y)+G(y))\left(x\left(\frac{-1}{y^2}(1+F(y)) + \frac{F'(y)}{y}\right) - \frac{1}{2}\frac{F'(y)}{1+F(y)} + \frac{1}{2}\frac{1}{y} + \frac{E'\left(\frac{x}{y}(1+F(y))\right)\left(\frac{-x}{y^2}(1+F(y)) + \frac{x}{y}F'(y)\right)}{E\left(\frac{x}{y}(1+F(y)\right)}\right.$$

$$= C_{3,3}\, x + O(\log x).$$

Similarly, using (84), we have

(89) $$K_4 << \int_x^\infty \frac{x^2}{y^2}\, d(S(y) + G(y)) << \log x.$$

Combining (85), (86), (87), (88), and (89), we obtain

(90) $$f(x) = \frac{x}{4\pi}\log x + C_{5,1}\, x\log x + C_{5,2}\, x - xu/2 + O(\log x)$$

$$= g(x) + O(\log x),$$

say. We have

$$g'(x) = \frac{1}{4\pi}\log^2 x + \left(\frac{1}{2\pi} + C_{5,1}\right)\log x + (C_{5,1} + C_{5,2} - u/2),$$

so the minimum of g(x) occurs at

$$X_0 = e^{-1 - 2\pi C_{5,1} + \sqrt{2\pi u + 1 + 4\pi^2 C_{5,1}^2 - 4\pi C_{5,2}}}\ .$$

Since there are constants $C_{6,1}$ and $C_{6,2}$ such that

$$g(x) + C_{6,1}\log x \le f(x) \le g(x) + C_{6,2}\log x,$$

we have

$$\lim_{x \to \infty} g'(x) = f'(x)$$

and

$$\lim_{x \to \infty} X_0 = T,$$

where $f'(T) = 0$.

Taking $T = X_0$, we have

$$f(T) = -\frac{1}{2\pi}\left(\sqrt{2\pi u + 4\pi^2 C_{5,1}^2 + 1 - 4\pi C_{5,2}} - 1\right) T + O(\log T).$$

Since $\frac{\log T}{T} << f''(T) << \frac{\log T}{T}$, there are constants a_1, a_2, a_3 and a_4 such that

$$G'(u) = \exp\left(-\left(\sqrt{a_1 u + a_2} \; -a_3\right) e^{\sqrt{2\pi u + a_4}} + O(\sqrt{u})\right).$$

CHAPTER III

THE ZEROS OF $\zeta_N(s)$

1. Introduction. Let $\zeta_N(s) = \sum_{n=1}^{N} n^{-s}$. We show that $\zeta_N(s)$ has a zero $s = \sigma + it$ with $\sigma > 1$ for all $N > 30$. The question is of particular interest because Turán [27] showed that the Riemann hypothesis is true if there exists an N_0 such that for all $N > N_0$, $\zeta_N(s)$ has no zero with $\sigma > 1$. The question has been considered by many others, including Haselgrove [11], Spira [25] and Voronin [29]. Montgomery [17] has shown that if $0 < c < \frac{4}{\pi} - 1$, then for all $N > N_0(c)$, $\zeta_N(s)$ has a zero with $\sigma > 1 + c(\log\log N)/\log N$. The following table summarizes the current knowledge concerning zeros of $\zeta_N(s)$ with $\sigma > 1$:

N	zero with $\sigma > 1$?
1	No (Turán)
2	No (Turán)
3	No (Turán)
4	No (Turán)
5	No (Turán)
6	No (Spira)
7	No (Spira)
8	No (Spira)
9	No (Spira)
10	?
11	?
12	?
13	?
14	?

N	zero with $\sigma > 1$?
15	?
16	?
17	?
18	?
19	Yes (Spira)
20	?
21	?
22	Yes (Spira)
23	Yes (Spira)
24	Yes (Spira)
25	Yes (Spira)
26	Yes (Spira)
27	Yes (Spira)
28	?
29–50	Yes (Spira)
51–N_0	Yes (Monach)
$>N_0$	Yes (Montgomery)

Van de Lune and Te Riele [15] have explicitly computed zeros of $\zeta_N(s)$ with $\sigma > 1$ for small values of N.

By the classical work of Bohr [4], $\zeta_N(s)$ takes the same values in the half-plane Re $s > \sigma_0$ as $f_N(s) = \sum_{n=1}^{N} a(n)\, n^{-s}$, where $|a(n)| = 1$ for all n and a(n) is totally multiplicative. Hence, $f_N(s)$ has a zero with $\sigma > 1$ if and only if $\zeta_N(s)$ has such a zero. We write

$$f_N(s) = \sum_{n=1}^{N}{}^{*} a(n)n^{-s} + \sum_{N^{\frac{1}{2}}<p\le N^{19/20}} a(p)p^{-s} \sum_{m\le N/p} a(m)m^{-s}$$

where $\sum_{n=1}^{N}{}^{*}$ is the sum over all n such that $p|n \rightarrow p \le N^{\frac{1}{2}}$ or $p > N^{19/20}$. We choose $a(p) = p^{i\tau}$ for $p \le N^{1/2}$ and $p > N^{19/20}$, leaving the a(p) for $N^{1/2} < p \le N^{19/20}$ to be chosen later. Thus, $f_N(\sigma + i2\tau)$ has a zero with $\sigma > 1$ if for some τ

(1) $$G(1+i\tau) > |F(1+i\tau)|$$

and there is no prime q, $N^{1/2} < q \le N^{19/20}$, for which

(2)
$$q^{-1}\Big|\sum_{m\le N/q} m^{-1-i\tau}\Big| > \sum_{\substack{N^{1/2}<p\le N^{19/20}\\ p\ne q}} p^{-1}\Big|\sum_{m\le N/p} m^{-1-i\tau}\Big|.$$

Here,

$$G(s) = \sum_{N^{1/2}<p\le N^{19/20}} p^{-1}\Big|\sum_{m\le N/p} m^{-s}\Big|$$

and

$$F(s) = \sum_{n=1}^{N}{}^{*}\, n^{-s}.$$

We will show that (1) and (2) are true for $\tau = \frac{2\pi}{\log N}$ for every $N > 549798$. The choice of τ is inspired by Levinson's work [14] on the zeros of $\zeta_N(s)$ near 1.

2. An upper bound for $|F(1+i\tau)|$. In this section, we prove

Theorem 3.1. Let $\tau = \frac{2\pi}{\log N}$. Let $a = N^{1/2}$ and $b = N^{19/20}$. Let

$$F_{r,0} = \frac{1}{\tau}(-0.4257265+K+L) - 0.05177\,\epsilon(a) - I + \left(\frac{1}{2N} - \frac{1}{12N^2} + \frac{1-\tau^2}{120N^4}\right)$$
$$+\left(0.575 + \frac{\tau^2}{120}\right)\left(1.1466715 - \frac{1.0011126\,\epsilon(b)}{\log N} - \frac{2\epsilon(a)}{\log N} - I - J + \tau K\right)$$
$$+\left(\frac{49\tau+\tau^3}{720}\right)\left(0.4257265 + \frac{0.325281\,\epsilon(b)}{\log N} - K - L + \tau I\right)$$

$$-\frac{1}{2N}\left(\int_a^b \frac{du}{\log u} + \frac{T(b)}{\log b} - \frac{T(a)}{\log a} + \int_{a^+}^{b^+} \frac{\epsilon(u)du}{\log^2 u}\right)$$

and

$$F_{i,0} = \frac{1}{\tau}\left(-0.7885255+I+J - \int_{a^+}^{b^+} \frac{\epsilon(u)(1+\frac{1}{\log u})du}{u \log u}\right) + 0.6366197\ \epsilon(a)$$

$$+ K-0.0081994\ \epsilon(b) + (0.575 + \frac{\tau^2}{120})\ (-0.4257265 - \frac{0.325281\epsilon(b)}{\log N}$$

$$+ K+L-\tau I) + \left(\frac{49\tau+\tau^3}{720}\right)\left(1.1466715 - \frac{1.0011126\,\epsilon(b)}{\log N}\right.$$

$$\left. - \frac{2\epsilon(a)}{\log N} - I-J-\tau K\right) - \frac{\tau}{12N^2} + \frac{11\tau-\tau^3}{720N^4}$$

$$-\frac{\tau}{2N^2}\left(\int_a^b \frac{udu}{\log u} + \frac{bT(b)}{\log b} - \frac{aT(a)}{\log a} - \int_{a^+}^{b^+} \frac{T(u)(1-\frac{1}{\log u})du}{\log u}\right),$$

where $T(u) = \theta(u)-u$, $\epsilon(u) = \frac{T(u)}{u}$,

$$I = \int_{a^+}^{b^+} \frac{\epsilon(u)\cos\tau\ \log u\,du}{u \log u},$$

$$J = \int_{a^+}^{b^+} \frac{\epsilon(u)\cos\tau\ \log u\,du}{u \log^2 u},$$

$$K = \int_{a^+}^{b^+} \frac{\epsilon(u)\sin\tau\ \log u\,du}{u \log u},$$

and

$$L = \int_{a^+}^{b^+} \frac{\epsilon(u)\sin\tau\ \log u\,du}{u \log^2 u}.$$

Let

$$E(u, s) = \frac{-s(s+1)(s+2)(s+3)(s+4)}{120} \int_u^\infty \hat{B}_5(x) x^{-s-5} dx$$

and $r(p) = N/p$. Let θ represent any number such that $|\theta| \leq 1$. If $N \geq 549798$, then

$$|F(1+i\tau)| \leq (F_{r,0}^2 + F_{i,0}^2)^{\frac{1}{2}} + |E(1, 1+i\tau) - E(N, 1+i\tau)|$$

$$+ \left| \sum_{N^{1/2} < p < N^{19/20}} p^{-1-i\tau} (E(1, 1+i\tau) - E(r(p)^+, 1+i\tau)) \right|$$

$$+ 5 \cdot 10^{-7} (\log N)\theta .$$

We need several preliminary results. From Edwards [7,pp.114-115], we have

Lemma 3.2. If $\sigma \geq 1$, then

$$\zeta(s) = \zeta_N(s) + \frac{N^{1-s}}{s-1} - \frac{1}{2} N^{-s} + \frac{s}{12} N^{-s-1} - \frac{s(s+1)(s+2)}{720} N^{-s-3}$$

$$+ E(N, s).$$

Using this lemma, we prove

Lemma 3.3. If N is an integer > 1, then

$$\zeta_N(1+i\tau) = \left(0.575 + \frac{\tau^2}{120}\right) + \left(\frac{1}{2N} - \frac{1}{12N^2} + \frac{(1-\tau^2)}{120N^4}\right)$$

$$+ i\left(\frac{49\tau + \tau^3}{720} - \frac{\tau}{12N^2} + \frac{11\tau - \tau^3}{720N^4}\right) + E(1, 1+i\tau) - E(N, 1+i\tau).$$

Proof. Using the previous lemma, we write

$$\zeta_N(1+i\tau) = \zeta(1+i\tau) - \frac{1}{i\tau} N^{-i\tau} + \frac{1}{2} N^{-1-i\tau} - \frac{(1+i\tau)}{12} N^{-2-i\tau}$$

$$+ \frac{(1+i\tau)(2+i\tau)(3+i\tau)}{720} N^{-4-i\tau} - E(N, 1+i\tau)$$

and

$$\zeta(1+i\tau) = 1 + \frac{1}{i\tau} - \frac{1}{2} + \frac{1+i\tau}{12} - \frac{(1+i\tau)(2+i\tau)(3+i\tau)}{720} + E(1, 1. i\tau).$$

Combining these equalities with the fact that $N^{-i\tau} = 1$, we complete the proof of the lemma.

