A common combinatorial principle underlies Riemann's formula, the Chebyshev phenomenon, and other subtle effects in comparative prime number theory. I

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Dedicated to Alfred T. Brauer on the occasion of his 85-th birthday

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1. Notation and history

Throughout $\pi_{b,c}(t)$ will denote the number of positive primes $\leqq t$ which are $\equiv c \pmod{b}$, $1 \leqq c \leqq b-1$, $(c, b)=1$, t will always denote a positive integer and x a positive real number. Moreover, $\operatorname{li} x$ will be the usual integral logarithm of x,

$$(1.0) \qquad \operatorname{li} x = \lim_{\eta \to 0^+} \left(\int_0^{1-\eta} + \int_{1+\eta}^{x} \right) \frac{du}{\log u}.$$

Finally our $\Omega_\pm$ notation is that of Ingham [9].

The approximation to which Riemann [13] attached the central role in his 1859 memoir is

$$(1.1) \qquad \pi(x) \cong \operatorname{li} x + \sum_{k=2}^{\infty} \frac{u(k)}{k} \operatorname{li} x^{1/k} = \operatorname{li} x - \frac{\operatorname{li} x^{1/2}}{2} - \frac{\operatorname{li} x^{1/3}}{3} - \frac{\operatorname{li} x^{1/5}}{5} + \cdots;$$

the coefficients $u(k)/k$ for $k > 5$ are defined on p. 34 of [2].

As we now know from J. E. Littlewood [12], the second term in Riemann's formula is overcome infinitely often since

$$(1.2) \qquad \pi(x) - \operatorname{li} x = \Omega_\pm \left(\frac{x^{1/2}}{\log x} \log \log \log x \right).$$

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Nonetheless, Riemann not only placed (1. 1) at the center of his memoir but specifically directed the reader's attention to the need to diminish $\operatorname{li} x$ by 1/2 the density of prime squares, 1/3 the density of prime cubes, and so forth. Certainly, in light of Riemann's interest in these terms it seems of some interest to search for a context in which they are meaningful.

Comparative prime number theory, often contrasted with the "uniformity trend" launched by Riemann's memoir, see [10], p. 24, had its roots in an 1853 letter of P. L. Chebyshev [1]. This letter was popularly interpreted as asserting that there are "many more primes" of form $4n+3$ than of form $4n+1$ [11], p. 23.

If Chebyshev had depicted $\pi_{4,3}(t)-\pi_{4,1}(t)$ as approximated by 1/2 the density of prime squares, rather than by $t^{1/2}/\log t$, the common combinatorial principle underlying all quadratic effects would probably have been uncovered much earlier (the effects were certainly known to be analogous).

2. A combinatorial explanation for the fact that $\pi_{4,3}(t)-\pi_{4,1}(t)$ is approximated by $\pi(t^{1/2})/2$ for small t

In [7] the author has derived a combinatorial model for arbitrary modulus in the case of quadratic effects.

For completeness, a *sketch* of the argument for the modulus 4 runs as follows. Let $\psi_{b,c}(x,k)$ denote the number of integers >1 and $\leqq x$ which are $\equiv c \pmod b$ and relatively prime to the first k primes. Let $m=\pi(x^{1/3})$, $n=\pi(x^{1/2})$, $s=n-m$, and let c' be the unique positive integer less than b such that $c'p_{m+\sigma}\equiv c \pmod b$ for each σ between 1 and $s=n-m$.

Generalizing the well-known formula of E. Meissel, the author [6] (see, also, [5]) proved that for each $x>8$,

$$\pi_{b,c}(x)=\psi_{b,c}(x,m)-\sum_{\substack{1\leq\sigma\leq s\\(p_{m+\sigma},b)=1}}\pi_{b,c'}\left(\frac{x}{p_{m+\sigma}}\right)+\sum_{\substack{1\leq\sigma\leq s\\(p_{m+\sigma},b)=1}}\pi_{b,c'}(p_{m+\sigma})+\pi_{b,c}(p_m)-\sum_{\substack{1\leq\sigma\leq s\\p_{m+\sigma}\equiv c' \pmod b}}1. \tag{2. 1}$$

Consequently, for values of x with $|\psi_{4,3}(x,m)-\psi_{4,1}(x,m)|$ small relative to $|\pi_{4,3}(x)-\pi_{4,1}(x)|$, an "excess" of $\cong\pi(x^{1/2})/2$ among integers $\equiv 1 \pmod 4$ which are the product of exactly two primes p, q with $x^{1/3}\leqq p\leqq q\leqq x^{1/2}$ must result in a "deficiency" of $\cong\pi(x^{1/2})/2$ among primes $\equiv 1 \pmod 4$.

A simple combinatorial explanation for such an "excess" exists. Namely, let S_2 denote the set of positive integers $\leqq x$ which are the product of exactly two prime factors p, q with $x^{1/3} \leqq p \leqq q \leqq x^{1/2}$. Let V_1 and V_2 denote respectively the number of elements of S_2 which are $\equiv 1$ or $3 \pmod 4$, and let T_1 and T_2 denote the number of primes $\equiv 1$ or $3 \pmod 4$, $x^{1/3} \leqq p \leqq x^{1/2}$.

Then $V_2 = T_1 T_2$, but

$$V_1 = \frac{T_1(T_1+1)}{2} + \frac{T_2(T_2+1)}{2} = \tfrac{1}{2}(T_1^2 + T_2^2) + \tfrac{1}{2}(T_1 + T_2), \tag{2. 2}$$

the "extra" $\frac{1}{2}(T_1 + T_2)$ coming from the simple combinatorial fact that the products of distinct primes are counted twice (as pq and as qp) but the prime squares are counted only once.

Now,

$$\tfrac{1}{2}(T_1 + T_2) = \tfrac{1}{2}(\pi(x^{1/2}) - \pi(x^{1/3})), \tag{2. 3}$$

so that if the primes $p \equiv 1 \pmod 4$ and $\equiv 3 \pmod 4$ are "about" equally common for p with $x^{1/3} \leqq p \leqq x^{1/2}$, then S_2 has an excess of $\cong \pi(x^{1/2})/2$ integers $\equiv 1 \pmod 4$.

Remarks. For abitrary modulus the "deficiency" is entirely amassed in the progressions $bn + c$ with c a quadratic residue of b. The "deficiency" of $\pi(x)$ relative to $\mathrm{li}\, x$ can (excluding the prime 2) be thought of as simply the special case $b = 2$, $c = 1$. Although no $\pi_{b,c}(x)$ with c a quadratic non-residue of b exists with which to compare $\pi(x)$, the function naturally associated with $\log \zeta(s)$, namely

$$\Pi(x) = \sum_{p^m \leqq x} \frac{1}{m} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) x^s \frac{ds}{s} \qquad (a > 1), \tag{2. 4}$$

plays this role; the coefficients in (1. 1) result from Moebius inversion [2], p. 34.

3. Degeneration of the quadratic effect for large x

On the basis of limited numerical and theoretical evidence, Shanks [13] conjectured that

$$\frac{1}{x} \sum_{t=1}^{x} \frac{\pi_{4,3}(t) - \pi_{4,1}(t)}{t^{1/2}/\log t} \to 1 \quad \text{as } x \to \infty. \tag{3. 1}$$

Recently, Ellison [3] has shown that this is false.

Let

$$A_4^{(1)}(x, 2) = \frac{1}{x} \sum_{t=1}^{x} (\pi_{4,3}(t) - \pi_{4,1}(t)), \quad h^{(1)}(x, 2) = \frac{1}{x} \sum_{t=1}^{x} \frac{\pi(t^{1/2})}{2}. \tag{3. 2}$$

For $k \geqq 1$, let

$$A_4^{(k+1)}(x, 2) = \frac{1}{x} \sum_{t=1}^{x} A_4^{(k)}(t, 2), \quad h^{(k+1)}(x, 2) = \frac{1}{x} \sum_{t=1}^{x} h_k(t, 2). \tag{3. 3}$$

In a letter to the author, see [8], A. Schinzel proved that it is not true that

$$\frac{A_4^{(1)}(x,2)}{h^{(1)}(x,2)} \to 1 \quad \text{as } x \to \infty, \tag{3.4}$$

and he provided a heuristic argument that it is not true that

$$\frac{A_4^{(k)}(x,2)}{h^{(k)}(x,2)} \to 1 \quad \text{as } x \to \infty \tag{3.5}$$

for any k (however large).