Next we have

Lemma 3.4. Let $h(p, s) = \sum_{m \leq N/p} m^{-s}$,

$$C(p) = -\hat{B}_1(r(p)) \frac{p}{N} - \hat{B}_2(r(p)) \frac{p^2}{2N^2} - \hat{B}_3(r(p)) \frac{p^3(2-\tau^2)}{6N^3}$$

$$- \hat{B}_4(r(p)) \frac{p^4(1-\tau^2)}{4N^4} - \hat{B}_5(r(p)) \frac{p^5(24-35\tau^2+\tau^4)}{120N^4}$$

and

$$D(p) = -\hat{B}_2(r(p)) \frac{p^2\tau}{2N^2} - \hat{B}_3(r(p)) \frac{p^3\tau}{2N^3} - \hat{B}_4(r(p)) \frac{p^4(11\tau-\tau^3)}{24N^4}$$

$$- \hat{B}_5(r(p)) \frac{p^5(5\tau-\tau^3)}{12N^5}.$$

If $N \geq 549798$, then

$$h(p, 1+i\tau) = \left(0.575 + \frac{\tau^2}{120}\right) - \frac{1}{\tau} \sin\tau \log p + C(p) \cos\tau \log p$$

$$-D(p) \sin\tau \log p + i\left(\frac{1}{\tau}(\cos\tau \log p - 1) + \left(\frac{49\tau+\tau^3}{720}\right)\right.$$

$$\left. + D(p)\cos\tau \log p + C(p) \sin\tau \log p\right) + E(1, 1+i\tau) - E(r(p)^+, 1+i\tau).$$

Proof. For $\sigma > 1$, we have

$$h(p, s) = \zeta(s) - \sum_{n>N/p} n^{-s}$$

$$= \zeta(s) - \frac{\left(\frac{N}{p}\right)^{1-s}}{s-1} - \hat{B}_1(r(p))\left(\frac{N}{p}\right)^{-s} + s\int_{r(p)^+}^{\infty} \hat{B}_1(x)x^{-s-1}dx.$$

Integrating by parts, we see that this

$$= \zeta(s) - \frac{\left(\frac{N}{p}\right)^{1-s}}{s-1} - \hat{B}_1(r(p))\left(\frac{N}{p}\right)^{-s} - \frac{s}{2}\hat{B}_2(r(p))\left(\frac{N}{p}\right)^{-s-1}$$

$$- \frac{s(s+1)}{6}\hat{B}_3(r(p))\left(\frac{N}{p}\right)^{-s-2} - \frac{s(s+1)(s+2)}{24}\hat{B}_4(r(p))\left(\frac{N}{p}\right)^{-s-3}$$

$$- \frac{s(s+1)(s+2)(s+3)}{120}\hat{B}_5(r(p))\left(\frac{N}{p}\right)^{-s-4}$$

$$+ \frac{s(s+1)(s+2)(s+3)(s+4)}{120}\int_{r(p)^+}^{\infty} \hat{B}_5(x)x^{-s-5}dx.$$

Since this expression remains valid for $\sigma = 1$, we set $s = 1+i\tau$ and obtain the desired result.

To bound the $C(p)$ and $D(p)$ in the previous lemma, we need

Lemma 3.5. If $N \geq 549798$ and $p \leq N^{19/20}$, then

$$\frac{p}{N}\left(-\frac{1}{2}-\frac{p}{12N}+\frac{p^3(1-\tau^2)}{120N^3}\right) \leq C(p) \leq \frac{p}{N}\left(\frac{1}{2}-\frac{p}{12N}+\frac{p^3(1-\tau^2)}{120N^3}\right)$$

and

$$-\frac{p^2\tau}{N^2}\left(\frac{1}{12}+\frac{p}{4N}\right) \leq D(p) \leq \frac{p^2\tau}{2N^2}.$$

Proof. We have $t \leq 0.4753755$ and $\frac{p}{N} \leq 0.5164054$. Replace $\{r(p)\}$ by x in the definition of $C(p)$. We have

$$\frac{\partial C(p)}{\partial x} = \frac{p}{N}\left(\frac{-p^4(24-35\tau^2+\tau^4)}{24N^4}\,x^4+\left(\frac{p^4(24-35\tau^2+\tau^4)}{12N^4}-\frac{p^3(1-\tau^2)}{N^3}\right)x^3\right.$$

$$+\left(\frac{-p^4(24-35\tau^2+\tau^4)}{24N^4}+\frac{3p^3(1-\tau^2)}{2N^3}-\frac{p^2(2-\tau^2)}{2N^2}\right)x^2$$

$$+\left(\frac{-p^3(1-\tau^2)}{2N^3}+\frac{p^2(2-\tau^2)}{2N^2}-\frac{p}{N}\right)x+\left(\frac{p^4(24-35\tau^2+\tau^4)}{720N^4}-\frac{p^2(2-\tau^2)}{12N^2}\right.$$

$$\left.\left.+\frac{p}{2N}-1\right)\right).$$

Since this is < 0 for $0 \leq x < 1$, we have our desired result for $C(p)$. In the case of $D(p)$, we replace $\{r(p)\}$ by x and observe that our desired result is true since $x^5 \leq x^4 \leq x^3 \leq x^2 \leq x$ for $0 \leq x \leq 1$.

We now prove the theorem. Write

$$F(1+i\tau) = \zeta_N(1+i\tau) - \sum_{N^{1/2}<p\leq N^{19/20}} p^{-1-i\tau}\, h(p, 1+i\tau).$$

Using Lemma 3.3 and Lemma 3.4, we obtain

(3)

$$F(1+i\tau) = \left(.575+\frac{\tau^2}{120}\right)+\left(\frac{1}{2N}-\frac{1}{12N^2}+\frac{(1-\tau^2)}{120N^4}\right)+i\left(\frac{49\tau+\tau^3}{720}-\frac{\tau}{12N^2}+\frac{11\tau-\tau^3}{720N^4}\right)$$

$$+ E(1, 1+i\tau)-E(N, 1+i\tau) - \sum_{N^{1/2}<p\leq N^{19/20}} p^{-1-i\tau}\left(\left(.575+\frac{\tau^2}{120}\right)\right.$$

$$\left.+ i\left(\frac{-1}{\tau}+\frac{49\tau+\tau^3}{720}\right)+ E(1, 1+i\tau)+\left(\frac{-1}{i\tau}+C(p)+iD(p)\right)p^{i\tau} - E(r(p)^+, 1+i\tau)\right).$$

We need to evaluate several summations over primes in this expression. Using Lebesgue-Stieltjes integration, we obtain

(4)

$$\sum_{N^{1/2}<p\leq N^{19/20}} p^{-1-i\tau} = \int_{a^+}^{b^+} \frac{u^{-1-i\tau}\, d\theta(u)}{\log u}$$

$$= -0.1466715 + 0.4257265i$$

$$+ \frac{1.0011126\epsilon(b)}{\log N} + \frac{0.325281\epsilon(b)i}{\log N}$$

$$+ \frac{2\epsilon(a)}{\log N} + I+J+ K-iK-iL+i\tau I+2\cdot 10^{-7}\theta.$$

Similarly, we have

(5)

$$\sum_{N^{1/2}<p\leq N^{19/20}} p^{-1} = 0.641854 + \frac{\epsilon(b)}{\log b} - \frac{\epsilon(a)}{\log a} + \int_{a^+}^{b^+} \frac{\epsilon(u)(1+\frac{1}{\log u})du}{u \log u} + 10^{-6}\theta,$$

(6)

$$\sum_{N^{1/2}<p\leq N^{19/20}} 1 = \int_a^b \frac{du}{\log u} + \frac{T(b)}{\log b} - \frac{T(a)}{\log a} + \int_{a^+}^{b^+} \frac{\epsilon(u)du}{\log^2 u}$$

and

(7)

$$\sum_{N^{1/2}<p\leq N^{19/20}} p = \int_a^b \frac{u\,du}{\log u} + \frac{bT(b)}{\log b} - \frac{aT(a)}{\log a} + \int_{a^+}^{b^+} \frac{T(u)(\frac{1}{\log u}-1)du}{\log u}.$$

Using Lemma 3.5, (6) and (7), we obtain

(8)

$$\sum_{N^{1/2}<p\leq N^{19/20}} p^{-1}C(p) \leq \frac{1}{2N}\left(\int_a^b \frac{du}{\log u} + \frac{T(b)}{\log b} - \frac{T(a)}{\log a} + \int_{a^+}^{b^+} \frac{\epsilon(u)du}{\log^2 u}\right)$$

and

(9)

$$\sum_{N^{1/2}<p\leq N^{19/20}} p^{-1}D(p) \leq \frac{\tau}{2N^2}\left(\int_a^b \frac{u\,du}{\log u} + \frac{bT(b)}{\log b} - \frac{aT(a)}{\log a} - \int_{a^+}^{b^+} \frac{T(u)(1-\frac{1}{\log u})du}{\log u}\right).$$

We note that Re $F(1+i\tau)$ and Im$F(1+i\tau)$ are both negative, so that $|F(1+i\tau)|$ is maximized when $\sum_{N^{1/2}<p\leq N^{19/20}} p^{-1}C(p)$ and $\sum_{N^{1/2}<p\leq N^{19/20}} p^{-1}D(p)$ are maximized. Combining (3), (4), (5), (8), and (9), we complete the proof of the theorem.

3. A lower bound for $G(1+i\tau)$. In this section, we prove

Theorem 3.6. Let

$$P = \int_{a^+}^{b^+} \frac{\epsilon(u)\cos\frac{\tau}{2}\log u\,du}{u\log u},$$

$$Q = \int_{a^+}^{b^+} \frac{\epsilon(u)\sin\frac{\tau}{2}\log u\,du}{u\log u} \quad \text{and}$$

$$R = \int_{a^+}^{b^+} \frac{\epsilon(u)\sin\frac{\tau}{2}\log u\,du}{u\log^2 u}.$$

If $N \geq 549798$, then

$$G(1+i\tau) \geq 0.1518588\log N + 0.0524155\epsilon(b) - 0.6366197\epsilon(a)$$

$$-P + \frac{\log N}{\pi}Q + \frac{\log N}{\pi}R - \sum_{N^{1/2}< p\leq N^{19/20}} p^{-1}|E(1,1+i\tau)$$

$$-E(r(p),1+i\tau)| + 5\cdot 10^{-7}(\log N)\theta.$$

We need a lower bound for $|h(p,1+i\tau)|$. Using Lemma 3.4, we obtain

(10)

$$|h(1,1+i\tau)| \geq \Bigg(\frac{2}{\tau^2}(1-\cos\tau\ \log p) + (0.575 + \frac{\tau^2}{120})^2$$

$$+ \left(\frac{49\tau+\tau^3}{720}\right)^2 - \frac{49+\tau^2}{360} + C(p)^2 + D(p)^2 + \frac{2D(p)}{\tau}$$

$$+ \sin\tau\ \log p\left(\frac{-1}{\tau}\left(1.15 + \frac{\tau^2}{60}\right) + C(p)\left(\frac{49\tau+\tau^3}{360}\right)\right.$$

$$\left. - D(p)\left(1.15 + \frac{\tau^2}{60}\right) - \frac{2C(p)}{\tau}\right) + \cos\tau\ \log p\left(\frac{49+\tau^2}{360}\right.$$

$$\left. + C(p)\left(1.15 + \frac{\tau^2}{60}\right) + D(p)\left(\frac{49\tau+\tau^3}{360}\right) - \frac{2D(p)}{\tau}\right)\Bigg)^{1/2}$$

$$- |E(1, 1+i\tau) - E(r(p)^+, 1+i\tau)|.$$

We simplify this result in

Lemma 3.7. If $N \geq 549798$ and $p \leq N^{19/20}$, then

$$|h(p, 1+i\tau)| \geq \left(\frac{2}{\tau^2}(1-\cos\tau\ \log p)\right)^{1/2} - |E(1, 1+i\tau) - E(r(p)^+, 1+i\tau)|.$$

Proof. We first assume that $N^{1/2} < p \leq N^{3/4}$. We have $\cos\tau\ \log p \leq 0$, $\sin\tau\ \log p \leq 0$, $\frac{p}{N} \leq 0.0367239$, $\frac{p^2}{N^2} \leq 0.001349$ and $\tau \leq 0.4753755$. From Lemma 3.5, we have

$$-\frac{p}{N}\left(\frac{1}{2} + \frac{p}{12N}\right) \leq C(p) \leq \frac{p}{2N}$$

and

$$-\frac{p^2\tau}{N^2}\left(\frac{1}{12}+\frac{p}{4N}\right) \le D(p) \le \frac{p^2\tau}{2N^2} .$$

Thus, the coefficient of $\sin \tau \log p$ in (10) is negative and the right-hand side of (10) is minimized when this coefficient is replaced by 0 and we choose $C(p) = \frac{-p}{N}\left(\frac{1}{2}+\frac{p}{12N}\right)$ and $D(p) = \frac{-p^2\tau}{N^2}\left(\frac{1}{12}+\frac{p}{4N}\right)$. The right-hand side of (10) is obviously still

$$\ge \left(\frac{2}{\tau^2}(1-\cos \tau \log p)\right)^{1/2} - |E(1, 1+i\tau) - E(r(p)^+, 1+i\tau)| ,$$

which proves the lemma for $N^{1/2} < p \le N^{3/4}$.

We now assume that $N^{3/4} < p \le N^{19/20}$. We have $0 \le \cos \tau \log p \le 0.951057$, $-0.309017 \le \sin \tau \log p \le 0$, $\frac{p}{N} \le 0.5164043$, $\frac{p^2}{N^2} \le 0.266674$, and $\tau \le 0.4753755$. As before, the coefficient of $\sin \tau \log p$ in (10) is negative. Thus, the right-hand side of (10) is minimized when this coefficient is replaced by 0 and we choose $\cos \tau \log p = 0.951057$, $C(p) = \frac{-p}{N}\left(\frac{1}{2}+\frac{p}{12N}\right)$ and $D(p) = \frac{-p^2\tau}{N^2}\left(\frac{1}{12}+\frac{p}{4N}\right)$. The right-hand side of (10) is obviously still

$$\ge \left(\frac{2}{\tau^2}(1-\cos \tau \log p)\right)^{1/2} - |E(1, 1+i\tau) - E(r(p)^+, 1+i\tau)| ,$$

which completes the proof of the lemma.