On the other hand, the author and Schinzel have shown that if the generalized Riemann hypothesis is true for $L(S,\chi)$, χ the nonprincipal character mod 4, then

$$\lim_{k\to\infty} \overline{\lim_{x\to\infty}} \frac{A_4^{(k)}(x,2)}{h^{(k)}(x,2)} = 1 = \lim_{k\to\infty} \underline{\lim_{x\to\infty}} \frac{A_4^{(k)}(x,2)}{h^{(k)}(x,2)}. \tag{3.6}$$

Assuming then, the truth of the generalized Riemann hypothesis, it appears that $A_4^{(1)}(x,2)$ should have values distributed fairly evenly about $h^{(1)}(x,2)$; that this is, in fact, the case for $x<2\cdot 10^{10}$, see [7], Fig. 1. For every large (fixed) x, however, it is clearly necessary that k be taken >1 in order that $h^{(k)}(x,2)$ reasonably approximate $A_4^{(k)}(x,2)$. In other words, to preserve the "integrity" of the quadratic effect it is necessary to fix x and let k grow large, an obvious reversal of the ordinary procedure.

4. The third term in Riemann's formula

Let

$$F_1(x,3)=\frac{1}{x}\sum_{t=1}^{x}\left(\mathrm{li}\, t-\pi(t)-\frac{\mathrm{li}\, t^{1/2}}{2}\right),\quad G_1(x,3)=\frac{1}{x}\sum_{t=1}^{x}\frac{\mathrm{li}\, t^{1/3}}{3}, \tag{4.1}$$

and for $k\geqq 1$ let

$$F_{k+1}(x,3)=\frac{1}{x}\sum_{t=1}^{x}F_k(t,3),\quad G_{k+1}(x,3)=\frac{1}{x}\sum_{t=1}^{x}G_k(t,3). \tag{4.2}$$

Now $A_4^{(1)}(x,2)$ remains positive for large x if (but only if) the generalized Riemann hypothesis is true, whereas for each fixed k, however large, $F_k(x,3)$ takes on positive and negative values infinitely often. Ingham [9], p. 106 concludes from this fact that "no importance can be attached to the third term in Riemann's formula even by repeated averaging". This is clearly the case if k is fixed and x is allowed to increase without bound.

However, even before my talk in San Antonio [4], I sensed that the third term in Riemann's formula (or more directly the third term in $\Pi(x)$) was connected with a cubic residue effect in arithmetic progressions having cubic non-residues. For the most part this subtle effect has gone totally unnoticed in the literature, although it is alluded to by Shanks in [15].

Before developing the principle governing the cubic effect for general progressions, I briefly investigate, for fixed x_0 in computer range, whether one can find a small $k=k(x_0)$ with the property that

$$0<G_k(x,3)-F_k(x,3)<2G_k(x,3)\,\forall x<x_0. \tag{4.3}$$

This seems worth checking since failure to find any k such that (4. 3) is satisfied, even when x_0 is small, would imply that the cubic effect is entirely destroyed by chance fluctuations in the primes. Fortunately, this is not the case and, in fact, one finds that x may well exceed 10^{11} before $F_1(x, 3) < 0$ ($G_1(x, 3) - F_1(x, 3) < 0$ occurs sooner; checks were performed at steps of 4,000,000 and unless values of $F_1(x, 3)$ dive swiftly between checkpoints, we have $F_1(x, 3) > 0$ until x is about $1.02 \cdot 10^{11}$). Moreover, $F_2(x, 3)$ lies comfortably between $\frac{1}{2} G_2(x, 3)$ and $\frac{3}{2} G_2(x, 3)$ for all values in Tables 2 and 3 so that the first value of x_0 for which it is necessary to take $k = 3$, in order that (4. 3) be satisfied, must be quite large (probably, not computable).

Remarks. To full appreciate the subtlety of the cubic effect one should consider that at $t = 2.5 \cdot 10^9$, $\pi(t)$ exceeds 120 million, and even the much smaller (mean) quadratic term $\frac{1}{x} \sum_{t=1}^{x} (\operatorname{li} t - \pi(t))$ exceeds 1870, whereas $G_1(x)$ is about 60.

$$(4.\ 4) \qquad F_1(x, n) = \frac{1}{x} \left(\sum_{t=1}^{x} \operatorname{li} t - \pi(t) - \sum_{j=2}^{n-1} \frac{u(j)}{j} \operatorname{li} t^{1/j} \right).$$

Unfortunately, I have gathered no data for $n > 3$. It is, however, highly probable that higher order effects exist and are extendible to progressions having higher order non-residues (see section 7).

Values for $F_2(x, 3)$ and $G_2(x, 3)$ were obtained by averaging $F_1(x, 3)$ and $G_1(x, 3)$ at intervals of 800,000 in Table 2, and at intervals of 4,000,000 in Table 3. This introduces an error (to the high side) which is not large after about 30 averagings (this was tested by averaging at much smaller intervals with the result that values differed by less than 3). Obviously averaging at each prime is desirable but computer time necessary to complete this task over the ranges considered was not available. Since the relative sizes of $F_k(x, 3)$ and $G_k(x, 3)$ are of much greater interest to us then precise values for these quantities, and since the Tables are important for this paper only in that they indicate that the cubic effect is not entirely annihilated by chance fluctuations in the primes if x_0 is small, we will not apologize further for our crude form of averaging but will request that the reader not reproduce the values in Tables 1, 2, and 3 elsewhere as though they were exact.

In particular, it would be a misconception to interpret the apparent elevation of values of $F_2(x, 3)$ relative to $G_2(x, 3)$ as evidence of later terms in Riemann's formula. It is more probable that error is introduced in the computation of $F_1(x, 3)$ and $G_1(x, 3)$ through the fact that $h^{(1)}(x, 2)$ is approximated using $[\log x]$ terms of the asymptotic expansion for each fixed k (see [8],

$$(4.\ 5) \qquad h^{(1)}(x, 2) \cong \frac{1}{x} \sum_{t=1}^{x} \frac{\operatorname{li} t^{1/2}}{2} = \sum_{j=1}^{k} \frac{2^{j-1}(j-1)!\,(3^j - 1) x^{1/2}}{3^j \log^j x} + O\left(\frac{x^{1/2}}{\log^{k+1} x} \right)$$

whereas $G_1(x, 3)$ is approximated using only seven terms of the asymptotic expansion for each fixed k,

$$(4.\ 6) \qquad \frac{\operatorname{li} x^{1/3}}{3} = \sum_{j=1}^{k} \frac{3^{j-1}(j-1)!\, x^{1/3}}{\log^j x} + O\left(\frac{x^{1/3}}{\log^{k+1} x} \right).$$

A program sufficiently refined to accurately compare the relative sizes of $F_k(x, 5)$ and $G_k(x, 5)$ over large ranges would constitute a major numerical undertaking.

Table 1

Values of $F_1(x)$ for $8 \cdot 10^7 \leqq x \leqq 320 \cdot 10^7$ (x in ten millions)

x	$F_1(x)$	x	$F_1(x)$	x	$F_1(x)$	x	$F_1(x)$
8	26	88	43	168	52	248	85
16	38	96	70	176	50	256	91
24	51	104	64	184	39	264	60
32	35	112	33	192	49	272	47
40	47	120	31	200	73	280	51
48	34	128	57	208	70	288	45
56	57	136	71	216	74	296	37
64	53	144	78	224	85	304	43
72	34	152	78	232	76	312	52
80	57	160	58	240	73	320	61

Remark. The values of $F_1(x, 3)$ may be contrasted with values of

$$A_4^{(1)}(x, 2) - h^{(1)}(x, 2)$$

where a cubic effect does not take place. Values of the latter are distributed more or less symmetrically about zero — see [7], Fig. 1.

Table 2

Approximate Values for $F_2(x)$ and $G_2(x)$ for $30 \cdot 10^6 < x \leqq 262 \cdot 10^6$ in Steps of 4,000,000*