Using this lemma, we have

$$G(1+i\tau) = \sum_{N^{1/2}<p\leq N^{19/20}} p^{-1} \left| h(p, 1+i\tau) \right|$$

$$\geq \sum_{N^{1/2}<p\leq N^{19/20}} p^{-1}\left(\frac{2}{\tau^2}(1-\cos\tau \log p)\right)^{1/2} -$$

$$- \sum_{N^{1/2}<p\leq N^{19/20}} p^{-1} \left| E(1, 1+i\tau) - E(r(p)^+, 1+i\tau) \right| .$$

Using Lebesque-Stieltjes integration, we obtain

$$\sum_{N^{1/2}<p\leq N^{19/20}} p^{-1}\left(\frac{2}{\tau^2}(1-\cos\tau \log p)\right)^{1/2} = \frac{2}{\tau} \sum_{N^{1/2}<p\leq N^{19/20}} p^{-1} \sin\frac{\tau}{2} \log p$$

$$= \frac{2}{\tau} \int_a^b \frac{\sin\frac{\tau}{2} \log u \, d\theta(u)}{u \log u}$$

$$= 0.1518588 \log N + 0.0524155\epsilon(b)$$

$$- 0.6366197\epsilon(a)$$

$$- P + \frac{\log N}{\pi} Q + \frac{\log N}{\pi} R$$

$$+ 5\cdot 10^{-7} (\log N)\theta ,$$

which completes the proof of the theorem.

4. The zeros of $\zeta_N(s)$. In this section we prove

Theorem 3.8. For every $N \geq 30$, $\zeta_N(s)$ has a zero with $\sigma > 1$.

We first establish

Lemma 3.9. If $N \geq 549798$, then

$$|E(1, 1+i\tau)-E(N, 1+i\tau)| \leq 0.0040729$$

and

$$\sum_{N^{1/2}<p\leq N^{19/20}} p^{-1}| E(1, 1+i\tau)-E(r(p)^{+}, 1+i\tau)| \leq 0.002627.$$

Proof. We have

$$|E(1, 1+i\tau)-E(N, 1+i\tau)| \leq \left|\frac{(1+i\tau)(2+i\tau)(3+i\tau)(4+i\tau)(5+i\tau)}{120}\right| \int_1^N |\hat{B}_5(x)| x^{-6} dx.$$

Since $\tau \leq 0.4753755$, the first factor above is ≤ 1.1656289. In addition, the minimum of $\hat{B}_5(x)$ occurs at 0.2403353 ± 10^{-7} and the maximum of $\hat{B}_5(x)$ occurs at 0.7596647 ± 10^{-7}. Thus, $|\hat{B}_5(x)| \leq 0.0244588$. Using these results, we obtain

$$|E(1, 1+i\tau)-E(N, 1+i\tau)| \leq 0.0040729.$$

Similarly, we have

$$| E(1, 1+i\tau)-E(r(p)^{+}, 1+i\tau)| \leq 0.0040729.$$

Using the formula for $\sum_{N^{1/2}<p\leq N^{19/20}} p^{-1}$ in (8), we obtain

$$\sum_{N^{1/2}<p\le N^{19/20}} p^{-1} \left| E(1,1+i\tau)-E(r(p)^+,1+i\tau) \right| \le 0.002627.$$

This completes the proof of the lemma.

Combining the results of Theorem 3.1, Theorem 3.6, and Lemma 3.8, we see that $G(1+i\tau) > |F(1+i\tau)|$ for a given $N > 549798$ if

$$D_0 = G_0 + \frac{\log N}{\pi} Q - (F_{r,0}^2 + F_{i,0}^2)^{1/2} > 0.0093269 + 10^{-6}(\log N),$$

where

$$G_0 = 0.1518588 \log N + 0.0524155\epsilon(b) - 0.6366197\epsilon(a)$$
$$- P + \frac{\log N}{\pi} R.$$

We first examine the case when $N \ge 10^{11}$.

Lemma 3.10. If $N \ge 10^{11}$, then

$$G(1+i\tau) > |F(1+i\tau)|.$$

Proof. We have $N^{1/2} \ge 316226$, $\log N \ge 25.32835$ and $t \le 0.2480684$. Schoenfeld [24, p. 360] has shown that $\theta(u) < 1.000081$, for all $u > 0$. Hence, $\epsilon(u) < 0.000081$ for all $u > 0$. Schoenfeld [24, Corollary 2*] has determined values of c and d such that if $d \le u$, then

$$u - u/(c \log u) < \theta(u).$$

Thus,

(11)

$$- \frac{1}{c \log u} < \epsilon (u).$$

In particular, (11) is true for d = 315437 and c = 29. Hence, for $a \leq u \leq b$

$$-0.00273 < \epsilon(u).$$

From Schoenfeld [24, 5.66*], we have, for $u > e^{22}$,

$$|\theta(u)-u| < 0.0077629\ u/\log u.$$

Hence, for $u > e^{22}$

$$-0.0077629/\log u < \epsilon(u).$$

Therefore,

$$-0.000030649 < \epsilon(b).$$

Combining the above results, we obtain

$$-0.00273 < \epsilon(u) < 0.000081 \qquad (a \leq u \leq b)$$

and

$$-0.000030649 < \epsilon(b) < 0.000081.$$

Thus,

$$0.0524155\epsilon(b) > -0.000017$$

and

$$-0.6366197\epsilon(a) > -0.000052.$$

In addition, we have

$$-P = -\int_{\left(\frac{\pi}{2}\right)^+}^{\left(\frac{19\pi}{20}\right)^+} \frac{\epsilon(e^{2v/\tau})\cos v\, dv}{v} > -0.00273\int_{\left(\frac{\pi}{2}\right)}^{\left(\frac{19\pi}{20}\right)} \frac{\cos v\, dv}{v} > -0.001726.$$

In a similar manner, we obtain

$$\frac{\log N}{\pi} Q \geq -0.0007593 \log N$$

and

$$\frac{\log N}{\pi} R \geq -0.0023854.$$

Combining these results, we have

$$G_0 + \frac{\log N}{\pi} Q \geq 0.1510995 \log N - 0.0041804.$$

To bound $F_{r,0}$ and $F_{i,0}$, we need bounds on I, J, K, and L. Using the same method as we used to bound P, we obtain

$$|I| \le 0.0019875,$$

$$|J| \le 0.0004931,$$

$$-0.0000345 \le K \le 0.0021287$$

and

$$-0.0000085 \le L \le 0.0005217.$$

Combining these results with those previously obtained, we have

$$F_{r,0} \ge -0.0677634 \log N + 0.6450713$$

and

$$F_{i,0} \ge -0.125909 \log N - .2483715.$$

Hence,

$$D_0 \ge 0.0091103 \log N - 0.0041804$$

$$\ge .2012154 + 10^{-6} \log N.$$

Since $D_0 > .0093269 + 10^{-6}(\log N)$, we have $G(1+i\tau) > |F(1+i\tau)|$ for every $N \ge 10^{11}$, which completes the proof of the lemma.

Next, we examine the case when $549798 \le N \le 10^{11}$. In this range, we use Schoenfeld's result [24, p. 360] that $\theta(u) < u$ for $0 < u < 10^{11}$. This result implies that $\epsilon(u) < 0$ for $0 < u \le 10^{11}$.

Treat

$$H_0 = (F^2_{r,0}+F^2_{i,0})^{1/2}$$

as a function of K and L. Since $F_{r,0}$ and $F_{i,0}$ are negative for $N \geq 549798$, it is obvious that H_0 is maximized when K and L are minimized. But $K \geq 0$ and $L \geq 0$, so $H_0 \leq H_1$, where

$$H_1 = (F^2_{r,1}+F^2_{i,1})^{1/2}.$$

and $F_{r,1}$ and $F_{i,1}$ are defined by replacing K and L by 0 in the definitions of $F_{r,0}$ and $F_{i,0}$.

Treat H_1 as a function of I. Let

$$H_2 = (F^2_{r,2}+F^2_{i,2})^{1/2},$$

where $F_{r,2}$ and $F_{i,2}$ are defined by replacing I by 0 in the definitions of $F_{r,1}$ and $F_{i,1}$. We have

$$\frac{\partial H_1}{\partial I} = (F^2_{r,1}+F^2_{i,1})^{-\frac{1}{2}}\left(F_{r,1}\left(-1-\left(0.575+\frac{\tau^2}{120}\right)+\tau\left(\frac{49\tau+\tau^3}{720}\right)\right)\right.$$

$$\left.+F_{i,1}\left(\frac{1}{\tau}-\tau\left(0.575+\frac{\tau^2}{120}\right)-\left(\frac{49\tau+\tau^3}{720}\right)\right)\right)$$

$$\leq 0.$$

Thus, H_1 is a decreasing function of I. Hence, if $I > 0$, then $H_1 \leq H_2$. If $I \leq 0$, then

$$d(I) = H_2 - \frac{1}{\tau} I - H_1$$

is a decreasing function of I which is 0 when $I = 0$. Thus, $H_1 \leq H_2 - \frac{1}{\tau} I$ if $I \leq 0$. We now examine D_0. If $I \geq 0$, then

(12) $$D_0 \geq G_0 + \frac{\log N}{\pi} Q - H_2.$$

If $I \leq 0$, then

$$D_0 \geq G_0 + \frac{\log N}{\pi} Q - H_2 + \frac{1}{\tau} I.$$

But

$$\frac{\log N}{2\pi} Q + \frac{1}{\tau} I = \frac{1}{\tau} \int_{a^+}^{b^+} \frac{\epsilon(u)(\sin \frac{\tau}{2} \log u + \cos \tau \log u) du}{u \log u}$$

and

$$\sin \frac{\tau}{2} \log u + \cos \tau \log u \geq 0$$

for $a \leq u \leq b$. Thus, if we let

$$m(j) = \min_{x_{j-1} \leq u \leq b} \epsilon(u),$$

$$Q_1 = \sum_{j=1}^{n} m(j) \int_{x_{j-1}}^{x_j} \frac{\sin \frac{\tau}{2} \log u \, du}{u \log u}$$

and

$$I_1 = \sum_{j=1}^{n} m(j) \int_{x_{j-1}}^{x_j} \frac{\cos \tau \log u \, du}{u \log u}$$

for $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, then

$$D_0 \geq G_0 + \frac{\log N}{2\pi} Q - H_2 + \frac{1}{\tau}(Q_1 - I_1). \tag{13}$$

But m(j) is an increasing function of j, so $I_1 \geq 0$. Hence, for $549798 \leq N \leq 10^{11}$, we have

$$D_0 \geq G_0 + \frac{\log N}{2\pi} Q - H_2 + \frac{1}{\tau} Q_1.$$

Treat H_2 as a function of J, and let

$$H_3 = (F^2_{r,3} + F^2_{i,3})^{1/2},$$

where $F_{r,3}$ and $F_{i,3}$ are defined by replacing J by 0 in the definitions of $F_{r,2}$ and $F_{i,2}$. Using the same methods we used above, we obtain, for $549798 \leq N \leq 10^{11}$,

$$D_0 \geq G_0 - H_3 + \frac{\log N}{\pi} Q_1.$$

Let

$$P_1 = \sum_{j=1}^{n} m(j) \int_{x_{j-1}}^{x_j} \frac{\cos \frac{\tau}{2} \log u \, du}{u \log u}$$

and

$$R_1 = \sum_{j=1}^{n} m(j) \int_{x_{j-1}}^{x_j} \frac{\sin \frac{\tau}{2} \log u \, du}{u \log^2 u} .$$

Then

(14) $$D_0 \geq D_1 = G_1 - H_3,$$

where

$$G_1 = 0.1518588 \log N + 0.0524155\epsilon(b) - 0.6366197\epsilon(a)$$
$$- P_1 + \frac{\log N}{\pi} Q_1 + \frac{\log N}{\pi} R_1.$$

Let D_3 be defined by replacing $\epsilon(a)$ by ϵ^* in D_1, where

$$0 \geq \epsilon^* \geq \max_{N_1^{1/2} \leq u \leq N_2^{1/2}} \epsilon(u).$$

Let D_2 be defined by replacing $\epsilon(a)$ by v in D_1. We have

$$\frac{\partial D_2}{\partial v} = -0.636619 - (F_{r,3}^2 + F_{i,3}^2)^{-\frac{1}{2}} \left(F_{r,3} \left(-0.05177 - \left(0.575 + \frac{\tau^2}{120}\right)\left(\frac{2}{\log N}\right) - \frac{a}{2N \log a} \right) + F_{i,3} \left(0.6366197 - \left(\frac{49\tau + \tau^3}{720}\right)\left(\frac{2}{\log N}\right) + \frac{\tau a^2}{2n^2} \right) \right) \leq 0.$$

Thus, $D_1 \geq D_3$ for $549798 \leq N_1 \leq N \leq N_2 \leq 10^{11}$, where

$$D_3 = 0.1518588 \log N + 0.0524155\epsilon(b) - 0.6366197 \epsilon^*$$
$$- P_1 + \frac{\log N}{\pi} Q_1 + \frac{\log N}{\pi} R_1 - \Bigg((-0.0677564 \log N - 0.05177 \epsilon^*)$$
$$+ \left(0.575 + \frac{\tau^2}{120}\right)\left(1.1466715 - \left(\frac{1.0011126\epsilon(b)}{\log N} - \frac{2\epsilon^*}{\log N}\right)\right.$$

$$+\left(\frac{49\tau+\tau^3}{720}\right)\left(0.4257265+\frac{0.325281\epsilon(b)}{\log N}\right)-\frac{1}{2N}\left(\int_a^b\frac{du}{\log u}+\frac{b\epsilon(b)}{\log b}-\frac{a\epsilon^*}{\log a}\right)\Bigg)^2$$

$$+\Bigg(-0.1254977\log N+0.6366197\,\epsilon^*-0.0081994\,\epsilon(b)$$

$$+\left(0.575+\frac{\tau^2}{120}\right)\left(-0.4257265-\frac{0.325281\epsilon(b)}{\log N}\right)$$

$$+\left(\frac{49\tau+\tau^3}{720}\right)\left(1.1466715-\frac{1.0011126\epsilon(b)}{\log N}-\frac{2\epsilon^*}{\log N}\right)$$

$$-\frac{\tau}{2N^2}\left(\int_a^b\frac{u\,du}{\log u}+\frac{b^2\epsilon(b)}{\log b}-\frac{a^2\epsilon^*}{\log a}-\int_{a^+}^{b^+}\frac{\epsilon(u)u(1-\frac{1}{\log u})du}{\log u}\right)\Bigg)^2\Bigg)^{1/2}.$$

Replacing τ in D_3 by W, we have $\frac{\partial D_3}{\partial W}\geq 0$. Thus, $D_0\geq D_4$ for $549798\leq N_1\leq N\leq N_2\leq 10^{11}$, where D_4 is defined by replacing τ by τ_* in D_3, with $0\leq\tau_*\leq\frac{2\pi}{\log N_2}$.

Hence, for $549798\leq N_1\leq N<N_2\leq 10^{11}$, we have

(15) $$D_0>D_4=G_4-(F_{r,4}^2+F_{i,4}^2)^{1/2}$$

where

$$G_4=0.1518588\log N+0.0524155\epsilon(b)-0.6366197\epsilon^*-P_1$$
$$+\frac{\log N}{\pi}Q_1+\frac{\log N}{\pi}R_1,$$

$$F_{r,4}=-0.0677564\log N-0.05177\epsilon^*+\left(0.575+\frac{\tau_*^2}{120}\right)$$
$$\left(1.1466715-\frac{1.0011126\epsilon(b)}{\log N}-\frac{2\epsilon^*}{\log N}+\left(\frac{49\tau_*+\tau_*^3}{720}\right)\right)\Bigg(0.4257265$$
$$+\frac{0.325281\epsilon(b)}{\log N}\Bigg)-\frac{1}{2N}\left(\int_a^b\frac{du}{\log u}+\frac{b\epsilon(b)}{\log b}-\frac{a\epsilon^*}{\log a}\right)$$

and

$$F_{i,4} = -0.1254977 \log N + 0.6366197\epsilon^* - 0.0081994\epsilon(b)$$
$$+ \left(0.575 + \frac{\tau_*^2}{120}\right)\left(-0.4257265 - \frac{0.325281\epsilon(b)}{\log N}\right)$$
$$+ \left(\frac{49\tau_* + \tau_*^3}{720}\right)\left(1.466715 - \frac{1.0011126\epsilon(b)}{\log N} - \frac{2\epsilon^*}{\log N}\right)$$
$$- \frac{\tau_*}{2N^2}\left(\int_a^b \frac{udu}{\log u} + \frac{b^2\epsilon(b)}{\log b} - \frac{a^2\epsilon^*}{\log a} - \int_{a^+}^{b^+} \frac{\epsilon(u)u(1 - \frac{1}{\log u})du}{\log u}\right) .$$

We use the following upper bounds when calculating D_4:

$$\frac{1}{2N}\int_a^b \frac{du}{\log u} \leq \frac{1}{2 \log N}\left(\frac{1.11111112}{N^{1/20}} + \frac{0.1388889}{N^{1/10}} + \frac{0.1785714}{N^{1/5}}\right.$$
$$\left. + \frac{0.2380952}{N^{3/10}} + \frac{0.3333334}{N^{2/5}} - \frac{2}{N^{1/2}}\right) = g(N)$$

and

$$\frac{1}{2N^2}\int_a^b \frac{udu}{\log u} \leq \frac{1}{N^{1/20}} g(N).$$

The x_i used in computing P_1, Q_1, and R_1 are

$$x_0 = a = N^{1/2},\ x_1 = N^{1.58/\pi},\ x_2 = N^{1.59/\pi},\ x_3 = N^{1.60/\pi}$$

$$x_4 = N^{1.65/\pi},\ x_5 = N^{1.70/\pi},\ x_6 = N^{1.75/\pi},\ x_7 = N^{1.8/\pi}$$

$$x_8 = N^{1.85/\pi},\ x_9 = N^{1.90/\pi},\ x_{10} = N^{1.95/\pi},\ x_{11} = N^{2/\pi}$$

$$x_{12} = N^{2.1/\pi},\ x_{13} = N^{2.2/\pi},\ x_{14} = N^{2.3/\pi},\ x_{15} = N^{2.4/\pi}$$

$$x_{16} = N^{2.5/\pi},\ x_{17} = b = N^{19/20}.$$

We now prove

Lemma 3.11. If $2193361 \le N \le 10^{11}$, then

$$G(1+i\tau) > |F(1+i\tau)|.$$

Proof. We have $-P_1 \ge -0.0058212$, $\frac{\log N}{\pi} Q_1 \ge -0.0034335 \log N$ and $\frac{\log N}{\pi} R_1 \ge -0.006867$. We choose $\epsilon^* = 0$. From Schoenfeld [24, 5.2*], we have $0.998697\, u < \theta(u)$ for $u \ge 1155901$, so that $-0.001303 \le \epsilon(b) \le 0$. Hence, $G_1 \ge 0.1484253 \log N - 0.0127565$.

In evaluating the rest of D_4, we choose $\tau_* = 0$. Since $g(N) \le 0.0164211$, we have $D_4 \ge 0.0058052 \log N - 0.0362244 \ge 0.0485223 + 10^{-6} \log N$, which completes the proof of the lemma.

Continuing, we have

Lemma 3.12. If $549798 \le N \le 2193361$, then

$$G(1+i\tau) > |F(1+i\tau)|.$$

Proof. Schoenfeld [24, Corollary 2*] has shown that (11) holds with $d = 8623$ and $c = 6$ as well as with $d = 486377$ and $c = 37$. Using (11), we obtain

$$-0.0178032 \le \epsilon(u) \qquad (u \ge 11633)$$

and

$$-0.0020448 \le \epsilon(u) \qquad (u \ge 549798).$$

Supplementing these results with our own calculations of minimum values of $\epsilon(u)$ for $1 \le u \le 11633$, we obtain

$$\epsilon(u) \le -0.01872009 \qquad (549798)^{1/2} \le u \le (2193361)^{1/2}$$

and

$$-0.0020448 \le \epsilon(u) \le 0 \qquad 549798 \le u \le 2193361.$$

Hence, we choose $\epsilon^* = -0.01872009$ and we have $-0.0020448 \le \epsilon(b) \le 0$. We have $-P_1 \ge -0.0068286$, $\frac{\log N}{\pi} Q_1 \ge -0.004764 \log N$ and $\frac{\log N}{\pi} R_1 \ge -0.009528$, so that $G_4 \ge 0.1470948 \log N - 0.0052253$.

In evaluating the rest of D_4, we choose $\tau_* = 0.4303274$. Since $g(N) \le 0.0237707$, we have

$$D_4 \ge 0.0044747 \log N - 0.0221691 \ge 0.0369611 + 10^{-6} \log N,$$

which completes the proof of Lemma 3.12.

We have now shown that $G(1+i\tau) > |F(1+i\tau)|$ for $N \geq 549798$. To show that $\zeta_N(s)$ has a zero with $\sigma > 1$ for $N \geq 549798$, we need

<u>Lemma 3.13.</u> If $N \geq 549798$, then there is no prime q, $N^{1/2} < q \leq N^{19/20}$, such that

$$q^{-1}\Big|\sum_{m\leq N/q} m^{-1-i\tau}\Big| > \sum_{\substack{N^{1/2}<p\leq N^{19/20}\\ p\neq q}} p^{-1}\Big|\sum_{m\leq N/p} m^{-1-i\tau}\Big|.$$

<u>Proof.</u> It is sufficient to show that

$$q^{-1}|h(q, 1+i\tau)| \leq \frac{1}{2}G(1+i\tau)$$

for all primes q, $N^{1/2} < q \leq N^{19/20}$. Using Theorem 3.6 and Lemma 3.9, we obtain

$$G(1+i\tau) \geq 0.1518578 \log N + 0.0524155\epsilon(b) - 0.6366197\epsilon(a)$$
$$- P + \frac{\log N}{\pi} Q + \frac{\log N}{\pi} R - 0.002627$$
$$\geq 0.1470948 \log N - 0.0198213.$$

Thus, it is sufficient to show that for all primes q, $N^{1/2} < q \leq N^{19/20}$,

$$q^{-1}|h(q, 1+i\tau)| \leq 0.0735474 \log N - 0.0099106.$$

Using Lemma 3.4, Lemma 3.5, and Lemma 3.9, we obtain

$$|h(q, 1+i\tau)| \le \left(0.575 + \frac{\tau^2}{120}\right) + \frac{\log N}{2\pi} + \frac{p}{N}\left(\frac{1}{2} + \frac{p}{12N}\right) + \frac{p^2\tau}{2N^2}$$

$$+ \frac{\log N}{\pi} + \frac{49\tau + \tau^3}{720} + \frac{p^2\tau}{2N^2} + \frac{p}{N}\left(\frac{1}{2} + \frac{p}{12N}\right) + 0.002627 + 10^{-6}\log N$$

$$\le 0.4774657 \log N + 3.$$

Hence,

$$q^{-1}|h(q, 1+i\tau)| \le 0.07354741 \log N - 0.0099106,$$

which completes the proof of the lemma.

Combining the results of Lemma 3.10, Lemma 3.11, Lemma 3.12, and Lemma 3.13, we see that

$\zeta_N(s)$ has a zero with $\sigma > 1$ for every $N \ge 549798$.

Finally we have

Lemma 3.14. If $30 < N < 549798$, then

$\zeta_N(s)$ has a zero with $\sigma > 1$.

Proof. Using Bohr's result as before, we have a zero of $\zeta_N(s)$ with $\sigma > 1$ if for some τ

$$(16) \qquad \sum_{N^{1/2}<p\le N} p^{-1} \left| \sum_{m\le N/p} m^{-1-i\tau} \right| > \left| \sum_{\substack{n=1 \\ p/n \to p \le N^{1/2}}}^{N} n^{-1-i\tau} \right|$$

and there is no prime q, $N^{1/2} < q \leq N$, such that

$$(17) \qquad q^{-1} \Big| \sum_{m \leq N/q} m^{-1-i\tau} \Big| > \sum_{\substack{N^{1/2} < p \leq N \\ p \neq q}} p^{-1} \Big| \sum_{m \leq N/p} m^{-1-i\tau} \Big| .$$

Using a computer program (Appendix C), we showed that the above inequalities are true for $30 < N < 549798$. Thus, $\zeta_N(s)$ has a zero with $\sigma > 1$ for $30 < N < 549798$.

This completes the proof of the theorem.