x	$F_2(x)$	$G_2(x)$	x	$F_2(x)$	$G_2(x)$	x	$F_2(x)$	$G_2(x)$
30,444,449	19.3	15.8	110,444,449	25.8	21.0	186,444,449	29.1	23.9
34,444,449	19.3	16.2	114,444,449	25.9	21.2	190,444,449	29.0	24.0
38,444,449	19.9	16.6	118,444,449	25.8	21.4	194,444,449	28.9	24.1
42,444,449	20.6	16.9	122,444,449	25.8	21.5	198,444,449	28.8	24.2
46,444,449	21.1	17.2	126,444,449	26.0	21.7	202,444,449	28.8	24.4
50,444,449	21.3	17.6	130,444,449	26.2	21.9	206,444,449	28.8	24.5
54,444,449	21.4	17.9	134,444,449	26.3	22.0	210,444,449	28.8	24.6
58,444,449	21.6	18.1	138,444,449	26.4	22.2	214,444,449	28.9	24.7
62,444,449	21.9	18.4	142,444,449	26.6	22.4	218,444,449	29.0	24.8
66,444,449	22.5	18.7	146,444,449	26.8	22.5	222,444,449	29.1	24.9
70,444,449	23.1	18.9	150,444,449	26.9	22.7	226,444,449	29.3	25.1
74,444,449	23.4	19.2	154,444,449	27.1	22.8	230,444,449	29.5	25.2
78,444,449	23.4	19.4	158,444,449	27.3	22.9	234,444,449	29.8	25.3
82,444,449	23.5	19.6	162,444,449	27.6	23.1	238,444,449	30.1	25.4
86,444,449	23.5	19.9	166,444,449	28.0	23.2	242,444,449	30.5	25.5
90,444,449	23.7	20.1	170,444,449	28.3	23.4	246,444,449	30.7	25.6
94,444,449	24.0	20.3	174,444,449	28.7	23.5	250,444,449	31.0	25.7
98,444,449	24.5	20.5	178,444,449	28.9	23.6	254,444,449	31.2	25.8
102,444,449	25.0	20.7	182,444,449	29.0	23.7	258,444,449	31.4	25.9
106,444,449	25.5	20.8				262,444,449	31.5	26.0

* averaging at intervals of 800,000

Table 3

Approximate Values for $F_2(x)$ and $G_2(x)$ for
$20 \cdot 10^7 < x \leqq 320 \cdot 10^7$ in Steps of 40,000,000*

x	$F_2(x)$	$G_2(x)$	x	$F_2(x)$	$G_2(x)$
200,044,449	29.0	24.9	1,000,044,449	43.6	36.9
240,044,449	30.7	26.0	1,040,044,449	44.6	37.3
280,044,449	32.1	26.9	1,080,044,449	45.2	37.6
320,044,449	32.4	27.8	1,120,044,449	45.1	38.0
360,044,449	33.1	28.6	1,160,044,449	44.5	38.3
400,044,449	34.8	29.3	1,200,044,449	44.0	38.7
440,044,449	35.8	30.0	1,240,044,449	43.8	39.0
480,044,449	36.1	30.7	1,280,044,449	44.1	39.3
520,044,449	35.5	31.3	1,320,044,449	44.6	39.6
560,044,449	36.5	31.9	1,360,044,449	45.4	39.9
600,044,449	38.3	32.4	1,400,044,449	46.0	40.2
640,044,449	39.4	32.9	1,440,044,449	46.6	40.5
680,044,449	39.9	33.4	1,480,044,449	47.4	40.7
720,044,449	39.9	33.9	1,520,044,449	48.2	41.0
760,044,449	39.5	34.4	1,560,044,449	48.7	41.2
800,044,449	40.2	34.9	1,600,044,449	48.9	41.4
840,044,449	40.8	35.3	1,640,044,449	49.1	41.6
880,044,449	41.0	35.7	1,680,044,449	49.1	41.9
920,044,449	41.5	36.1	1,720,044,449	49.2	42.1
960,044,449	42.3	36.5	1,760,044,449	49.2	42.3

averaging at intervals of 4,000,000

x	$F_2(x)$	$G_2(x)$	x	$F_2(x)$	$G_2(x)$
1,800,044,449	49.2	42.5	2,520,044,449	54.3	45.9
1,840,044,449	49.0	42.7	2,560,044,449	54.8	46.1
1,880,044,449	48.9	42.9	2,600,044,449	55.2	46.3
1,920,044,449	48.8	43.1	2,640,044,449	55.3	46.4
1,960,044,449	49.0	43.3	2,680,044,449	55.3	46.6
2,000,044,449	49.3	43.5	2,720,044,449	55.3	46.8
2,040,044,449	49.6	43.7	2,760,044,449	55.2	46.9
2,080,044,449	50.0	43.9	2,800,044,449	55.1	47.1
2,120,044,449	50.3	44.1	2,840,044,449	55.1	47.3
2,160,044,449	50.6	44.3	2,880,044,449	55.0	47.4
2,200,044,449	51.0	44.5	2,920,044,449	54.8	47.6
2,240,044,449	51.5	44.7	2,960,044,449	54.7	47.7
2,280,044,449	52.0	44.8	3,000,044,449	54.4	47.9
2,320,044,449	52.5	45.0	3,040,044,449	54.3	48.1
2,360,044,449	52.8	45.2	3,080,044,449	54.2	48.2
2,400,044,449	53.1	45.4	3,120,044,449	54.1	48.4
2,440,044,449	53.4	45.6	3,160,044,449	54.1	48.5
2,480,044,449	53.8	45.7	3,200,044,449	54.2	48.7