APPENDICES

APPENDIX A

The following programs are used to obtain the results in Chapter 1. Our polynomials are stored as vectors of coefficients, with the first element equal to the position of the highest degree non-zero term, the second element equal to the value of the constant term, and the remaining elements equal to the values of the other coefficients. The program listings follow:

A program to add two polynomials

```
      SUBROUTINE ADD (P, Q, R)
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION P (38)
      DIMENSION Q (38)
      DIMENSION R (38)
      MDEGP = P(1)
      MDEGQ = Q(1)
      MDEGR = MDEGP
      IF  (MDEGQ  . GT.  MDEGP)  MDEGR = MDEGQ
      R(1)  = MDEGR
      DO 10 I = 2, 38
      R(I) = P(I) + Q(I)
10    CONTINUE
      RETURN
      END
```

A program to multiply a polynomial by a scalar

```
      SUBROUTINE SMULT (P, S, R)
      IMPLICIT REAL * 8 (A-H, O-Z)
      DIMENSION P(38)
      DIMENSION R(38)
      MDEG = P (1)
      DO  10  I = 2, 38
      R(I) = P(I) * S
10    CONTINUE
      R(1) = MDEG
      RETURN
      END
```

A program to multiply two polynomials

```
      SUBROUTINE MULT (P, Q, R)
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION P(38)
      DIMENSION Q(38)
      DIMENSION R(38)
      MDEGP = P(1)
      MDEGQ = Q(1)
      R(1) = MDEGP + MDEGQ - 2
      DO 5 K = 2, 38
      R(K) = 0.0D0
    5 CONTINUE
      DO 20 I = 2, MDEGP
      DO 10 J = 2, MDEGQ
      IF (P(I) .EQ. 0.0D0) GO TO 10
      IF (Q(J) .EQ. 0.0D0) GO TO 10
      R(I + J - 2) = R(I + J - 2) + (P(I) *Q(J))
   10 CONTINUE
   20 CONTINUE
      RETURN
      END
```

A program to evaluate polynomials

```
      SUBROUTINE EVAL (P, X, Y)
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION P(300)
      MDEGP = P(1)
      Y = P(MDEGP)
      MSTOP = MDEGP - 2
      IF (MSTOP .LT. 1) GO TO 10
      DO 10 I = 1, MSTOP
      INDEX = MDEGP - I
      Y = P (INDEX) + (X*Y)
   10 CONTINUE
      RETURN
      END
```

A program to bound $\prod_{p \leq p_o} \frac{2}{\pi} \sqrt{6 - \sqrt{36 - \frac{24\pi p}{B} - \frac{12\pi^2}{B^2}}}$

```
      IMPLICIT REAL * 8 (A-H, O-Z)
      DIMENSION MP(13)
      FPI = 3.14159265358973D0
      CONSQ = 2.0D0/FPI
      TOTAL = 1.0D0
      N = 0
      READ (4, 50)  T
 50   FORMAT  (F13.6)
 10   READ (5, 100)  (MP(I) , I = 1, 13)
100   FORMAT (13I6)
      I = 1
 15   IF (MP(I) .EQ. 0)  GO TO 20
      P = MP(I)
      BOUNDP = (2.0D0 * T)/FPI
      IF (P .GT.  BOUNDP)  GO TO 20
      BOUND1 = 36.0D0 - (((24.0D0*FPI*P)/T) + ((12.0D0*FPI*FPI)/(T*T)))
      BOUND2 = 6.0D0 - DSQRT (BOUND1)
      BOUND = CONSQ * DSQRT (BOUND2)
      IF (BOUND .GT.  1.0D0)  GO TO 20
      TOTAL = TOTAL * BOUND
      N = N + 1
      WRITE (6,200)  N, TOTAL
      I = I + 1
      IF (I .EQ.  14)  GO TO 10
      GO TO 15
 20   WRITE (6,200) N, TOTAL
200   FORMAT (I5, E15.6)
      RETURN
      END
```

A program to bound $\sum_{p \leq 1010} \frac{1}{\sqrt{p^2-1}}$

```
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION IP(169)
      READ  (5,100)  (IP(I) ,  I = 1, 169)
  100 FORMAT  (13I6)
      TOTAL  =  0.0
      DO  10  I  =  1, 169
      P  =  IP (I)
      TOTAL = TOTAL + (1.0D0/DSQRT ((P * P) - 1.0D0))
      WRITE  (6, 200)  IP(I),  TOTAL
  200 FORMAT  (I6, E20.6)
   10 CONTINUE
      RETURN
      END
```

A program to compute $\prod_{p \leq 101} \frac{2}{\pi} \left(\frac{\pi}{2} - \sin^{-1} p \sin\left(\sin^{-1} \frac{1}{p} - \frac{\pi}{nh}\right) - \sin^{-1} p \sin \frac{\pi}{2nh}\right)$

```
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION MP(26)
      FPI  =  3.14159265358973D0
      CONSQ  =  2.0D0/FPI
      READ (5,100)  (MP(I) ,  I = 1, 26)
  100 FORMAT (13I6)
      DO  40  IT = 401, 410
      T  = IT
      T  =  T*.5D0
      TOTAL  =  1.0D0
      DO  30  I = 1, 26
      P  =  MP(I)
      BOUNDP  =  (2.0D0 * T)/FPI
      IF  (P .GT.  BOUNDP)  GO TO 20
      BOUND1 = DARSIN (P * DSIN(DARSIN(1.0D0/P)  -  (FPI/T)))
      BOUND2 = DARSIN (P * DSIN(FPI/(2.0D0 * T)))
      BOUND  =  1.0D0 - (BOUND1 + BOUND2) * CONSQ
      IF  (BOUND  .GT.  1.0D0)  GO TO 20
      TOTAL  =  TOTAL*BOUND
```

```
 30   CONTINUE
 20   WRITE  (6, 200)  TOTAL
200   FORMAT  (D24.16)
 40   CONTINUE
      RETURN
      END
```

A program to compute $\prod_{p_0 \leq p \leq p_1} G(p, nh)$ using Lemma 1.15

```
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION MP(13)
      DIMENSION MPRIME (200)
      DIMENSION P(160)
      DIMENSION TOTAL (400)
      N = 0
 10   READ  (5,100)  (MP(I) ,  I = 1,13)
100   FORMAT  (13I6)
      I = 1
 15   IF (MP(I)  .EQ.  0)  GO TO 20
      N = N + 1
      MPRIME (N)  = MP(I)
      I = I + 1
      IF (I  .EQ.  14)  GO TO 10
      GO TO 15
 20   CONTINUE
      DO 40  I = 1, 400
      TOTAL(I)  = 1.0D0
      FI  = I
      FI  = FI*.5D0
      CALL COMFPT (43, FI, M)
      CALL PREPOL (FI, M, P)
      DO 30  J = 1, N
      CALL COMFPT (MPRIME(J),  FI,  IDEG)
      P(1) = IDEG + 2
      FP  = MPRIME (J)
      X  = 1.0D0/(FP*FP)
      CALL PREEVA (P, X, Y)
      TOTAL (I)  = TOTAL (I)*Y
 30   CONTINUE
      WRITE (6,150)  TOTAL (I)
150   FORMAT  (D24.16)
 40   CONTINUE
      RETURN
      END
```

A program to compute the number of terms of $\sum_{k=0}^{N} \frac{\prod_{j=0}^{k-1}\left(j^2 - \frac{(nh)^2}{4}\right)}{(k!)^2 p^{2k}}$

necessary to compute G(p, nh) using Lemma 1.15

```
      SUBROUTINE COMFPT (IP, T, N)
      IMPLICIT REAL*8 (A-H, O-Z)
      C0 = -8.0D0*DLOG(10.0D0)
      C1 = DLOG(2.0D0*3.14159265358973D0)/2.0D0
      C2 = DEXP(1.0D0)
      C3 = T/2.0D0
      K = T/DSQRT(2.0D0)
      FP = IP
      FK = K
      FK1 = FK + 1.0D0
      REMAIN = (FK1*DLOG((T*C2)/(2.0D0*FP*FK1)) - (C1 + (DLOG
     1(FK1)/2.0D0))
      N = K
      IF (REMAIN .EQ. C0) GO TO 90
      IF (REMAIN .LT. C0) GO TO 10
      N = ((FK1 + (FK1*DLOG(C3))) - (C0 + C1 + ((FK1 + .5D0)*DLOG
     1(FK1))))/DLOG(FP)
      GO TO 90
   10 NLOW = (C2*C3)/FP
      FNLOW1 = NLOW + 1
      REMAIN = (FNLOW1*DLOG((T*C2)/(2.0D0*FP*FNLOW1))) -
     1(C1 + (DLOG(FNLOW1)/2.0D0))
      IF (REMAIN .GT. C0) GO TO 15
      N = NLOW
      GO TO 90
   15 NHIGH = N
      DO 40 IBISEC = 1, 40
      NDIFF = NHIGH - NLOW
      IF (NDIFF .LT. 2) GO TO 50
      NMID = (NHIGH + NLOW)/2
      FNMID1 = NMID + 1
      REMAIN = (FNMID1*DLOG((T*C2)/(2.0D0*FP*FNMID1))) -
     1(C1 + DLOG(FNMID1)/2.0D0))
      IF (REMAIN .GT. C0) GO TO 30
      IF (REMAIN .EQ. C0) GO TO 20
      NHIGH = NMID
      GO TO 40
```

```
20   N = NMID
     GO TO 90
30   NLOW = NMID
40   CONTINUE
50   N = NHIGH
90   RETURN
     END
```

A program to compute $\sum_{k=0}^{N} \frac{\prod_{j=0}^{k-1}\left(j^2 - \frac{(nh)^2}{4}\right)}{(k!)^2\, p^{2k}}$ as a polynomial in p for given n

```
      SUBROUTINE PREPOL (T, N, P)
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION P(160)
      NPLUS2 = N + 2
      P(1) = NPLUS2
      P(2) = 1.0D0
      P(3) = -(T*T)/(4.0D0)
      DO 10 I = 4, NPLUS2
      FI = I
      P(I) = P(I - 1) * (FI - 3.0D0) * (FI - 3.0D0)/(FI - 2.0D0) *
     1(FI - 2.0D0))
      P(I) = P(I) * ((1.0D0) - ((T*T)/(4.0D0*(FI - 3.0D0) *
     1(FI - 3.0D0))))
10    CONTINUE
      NPLUS3 = N + 3
      DO 20 I = NPLUS3, 160
      P(I) = 0.0D0
20    CONTINUE
      RETURN
      END
```

A program to evaluate polynomials

```
      SUBROUTINE PREEVA (P, X, Y)
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION P(160)
      DIMENSION PPOS(160)
      DIMENSION PNEG(160)
      MDEGP = P(1)
      PPOS (1) = MDEGP
      PNEG(1) = MDEGP
      DO 20  I = 2, =DEGP
      PPOS(I) = 0.0D0
      PNEG(I) = 0.0D0
      IF  (P(I)  .GT.  0.0D0)  GO TO 10
      PNEG (I) = P(I)
      GO TO 20
10    PPOS (I) = P(I)
20    CONTINUE
      CALL EVAL (PPOS, X, YPOS)
      CALL EVAL (PNEG, X, YNEG)
      Y = YPOS + YNEG
      RETURN
      END
```

A program to compute $\prod_{p_0 \leq p \leq p_1} G(p, nh)$ in multiple precision using Lemma 1.15

```
      IMPLICIT REAL*8(A-H, O-Z)
      INTEGER B, T, R
      DIMENSION IP(32, 160)
      DIMENSION R(258)
      DIMENSION JP(32)
      DIMENSION KP(32)
      DIMENSION LP(32)
      DIMENSION MPRIME(13)
      COMMON B, T, M, LUN, MXR, R
      MXR = 258
      LUN = 8
      B = 10000
      T = 30
      M = 800
```

```
      READ (5, 100)  (MPRIME(I) , I = 1, 13)
  100 FORMAT (13I6)
      DO 40  I = 1, 400
      TOTAL = 1.0D0
      FI = I
      FI = FI*.5D0
      CALL COMFPT (2, FI, MDEG2)
      CALL PREPMP (FI, MDEG2, IP)
      DO 30  J = 1, 13
      CALL COMFPT(MPRIME (J), FI, IDEG)
      IDEG2 = IDEG + 2
      CALL MPCIM(IDEG2, IP (1, 1))
      MPS = MPRIME(J) * MPRIME(J)
      CALL EVALMP (IP, MPS, Y)
      TOTAL = TOTAL*Y
   30 CONTINUE
      WRITE (6, 200)  TOTAL
  200 FORMAT (D24.16)
   40 CONTINUE
      RETURN
      END
```

The MP routines are due to Brent and are available in the Michigan computer library.

A program to compute $\sum_{k=0}^{N} \frac{\prod_{j=0}^{k-1}\left(j^2 - \frac{(nh)^2}{4}\right)}{(k!)^2\, p^{2k}}$ as a polynomial in p for given n in multiple precision

```
      SUBROUTINE PREPMP(X, N, IP)
      IMPLICIT REAL*8 (A-H, O-Z)
      INTEGER B, T, R
      DIMENSION IP (32, 160)
      DIMENSION R(258)
      DIMENSION JP(32)
      DIMENSION KP(32)
      DIMENSION LP(32)
      COMMON B, T, M, LUN, MXR, R
      MXR = 258
      LUN = 8
```

```
      B = 10000
      T = 30
      M = 800
      Y = (X*X)/4.0D0
      I1 = 1
      NPLUS2 = N + 2
      CALL MPCIM (NPLUS2, IP (1,1))
      CALL MPCIM (I1, IP (1, 2))
      OPPY = -Y
      CALL MPCDM (OPPY, IP(1, 3))
      DO 10 I = 4, NPLUS2
      IL2SQ = (I - 2) * (I - 2)
      IL3SQ = (I - 3) * (I - 3)
      CALL MPMULQ (IP(1, I-1), IL3SQ, IL2SQ, JP)
      CALL MPDIVI (IP (1, 3), IL3SQ, KP)
      CALL MPADDI (KP, I1, LP)
      CALL MPMUL (JP, LP, IP (1, I))
   10 CONTINUE
      NPLUS3 = N + 3
      DO 20 I = NPLUS3, 160
      IP (1, I) = 0
   20 CONTINUE
      RETURN
      END
```

<u>A program to evaluate polynomials in multiple precision</u>

```
      SUBROUTINE EVALMP (IP, MP, Y)
      IMPLICIT REAL*8 (A-H, O-Z)
      INTEGER B, T, R
      DIMENSION IP(32, 160)
      DIMENSION R(258)
      DIMENSION JP(32)
      DIMENSION KP(32)
      DIMENSION LP(32)
      COMMON B, T, M, LUN, MXR, R
      MXR = 258
      LUN = 8
      B = 10000
      T = 30
      M = 800
      CALL MPCMI(IP (1, 1), MDEGP)
      CALL MPSTR(IP (1, MDEGP), JP)
      MSTOP = MDEG - 2
      IF (MSTOP .LT. 1) GO TO 10
```

```
      DO 10  I = 1, MSTOP
      INDEX = MDEGP - I
      CALL MPDIVI (JP, MP, JP)
      CALL MPADD (JP, IP (1, INDEX), JP)
   10 CONTINUE
      CALL MPCMD (JP, Y)
      RETURN
      END
```

<u>A program to compute $R_L(nh)$ and $R_H(nh)$ using Theorem 1.16</u>

```
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION S1(38)
      DIMENSION S2(38)
      DIMENSION S3(38)
      DIMENSION S4(38)
      DIMENSION S5(38)
      DIMENSION S6(38)
      DIMENSION BL(38)
      DIMENSION BH(38)
      DIMENSION H(38)
      DIMENSION HL(38)
      DIMENSION HH(38)
      U = 1155901.0D0
      THETAU = 1150824.716334041D0/U
      DLOGU = DLOG(U)
      DLOG11 = DLOG(1155901.0D0)
      DO 60  IT = 1, 400
      T = IT
      T = T*.5D0
      T5 = 1.0D0/(((144.0D0*U*U)/(T*T)) - 1.0D0)
      IF  (T  .LT.  12.0D0)  T5 = 1.0D0/((U*U) - 1.0D0)
      S1(1) = 8.0D0
      S1(2) = 0.0D0
      S1(3) = -(T*T)/(4.0D0*U*U)
      DO 3  I = 4, 8
      FI = I
      S1(I) = S1(I - 1)*(FI - 3.0D0)*(FI - 3.0D0)/((FI - 2.0D0)*
     1(FI - 2.0D0))
      S1(I) = S1(I)*((1.0D0) - ((T*T)/(4.0D0*(FI - 3.0D0)*(FI - 3.0D0)))
     1)/(U*U)
    3 CONTINUE
      SAVS18 = S1(8)
      S1(8) = SAVS18 - (DABS(SAVS18)*T5)
```

```
      DO 5  K = 9, 38
      S1(K) = 0.0D0
  5   CONTINUE
      CALL MULT (S1, S1, S2)
      CALL MULT (S1, S2, S3)
      CALL MULT (S1, S3, S4)
      CALL MULT (S1, S4, S5)
      CALL MULT (S1, S5, S6)
      S1(8) = SAVS18 + (DABS(SAVS18) * T5)
      CALL MULT (S1, S1, S2)
      CALL MULT (S1, S2, S3)
      CALL MULT (S1, S3, S4)
      CALL MULT (S1, S4, S5)
      CALL MULT (S1, S5, S6)
      DO 10  K = 1,38
      BL(K) = S1(K)
 10   CONTINUE
      D2 = -.5D0
      CALL SMULT(S2, D2, H)
      CALL ADD(BL, H, BL)
      D3 = 1.0D0/3.0D0
      CALL SMULT(S3,D3,H)
      CALL ADD(BL,H,BL)
      D4 = -1.0D0/4.0D0
      CALL SMULT(S4,D4,H)
      CALL ADD(BL,H,BL)
      D5 = 1.0D0/5.0D0
      CALL SMULT(S5,D5,H)
      CALL ADD(BL,H,BL)
      D6 = -1.0098775D0/6.0D0
      CALL SMULT(S6,D6,H)
      CALL ADD(BL,H,BL)
      DO 20  K = 1, 38
      BH(K) = S1(K)
 20   CONTINUE
      D2 = -.5D0
      CALL SMULT(S2,D2,H)
      CALL ADD(BH,H,BH)
      D3 = 1.0D0/3.0D0
      CALL SMULT(S3,D3,H)
      CALL ADD(BH,H,BH)
      D4 = -1.0D0/4.0D0
      CALL SMULT(S4,D4,H)
      CALL ADD(BH,H,BH)
```

```
      D5 = 1.0D0/5.0D0
      CALL SMULT(S5,D5,H)
      CALL ADD(BH,H,BH)
      D6 = -1.0D0/6.0D0
      CALL SMULT(S6,D6,H)
      CALL ADD(BH,H,BH)
      I = 41 - K
      IF (BL(I) .NE. 0.0D0) GO TO 22
      IF (BR(I) .NE. 0.0D0) GO TO 22
      GO TO 30
22    I2L1 = (I*2) - 5
      FI2L1 = I2L1
      FI = I - 2
      P1 = ((1.0D0 - (THETAU)) + (EI(FI2L1*DLOGU)/FI2L1))/DLOGU
      P2 = .00000003D0/(FI2L1*DLOGU)
      P3 = .001303D0*(1.0D0 + ((EI(FI2L1*DLOG11) + .0000000300)
     1/FI2L1))*((U/1155901.0D0)**I2L1)/DLOG11
      HL(I) = BL(I)*U*(P1 - (P2 + P3))
      IF (BL(I) .LT. 0.0D0) HL(I) = BL(I)*U*(P1 + P2)
      HH(I) = BH(I)*U*(P1 + P2)
      IF (BH(I) .LT. 0.0D0) HH(I) = BH(I)*U*(P1 - (P2 + P3))
      TOTHL = TOTHL + HL(I)
      TOTHH = TOTHH + HH(I)
30    CONTINUE
      BOUNDL = DEXP(TOTHL)
      BOUNDH = DEXP(TOTHH)
      WRITE (6,400) BOUNDL
      WRITE (7,400) BOUNDH
400   FORMAT (D24.16)
60    CONTINUE
      RETURN
      END
```

A program to compute exponential integrals using Lemma 1.20

```
      REAL FUNCTION EI*8(V)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Q1(6)
      DIMENSION Q2(6)
      Q1(1) = 6.0D0
      Q2(1) = 6.0D0
      Q1(2) = .2677737343D0
      Q2(2) = 3.9584969228D0
      Q1(3) = 8.6347608925D0
      Q2(3) = 21.0996530827D0
```

```
      Q1(4) = 18.0590159730D0
      Q2(4) = 25.6329561486D0
      Q1(5) = 8.5733287401D0
      Q2(5) = 9.5733223454D0
      Q1(6) = 1.0D0
      Q2(6) = 1.0D0
      CALL EVAL(Q1,V,Y1)
      CALL EVAL(Q2,V,Y2)
      EI = Y1/Y2
      RETURN
      END
```

<u>A program to compute θ (u)</u>

```
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION IP(9592)
      READ (5,100) (IP(I) , I = 1, 9592)
100   FORMAT (13I6)
      TOTAL = 0.0D0
      DO 10 I = 1, 9592
      P = IP(I)
      TOTAL = TOTAL + DLOG(P)
 10   CONTINUE
      WRITE (6,200) TOTAL
200   FORMAT (D30.16)
      RETURN
      END
```

<u>A program to compute upper and lower bounds for</u>

<u>F(x, A) using Theorem 1.16</u>

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION FUNVAL(400)
      DIMENSION BOUNDL(400)
      DIMENSION BOUNDH(400)
      DIMENSION TOTLOW(601)
      DIMENSION TOTHIG(601)
      DIMENSION TOTLOD(601)
      DIMENSION TOTHID(601)
      DIMENSION XDIST(601)
      NPOINT = 400
      READ (5,100) (FUNVAL(I), I = 1, NPOINT)
```

```
100   FORMAT (3D24.16)
      READ (2,200) (BOUNDI(I), I = 1, NPOINT)
      READ (3,200) (BOUNDH(I), I = 1, NPOINT)
200   FORMAT (D24.16)
      FPI = 3.14159265358973D0
      NDIST = 601
      DC 2 J = 1, DIST
      FJ = J - 1
      XDIST(J) = FJ*.001D0*FPI
  2   CONTINUE
      H = .5D0
  5   READ (4,50) INCR
 50   FORMAT (I6)
      IF (INCR .EQ. 0) GO TO 40
      NUSED = NPOINT/INCR
      DO 10 K = 1, NDIST
      TOTLOW(K) = 0.0D0
      TOTHIG(K) = 0.0D0
      TOTLOD(K) = 0.0D0
      TOTHID(K) = 0.0D0
 10   CONTINUE
      DO 30 J = 1, NUSED
      I = (NUSED + 1) - J
      INDEX = I*INCR
      FINDEX = INDEX
      X = FINDEX*H
      DO 20 K = 1, NDIST
      FUNC = DSIN(X*XDIST(K))/X
      F = FUNC*FUNVAL(INDEX)
      TRALOW = F*BOUNDL(INDEX)
      IF (F .LT. 0.0D0) TRALOW = F*BOUNDH(INDEX)
      TRAHIG = F*BOUNDH(INDEX)
      IF (F .LT. 0.0D0) TRAHIG = F*BOUNDL(INDEX)
      TOTLOW(K) = TOTLOW(K) + TRALOW
      TOTHIG(K) = TOTHIG(K) + TRAHIG
      FUNC = DCOS(X*XDIST(K))
      F = FUNC*FUNVAL(INDEX)
      TRALOD = F*BOUNDL(INDEX)
      IF (F .LT. 0.0D0) TRALOD = F*BOUNDH(INDEX)
      TRAHID = F*BOUNDH(INDEX)
      IF (F .LT. 0.0D0) TRAHID = F*BOUNDL(INDEX)
      TOTLOD(K) = TOTLOD(K) + TRALOD
      TOTHID(K) = TOTHID(K) + TRAHID
 20   CONTINUE
 30   CONTINUE
```

```
      FINCR = INCR
      DO 35  K = 1, NDIST
      TOTLOW(K) = TOTLOW(K) + (.5D0 * XDIST(K))
      TOTHIG(K) = TOTHIG(K) + (.5D0* XDIST(K))
      TOTLOD(K) = TOTLOD(K) + .5D0
      TOTHID(K) = TOTHID(K) + .5D0
      TOTLOW(K) = TOTLOW(K) * H * FINCR/FPI
      TOTHIG(K) = TOTHIG(K) * H *FINCR/FPI
      TOTLOD(K) = TOTLOD(K) * H * FINCR/FPI
      TOTHID(K) = TOTHID(K) * H * FINCR/FPI
      PRINT1 = TOTLOW(K) + .5D0
      PRINT2 = TOTHIG(K) + .5D0
      PRINT3 = .5D0 - TOTHIG(K)
      PRINT4 = .5D0 - TOTLOW(K)
      FK1 = K - 1
      PRINT0 = FK1 *.001D0
      WRITE (6,250) PRINT0, PRINT1, PRINT2, PRINT3, PRINT4
250   FORMAT (F13.3, 'PI', 4D24.16)
      WRITE (6,260) TOTLOD(K), TOTHID(K)
260   FORMAT (2D24.16)
 35   CONTINUE
      GO TO 5
 40   RETURN
      END
```

<u>A program to compute L(1, X) using (19) and (20)</u>

```
      IMPLICIT REAL * 8(A-H, O-Z)
      DIMENSION MPRIME(400)
      DIMENSION MPRIMR(400)
      DIMENSION FG(2000)
      DIMENSION FLSIN(2000)
      DIMENSION FRETAU(2000)
      DIMENSION FIMTAU(2000)
      PI = 3.14159265358979323D0
      TWOPI = PI * 2.0D0
      READ (4,100) M, N
100   FORMAT (13I6)
      READ (4,100) (MPRIME(I), I = 1, N)
      READ (5,100) (MPRIMR(I), I = 1, N)
      DO 40 I = M, N
         IPRIME = MPRIME(I)
         IPRIMR = MPRIMR(I)
```