5. The cubic residue effect for the modulus 7

The cubic residue effect described in this and the following section renders transparent the fact that the Chebyshev phenomenon is the simplest case of an n-th power principle which can be directly connected with the terms in Riemann's formula and described in terms of precise combinatorial formulae.

In order to facilitate the reader's appreciation of the basic simplicity of the combinatorial theory in the cubic case, it is presented in this section for $b=7$ and generalized to arbitrary prime modulus in section 6. We trust the sophisticated reader will forgive the redundancy.

Let S_3 denote the set of positive integers $\leqq x$ which are the product of exactly three prime factors, all three coming from the interval $x^{1/4} \leqq p \leqq x^{1/3}$. For each α, $\alpha=1,\ldots,6$, let T_α denote the number of primes in the interval $x^{1/4} \leqq p \leqq x^{1/3}$ which are $\equiv \alpha \pmod 7$. For each β, $\beta=1,2,\ldots,6$, let V_β denote the number of elements of S_3 which are $\equiv \beta \pmod 7$.

Then we have

$$V_1=\frac{T_1(T_1+1)(T_1+2)}{6}+\frac{T_2(T_2+1)(T_2+2)}{6}+\frac{T_4(T_4+1)(T_4+2)}{6}$$
$$+\frac{T_1(T_6)(T_6+1)}{2}+\frac{T_2(T_5)(T_5+1)}{2}+\frac{T_4 T_3(T_3+1)}{2}$$
$$+T_1T_2T_4+T_1T_3T_5+T_2T_3T_6+T_4T_5T_6 \tag{5. 1}$$

and

$$V_6=\frac{T_3(T_3+1)(T_3+2)}{6}+\frac{T_5(T_5+1)(T_5+2)}{6}+\frac{T_6(T_6+1)(T_6+2)}{6}$$
$$+\frac{T_3(T_4)(T_4+1)}{2}+\frac{T_5T_2(T_2+1)}{2}+\frac{T_6T_1(T_1+1)}{2}$$
$$+T_1T_2T_3+T_1T_4T_5+T_2T_4T_6+T_3T_5T_6. \tag{5. 2}$$

On the other hand, we have

$$V_2=\frac{T_1(T_1+1)T_2}{2}+\frac{T_2(T_2+1)T_4}{2}+\frac{T_3(T_3+1)T_1}{2}+\frac{T_4(T_4+1)T_1}{2}$$
$$+\frac{T_5(T_5+1)T_4}{2}+\frac{T_6(T_6+1)T_2}{2}+T_1T_5T_6+T_2T_3T_5+T_3T_4T_6, \tag{5. 3}$$

$$V_3=\frac{T_1(T_1+1)T_3}{2}+\frac{T_2(T_2+1)T_6}{2}+\frac{T_3(T_3+1)T_5}{2}+\frac{T_4(T_4+1)T_5}{2}$$
$$+\frac{T_5(T_5+1)T_6}{2}+\frac{T_6(T_6+1)T_3}{2}+T_1T_2T_5+T_2T_3T_4+T_1T_4T_6, \tag{5. 4}$$

$$(5.5)\qquad V_4=\frac{T_1(T_1+1)T_4}{2}+\frac{T_2(T_2+1)(T_1+2)}{2}+\frac{T_3(T_3+1)T_2}{2}+\frac{T_4(T_4+1)T_2}{2}$$
$$+\frac{T_5(T_5+1)T_1}{2}+\frac{T_6(T_6+1)T_4}{2}+T_1T_3T_6+T_2T_5T_6+T_3T_4T_5,$$

$$(5.6)\qquad V_5=\frac{T_1(T_1+1)T_5}{2}+\frac{T_2(T_2+1)T_3}{2}+\frac{T_3(T_3+1)T_6}{2}+\frac{T_4(T_4+1)T_6}{2}$$
$$+\frac{T_5(T_5+1)T_3}{2}+\frac{T_6(T_6+1)T_5}{2}+T_1T_2T_6+T_2T_4T_5+T_1T_3T_4.$$