```
      FPRIME = IPRIME
      IPHI = IPRIME - 1
      FIPHI = IPHI
      KSTOP = IPHI/2
      FM = TWOPI/FIPHI
      IPROD = 1
      DO 10  IR = 1, IPHI
          IPROD = IPROD * IPRIMR
          IF (IPROD  .GT.  IPRIME)  IPROD = IPROD -
     1    (IPRIME * (IPROD/IPRIME))
          FIPROD = IPROD
          FG(IR) = FIPROD
          FLSIN(IR) = DLOG(DSIN(PI * FIPROD/FPRIME))
          FRETAU(IR) = DCOS(TWOPI * FIPROD/FPRIME)
          FIMTAU(IR) = DSIN(TWOPI * FIPROD/FPRIME)
10    CONTINUE
      FREMIN = 1000.0D0
      ARGMAX = 0.0D0
      DO 30  K = 1, KSTOP
          FK = K
          FRE1 = DCOS(TWOPI * FK/FIPHI)
          FIM1 = -DSIN(TWOPI * FK/FIPHI)
          FRE = 1.0D0
          FIM = 0.0D0
          TOTRE = 0.0D0
          TOTIM = 0.0D0
          TAURE = 0.0D0
          TAUIM = 0.0D0
          KEY = -1
          QUO = K/2
          IF ((FK/2.0D0)  .EQ.  QUO)  KEY = 1
          DO 20  IR = 1, IPHI
              HRE = (FRE * FRE1) - (FIM * FIM1)
              HIM = (FRE * FIM1) + (FIM * FRE1)
              FRE = HRE
              FIM = HIM
              FMULT = FLSIN(IR)
              IF (KEY  .EQ.  -1)  FMULT = FG(IR)
              TOTRE = TOTRE + (FRE * FMULT)
              TOTIM = TOTIM + (FIM * FMULT)
              TAURE = TAURE + (FRE * FRETAU(IR)) - (FIM *
     1        FIMTAU(IR))
              TAUIM = TAUIM + (FRE * FIMTAU(IR)) + (FIM *
     1        FRETAU(IR))
```

```
 20          CONTINUE
             HRE = (TOTRE * TAURE) + (TOTIM * TAUIM)
             HIM = (TOTIM * TAURE) - (TOTRE * TAUIM)
             FMOD = (TAURE **2) + (TAUIM **2)
             FRE = -HRE/FMOD
             FIM = -HIM/FMOD
             IF (KEY .EQ. 1) GO TO 25
             HRE = -FIM * PI/FPRIME
             HIM = FRE * PI/FPRIME
             FRE = HRE
             FIM = HIM
 25          ARG = DATAN(FIM/FRE)
             ARG = DABS(ARG)
             IF (FRE .LT. FREMIN) FREMIN = FRE
             IF (ARG .GT. ARGMAX) ARGMAX = ARG
 30        CONTINUE
           WRITE (6,300) IPRIME, FREMIN, ARGMAX
300        FORMAT (' ', I4, 2F16.8)
 40     CONTINUE
        STOP
        END
```

APPENDIX B

The following programs are used to obtain the results in Chapter II. The polynomial manipulation routines are the same as those in Appendix A. The program listings follow.

A program to bound $\prod_{0 \leq y \leq 1005} \left(\frac{2|\rho|}{\pi c}\right)^{\frac{1}{2}} \left(1 + \frac{|\rho|}{8c}\right)$ using (31)

```
      DIMENSION GAMMA (6)
      CONSQ = 2. 0/3. 141593
      CONS = SQRT(CONSQ)
      TOTAL = 1. 0
      N = 0
      T = EXP(10. 0)
 10   READ (5, 100) (GAMMA(I), I = 1, 6)
100   FORMAT (6F13. 6)
      I = 1
 15   IF (GAMMA(I) .EQ. 0. 0) GO TO 20
      N = N+1
      RHO = SQRT(. 25 + (GAMMA(I) * GAMMA(I)))
      XRECIP = RHO/T
      TOTAL = TOTAL * CONS *(1. 0 + (XRECIP/8. 0)) *SQRT(XRECIP)
      WRITE (6, 200) N, TOTAL
      I = I + 1
      IF (I .EQ. 7) GO TO 10
      GO TO 15
 20   WRITE (6, 200) N, TOTAL
200   FORMAT (I5, E15. 6)
      RETURN
      END
```

A program to compute $\sum_{0<\gamma\leq 1005} \frac{1}{|\rho|}$ and related sums

```
      IMPLICIT REAL * 8 (A-H, O-Z)
      DIMENSION GAMMA (653)
      READ  (5,100)  (GAMMA(I),  I = 1, 653)
100   FORMAT  (6F13.6)
      TOTALL = 0.0D0
      TOTAL = 0.0D0
      TOTALH = 0.0D0
      DO 10  I = 1, 653
      RHOL = DSQRT(.25D0 + ((GAMMA(I) + .000002D0) * (GAMMA(I) +
     1.000002D0)))
      RHO = DSQRT(.25D0 + ((GAMMA(I)) * (GAMMA(I))))
      RHOH = DSQRT(.25D0 + ((GAMMA(I) - .000002D0) * (GAMMA(I) -
     1.000002D0)))
      TOTALL = TOTALL + (1.0D0/RHOL)
      TOTAL = TOTAL + (1.0D0/RHO)
      TOTALH = TOTALH + (1.0D0/RHOH)
      WRITE = (6,200)  TOTALL, TOTAL, TOTALH
200   FORMAT  (3D24.16)
 10   CONTINUE
      RETURN
      END
```

A program to calculate $\prod_{0<\gamma\leq 1005} J_o\left(\frac{nh}{|\rho|}\right)$

```
      IMPLICIT REAL * 8(A-H, O-Z)
      DIMENSION GAMMA(6)
      DIMENSION RHO(653)
      DIMENSION TOTAL(2688)
      N = 0
 10   READ  (5,100)  (GAMMA(I),  I = 1, 6)
100   FORMAT  (6F13.6)
      I = 1
 15   IF  (GAMMA(I)  .EQ.   0.0D0)  GO TO 20
      N = N + 1
      RHO(N) = DSQRT(.25D0 + GAMMA(I) * GAMMA(I)))
      I = I + 1
      IF  (I  .EQ.   7)  GO TO 10
      GO TO 15
 20   CONTINUE
```

```
      FPI = 3.14159265358973D0
      H = FPI/128.0D0
      DO 40  I = 1, 2688
      TOTAL(I) = 1.0D0
      FI = I
      FI = FI*H
      FI = FI + (21.0D0*FPI)
      DO 30  J = 1, N
      X = FI/RHO(J)
      F = BESJ0(X)
      TOTAL(I) = TOTAL(I)*F
   30 CONTINUE
   40 CONTINUE
      WRITE  (6,200)  (TOTAL(I),  I = 1, 2688)
  200 FORMAT  (3D26.16)
      RETURN
      END
```

The program BESJ0 is in the Michigan computer library.

A program to compute $C_L(|\rho|, t)$ and $C_H(|\rho|, t)$ using Theorem 2.12

```
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION S1(38)
      DIMENSION S2(38)
      DIMENSION S3(38)
      DIMENSION S4(38)
      DIEMNSION S5(38)
      DIMENSION S6(38)
      DIMENSION BL(38)
      DIMENSION BH(38)
      DIMENSION H(38)
      R = (1.0D0/(4.0D0*36.0D0*14745600.0D0))
      C1 = .9999121D0
      C2 = 1.0D0
      S1(1) = 8.0D0
      S1(2) = 0.0D0
      S1(3) = -(1.0D0/4.0D0)
      S1(4) = (1.0D0/64.0D0)
      S1(5) = -(1.0D0/2304.0D0)
      S1(6) = (1.0D0/147456.0D0)
      S1(7) = -(1.0D0/14745600.0D0)
      S1(8) = C1*R
```

```
      DO 5  K = 9, 38
      S1(K) = 0.0D0
    5 CONTINUE
      CALL MULT(S1,S1,S2)
      CALL MULT(S1,S2,S3)
      CALL MULT(S1,S3,S4)
      CALL MULT(S1,S4,S5)
      CALL MULT(S1,S5,S6)
      WRITE  (5,200)  (S1(I),  I = 1, 38)
      WRITE  (5,200)  (S2(I),  I = 1, 38)
      WRITE  (5,200)  (S3(I),  I = 1, 38)
      WRITE  (5,200)  (S4(I),  I = 1, 38)
      WRITE  (5,200)  (S5(I),  I = 1, 38)
      WRITE  (5,200)  (S6(I),  I = 1, 38)
      DO 10  K = 1, 38
      BL(K) = S1(K)
   10 CONTINUE
      D2 = -.5D0
      CALL SMULT(S2,D2,H)
      CALL ADD(BL,H,BL)
      D3 = 1.0D0/3.0D0
      CALL SMULT(S3,D3,H)
      CALL ADD(BL,H,BL)
      D4 = -1.0D0/4.0D0
      CALL SMULT (S4,D4,H)
      CALL ADD(BL,H,BL)
      D5 = 1.0D0/5.0D0
      CALL SMULT(S5,D5,H)
      CALL ADD(BL,H,BL)
      D6 = -1.0043233D0/6.0D0
      CALL SMULT(S6,D6,H)
      CALL ADD(BL,H,BL)
      WRITE  (6,200)  (BL(I),  I = 1, 38)
  200 FORMAT  (3D26.16)
      S1(8) = C2*R
      CALL MULT(S1,S1,S2)
      CALL MULT(S1,S2,S3)
      CALL MULT(S1,S3,S4)
      CALL MULT(S1,S4,S5)
      CALL MULT(S1,S5,S6)
      WRITE  (5,200)  (S1(I),  I = 1, 38)
      WRITE  (5,200)  (S2(I),  I = 1, 38)
      WRITE  (5,200)  (S3(I),  I = 1, 38)
      WRITE  (5,200)  (S4(I),  I = 1, 38)
      WRITE  (5,200)  (S5(I),  I = 1, 38)
      WRITE  (5,200)  (S6(I),  I = 1, 38)
```

```
      DO 20  K = 1, 38
      BH(K) = S1(K)
 20   CONTINUE
      D2 = -.5D0
      CALL SMULT(S2, D2, H)
      CALL ADD(BH, H, BH)
      D3 = 1.0D0/3.0D0
      CALL SMULT(S3, D3, H)
      CALL ADD(BH, H, BH)
      D4 = -1.0D0/4.0D0
      CALL SMULT(S4, D4, H)
      CALL ADD(BH, H, BH)
      D5 = 1.0D0/5.0D0
      CALL SMULT(S5, D5, H)
      CALL ADD(BH, H, BH)
      D6 = -1.0D0/6.0D0
      CALL SMULT(S6, D6, H)
      CALL ADD(BH, H, BH)
      WRITE  (7,200)  (BH(I),  I = 1, 38)
      RETURN
      END
```

A program to calculate $M_L(t)$ and $M_H(t)$ using Theorem 2.12

```
      IMPLICIT REAL*8(A-H, O-Z)
      DIMENSION BLVEC(38)
      DIMENSION BHVEC(38)
      DIMENSION QL(38)
      DIMENSION QH(38)
      T = 1005.0D0
      E = .34741D0
      READ  (2,200)  (BLVEC(I),  I = 1, 38)
      READ  (3,200)  (BHVEC(I),  I = 1, 38)
200   FORMAT  (3D26.16)
      A = 3.0767998D0
      B = 1.2046751D0
      C = (DLOG((T*T)/((T*T) - 4.0D0))*2.864789D0) + 7.2371007D0
      D = .1657863D0
      C0 = DLOG(T + 1.322875D0)
      C1 = (A*C0) + (B*DLOG(C0)) + C
      C2 = A + (B*(DLOG(C0)/C0))
      C3 = 2.0D0*3.14159265358973D0
      C4 = DLOG(T/C3)
```

```
      AL = -1.7412588D0
      BL = -2.864789D0
      CL = -.3936986
      DL = -3.8661494
      AH = .0198943
      BH = .4774648
      DH = -.666871
      IBOUND = BLVEC(1)
      QL(1) = IBOUND
      QH(1) = IBOUND
      DO 5  K = 2, 38
      QL(K) = 0.0D0
      QH(K) = 0.0D0
    5 CONTINUE
      DO 10  I = 3, IBOUND
      FI2 = 2 * (I - 2)
      FI2P1 = FI2 + 1.0D0
      P1 = T * (C4 + (1.0D0/(FI2 - 1.0D0)))/((C3 * (FI2 - 1.0D0)))
      P1 = P1 - E
      PH2 = D/FI2P1
      PH2 = PH2 + ((AH-AL) * DLOG(T)) + (AH/FI2P1)
      PH2 = PH2 + ((BH-BL) * DLOG(DLOG(T))) + (BH/FI2P1 * DLOG(T)))
      PH2 = PH2 + ((-CL) * DLOG(DLOG(T) + 1.322875D0))
      PH2 = PH2 + (DH-DL)
      PH2 = (PH2 * FI2)/T
      PL2 = D/FI2P1
      PL2 = PL2 + ((AH-AL) * DLOG(T)) - (AL/FI2P1)
      PL2 = PL2 + ((BH-BL) * DLOG(DLOG(T))) - (BL/(FI2P1 * DLOG(T)))
      PL2 = PL2 + ((-CL) * DLOG(DLOG(T) + 1.322875D0)) - (CL/(FI2P1 *
     1DLOG(T + 1.322875D0)))
      PL2 = PL2 + (DH-DL)
      PL2 = -(PL2 * FI2)/T
      FI2 = FI2 + 2.0D0
      FI2P1 = FI2 + 1.0D0
      PL3 = T * (C4 + (1.0D0/(FI2 - 1.0D0)))/((C3 * (FI2 - 1.0D0)))
      PL3 = PL3 - E
      PL3 = PL3 * ((FI2 - 2.0D0)/(T * T * 8.0D0))
      PL4 = D/FI2P1
      PL4 = PL4 + ((AH-AL) * DLOG(T)) - (AL/FI2P1)
      PL4 = PL4 + ((BH-BL) * DLOG(DLOG(T))) - (BL/(FI2P1 * DLOG(T)))
      PL4 = PL4 + ((-CL) * DLOG(DLOG(T) + 1.322875D0)) - (CL/(FI2P1 *
     1DLOG(T + 1.322875D0)))
      PL4 = PL4 + (DH-DL)
      PL4 = -PL4 * (((FI2 - 2.0D0) * FI2)/(T * T * T * 8.0D0)
      QL(I) = BLVEC(I) * (P1 + PL2 - (PL3 + PL4))
      IF (BLVEC(I) .LT. 0.0D0) QL(I) = BLVEC(I) * (P1 + PH2)
```

```
      QH(I) = BHVEC(I) * (P1 + PH2)
      IF (BHVEC(I) .LT. 0.0D0) QH(I) = BHVEC(I) * (P1 + PL2 - (PL3 +
     1 PL4))
   10 CONTINUE
      WRITE (6,200) (QL(I), I = 1, 38)
      WRITE (7,200) (QH(I), I = 1, 38)
      RETURN
      END
```

A program to compute upper and lower bounds for G(X) using Theorem 2.12

```
      IMPLICIT REAL * 8(A-H,O-Z)
      DIMENSION FUNVAL(5376)
      DIMENSION QL(38)
      DIMENSION QH(38)
      DIMENSION TOTLOW(40)
      DIMENSION TOTHIG(40)
      DIMENSION TOTLOD(40)
      DIMENSION TOTHID(40)
      DIMENSION XDIST(40)
      READ (2,200) (QL(I), I = 1, 38)
      READ (3,200) (QH(I), I = 1, 38)
  200 FORMAT (3D26.16)
      NPOINT = 5376
      READ (5,200) (FUNVAL(I), I = 1, NPOINT)
  100 FORMAT (2D30.16)
      FPI = 3.141592653589793D0
      READ (1, 150) NDIST
  150 FORMAT (I6)
      READ (1, 160) (XDIST(I), I = 1, NDIST)
  160 FORMAT (6F13.6)
      T = 1005.0D0
      H = FPI/128.0D0
    5 READ (4, 50) INCR
   50 FORMAT (I6)
      IF (INCR .EQ. 0) GO TO 40
      NUSED = NPOINT/INCR
      DO 10 K = 1, NDIST
      TOTLOW(K) = 0.0D0
      TOTHIG(K) = 0.0D0
      TOTLOD(K) = 0.0D0
      TOTHID(K) = 0.0D0
   10 CONTINUE
```

```
      DO 30  J = 1, NUSED
      I = (NUSED + 1) - J
      INDEX = I*INCR
      FINDEX = INDEX
      X = FINDEX*H
      X2 = (X*X)/(T*T)
      CALL EVAL(QL, X2, Y1)
      BOUNDL = DEXP(Y1)
      CALL EVAL (QH, X2, Y2)
      BOUNDH = DEXP(Y2)
      DO 20  K = 1, NDIST
      FUNC = DSIN(X*XDIST(K))/X
      F = FUNC * FUNVAL(INDEX)
      TRALOW = F*BOUNDL
      IF (F .LT. 0.0D0) TRALOW = F*BOUNDH
      TRAHIG = F*BOUNDH
      IF (F .LT. 0.0D0) TRAHIG = FBOUNDL
      TOTLOW(K) = TOTLOW(K) + TRALOW
      TOTHIG(K) = TOTHIG(K) + TRAHIG
      FUNC = DCOS(X*XDIST(K))
      F = FUNC * FUNVAL(INDEX)
      TRALOD = F*BOUNDL
      IF (F .LT. 0.0D0) TRALOD = F*BOUNDH
      TRAHID = F*BOUNDH
      IF (F .LT. 0.0D0) TRAHID = F*BOUNDL
      TOTLOD(K) = TOTLOD(K) + TRALOD
      TOTHID(K) = TOTHID(K) + TRAHID
   20 CONTINUE
   30 CONTINUE
      FINCR = INCR
      DO 35  K = 1, NDIST
      TOTLOW(K) = TOTLOW(K) + (.5D0*XDIST(K))
      TOTHIG(K) = TOTHIG(K) + (.5D0*XDIST(K))
      TOTLOD(K) = TOTLOD(K) + .5D0
      TOTHID(K) = TOTHID(K) + .5D0
      TOTLOW(K) = TOTLOW(K)*H*FINCR/FPI
      TOTHIG(K) = TOTHIG(K)*H*FINCR/FPI
      TOTLOD(K) = TOTLOD(K)*H*FINCR/FPI
      TOTHID(K) = TOTHID(K)*H*FINCR/FPI
      PRINT1 = TOTLOW(K) + .5D0
      PRINT2 = TOTHIG(K) + .5D0
      PRINT3 = .5D0 - TOTHIG(K)
      PRINT4 = .5D0 - TOTLOW(K)
      WRITE (6,250) XDIST(K), PRINT1, PRINT2, PRINT3, PRINT4
```

```
250  FORMAT  (F13.6, 4D24.16)
     WRITE  (6,260)  TOTLOD(K), TOTHID(K)
260  FORMAT  (2D24.16)
 35  CONTINUE
     GO TO 5
 40  RETURN
     END
```

APPENDIX C

The following program is used to obtain the results in Chapter 3. The program listing follows.

A program to verify (16) and (17) for N = NBEGIN, NEND

```
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION MPRIME(50000)
      DIMENSION SLHSRE(50000)
      DIMENSION SLHSIM(50000)
      DIMENSION AM(1000)
      READ (4, 51) MPRIME
 51   FORMAT (6I8)
      READ (5, 100, END = 6000) NBEGIN, NEND
100   FORMAT (2I8)
      N = NBEGIN
 10   IF (N .GT. NEND) GO TO 5000
      FLOATN = N
      TEE = 6.2831852/(DLOG(FLOATN))
      MBOUND = DSQRT(FLOATN)
      DO 50 I = 1, MBOUND
      AI = I
      AM(I) = TEE*DLOG(AI)
 50   CONTINUE
      DO 120 I = 1, N
      MPRIM = MPRIME(I)
      IF (MPRIM .GT. N) GO TO 121
      IF (MPRIM .GT. MBOUND) GO TO 120
      NLASTP = 1
120   CONTINUE
121   MSTART = I - 1
      TOTLHS = 0.0
      NSTART = 1
      TLHSRE = 0.0
      TLHSIM = 0.0
      TOTLRE = 0.0
      TOTLIM = 0.0
      DO 250 K = 1, MSTART
      I = (MSTART + 1) - K
```

```
      MPRIM = MPRIME(I)
      IF (.NOT. (MPRIM .GT. MBOUND)) GO TO 251
      NSTOP = N/MPRIM
      DO 225  J = NSTART, NSTOP
      IF (NSTART .GT. NSTOP) GO TO 225
      A = AM(J)
      AJ = J
      TLHSRE = TLHSRE + ((DCOS(A))/AJ)
      TLHSIM = TLHSIM - ((DSIN(A))/AJ)
225   CONTINUE
      FMPRIM = MPRIM
      TOTHOL = ((1.0/FMPRIM))
      TOTHOL = TOTHOL*DSQRT(TLHSRE**2 + TLHSIM**2)
      TOTLHS = TOTLHS + TOTHOL
      A = TEE*(DLOG(FMPRIM))
      PRE = (DCOS(A))/FMPRIM
      PIM = -(DSIN(A))/FMPRIM
      TOTLRE = TOTLRE + ((PRE*TLHSRE) - (PIM*TLHSIM))
      TOTLIM = TOTLIM + ((PRE*TLHSIM) + (PIM*TLHSRE))
235   SLHSRE(I) = TLHSRE
      SLHSIM(I) = TLHSIM
      NSTART = NSTOP + 1
250   CONTINUE
251   TRHSRE = TLHSRE
      TRHSIM = TLHSIM
      DO 200  I = NSTART, N
      AI = I
      A = TEE*DLOG(AI)
      TRHSRE = TRHSRE + ((DCOS(A))/AI)
      TRHSIM = TRHSIM - ((DSIN(A))/AI)
200   CONTINUE
      TRHSRE = TRHSRE - TOTLRE
      TRHSIM = TRHSIM - TOTLIM
      TOTRHS = DSQRT((TRHSRE*TRHSRE) + (TRHSIM*TRHSIM))
      DIFF = TOTLHS - TOTRHS
      WRITE (6,252) N, TOTLHS, TOTRHS, DIFF
252   FORMAT (I8, 3F16.8)
      WRITE (6,255) TRHSRE, TRHSIM
255   FORMAT (2F16.8)
      IC = 0
      NPLUS1 = N + 1
      NEXTP = NLASTP + 1
      DO 500  L = NPLUS1, NEND
      IC = IC + 1
      FLOATL = L
      DSQRTL = DSQRT(FLOATL)
```

```
      IF (MPRIME(NEXTP) .GT. DSQRTL) GO TO 305
      FMPRIM = MPRIME(NEXTP)
      HOLDTO = (DSQRT(SLHSRE(NEXTP) **2 + SLHSIM(NEXTP)**2))/
     1FMPRIM
      TOTLHS = TOTLHS - HOLDTO
      NLASTP = NEXTP
      NEXTP = NLASTP + 1
      MPRIM = MPRIME(NLASTP)
      DO 302  M = 1, MPRIM
      K = M *MPRIM
      FK = K
      A = TEE *DLOG(FK)
      IF (M .EQ. 1) AM(K) = A
      TRHSRE = TRHSRE + ((DCOS(A))/FK)
      TRHSIM = TRHSIM - ((DSIN(A))/FK)
302   CONTINUE
      GO TO 430
305   HOLDL = FLOATL
      DO 407  I = 1, NLASTP
      FMPRIM = MPRIME(I)
310   SAVE = HOLDL/FMPRIM
      ISAVE = SAVE
      IF (.NOT.(ISAVE .EQ. SAVE)) GO TO 405
      HOLDL = SAVE
      GO TO 310
405   IF (HOLDL .EQ. 1.0) GO TO 410
407   CONTINUE
410   IF (.NOT.(HOLDL .EQ. 1.0)) GO TO 413
      A = TEE *DLOG(FLOATL)
      TRHSRE = TRHSRE + ((DCOS(A))/FLOATL)
      TRHSIM = TRHSIM - ((DSIN(A))/FLOATL)
      GO TO 430
413   ITEST = MSTART + 1
      IF (MPRIME(ITEST) .GT. L) GO TO 900
      HOLD = MPRIME(ITEST)
      TOTLHS = TOTLHS + (1.0/HOLD)
      MSTART = ITEST
      SLHSRE(ITEST) = 1.0
      SLHSIM(ITEST) = 0.0
      GO TO 430
900   IHOLDL = HOLDL
      IBEGIN = NEXTP
      IEND = MSTART
      DO 1100  IBISEC = 1, 20
      ISEAR = (IBEGIN + IEND)/2
      MPRIM = MPRIME(ISEAR)
```

```
      IF (MPRIM .LT. IHOLDL) GO TO 1050
      IF (MPRIM .EQ. IHOLDL) TO TO 1040
      IEND = ISEAR - 1
      GO TO 1100
1040  J = ISEAR
      GO TO 418
1050  IBEGIN = ISEAR + 1
1100  CONTINUE
 415  DO 417 J = IBEGIN, IEND
      FMPRIM = MPRIME(J)
      IF (FMPRIM .EQ. HOLDL) GO TO 418
 417  CONTINUE
      WRITE (6,255) FMPRIM, HOLDL
      STOP
 418  NSTOP = L/IHOLDL
      A = AM(NSTOP)
      AJ = NSTOP
      HOLDTO = DSQRT(SLHSRE(J) **2 + SLHSIM(J) **2)/HOLDL
      SLHSRE(J) = SLHSRE(J) + ((DCOS(A))/AJ)
      SLHSIM(J) = SLHSIM(J) - ((DSIN(A))/AJ)
      TOTHOL = DSQRT(SLHSRE(J) **2 + SLHSIM(J) **2)/HOLDL
      TOTLHS = (TOTLHS + TOTHOL) - HOLDTO
 430  TOTRHS = DSQRT(TRHSRE **2 + TRHSIM **2)
      DIFF = TOTLHS - TOTRHS
      IF (TOTLHS .LT. 1.0) GO TO 999
      IF (IC .LT. 1000) GO TO 450
      IC = 0
 999  WRITE (6,252) L, TOTLHS, TOTRHS, DIFF
      WRITE (6, 255) TRHSRE, TRHSIM
 450  IF (DIFF .GT. .1) GO TO 500
      N = L
      GO TO 10
 500  CONTINUE
5000  CONTINUE
6000  CALL EXIT
      END
```

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