As in the quadratic case, the above are exact formulae arising from the simple combinatorial fact that in the counting of elements of S_3 the products ppq and pqq are counted three times (as ppq, pqp, and qpp in the former case and as pqq, qpq, and qqp in the latter) and the products pqr with $p \neq q \neq r$ are counted six times (as pqr, prq, qpr, qrp, rpq, and rqp) while the cubes are only counted once. This is easily visualized as a counting problem in the geometry of numbers by forming a one-to-one correspondence between the elements of S_3 and ordered triples in the first octant (in fact, the principle first occurred to the author in exactly this way).

Now the primes in the interval $x^{1/4} \leqq p \leqq x^{1/3}$ are approximately equidistributed so that if we let $T=(T_1+T_2+T_3+T_4+T_5+T_6)/6$, then

$$(5.7)\qquad V_1 \cong \frac{3T^3+9T^2+2(T_1+T_2+T_4)}{6}+\frac{3T^3+3T^2}{2}+4T^3$$
$$=6T^3+3T^2+(T_1+T_2+T_4)/3$$

and

$$(5.8)\qquad V_6 \cong \frac{3T^3+9T^2+2(T_3+T_5+T_6)}{6}+\frac{3T^3+3T^2}{2}+4T^3$$
$$=6T^3+3T^2+(T_3+T_5+T_6)/3.$$

However, V_α for $\alpha=2, 3, 4, 5$, is approximated by

$$(5.9)\qquad V_\alpha \cong \frac{6T^3+6T^2}{2}+3T^3=6T^3+3T^2.$$

Thus,

$$(5.10)\qquad V_1+V_6-\frac{(V_2+V_3+V_4+V_5)}{2} \cong (T_1+T_2+T_3+T_4+T_5+T_6)/3.$$

However,

$$(5.11)\qquad \frac{T_1+T_2+T_3+T_4+T_5+T_6}{3}=\frac{\pi(x^{1/3})-\pi(x^{1/4})}{3},$$

so that if the primes p with $x^{1/4} \leqq p \leqq x^{1/3}$ are about equally common for the progressions (mod 7), then S_3 has an excess of $\cong \pi(x^{1/3})/3$ integers $\equiv 1$ or 6 (mod 7).

Let $\pi_{7,c}(x, t)$ denote the number of integers >1 relatively prime to the first $t=\pi(x^{1/4})$ primes which are $\equiv c \pmod 7$, $c=1,\ldots,6$. Then, if p_t denotes the t-th prime, we have

$$(5.\ 12)\qquad \pi_{7,c}(x)=\psi_{7,c}(x,t)-\sum_{\substack{qr\leq x\\ q,r>p_t\\ qr\equiv c\,(\mathrm{mod}\,7)}}1-\sum_{\substack{qr\leq x\\ q,r>p_t\\ qr\equiv c\,(\mathrm{mod}\,7)}}1+\pi_{7,c}(p_t).$$

Now 3, 5, and 6 are quadratic non-residues and 1, 2, and 4 are quadratic residues resulting in a null quadratic effect (see remarks at the end of section 6) in the computation of

$$(5.\ 13)\qquad \frac{\pi_{7,2}(x)+\pi_{7,3}(x)+\pi_{7,4}(x)+\pi_{7,5}(x)}{2}-(\pi_{7,1}(x)+\pi_{7,6}(x)).$$

However, the "excess" in S_3 of approximately $\pi(x^{1/3})/3$ integers $\equiv 1$ or $6 \pmod 7$ should, for small x, result in a "deficiency" of approximately $\pi(x^{1/3})/3$ in the computation of $\pi_{7,1}(x)+\pi_{7,6}(x)$.

For $n\geqq 2$ let

$$(5.\ 14)\qquad A_7^{(1)}(x,3)=\frac{1}{x}\left(\sum_{t=1}^{x}\frac{\pi_{7,2}(t)+\pi_{7,3}(t)+\pi_{7,4}(t)+\pi_{7,5}(t)}{2}-(\pi_{7,1}(t)+\pi_{7,6}(t))\right),$$

$$h_1(x,n)=\frac{1}{x}\sum_{t=1}^{x}\pi(t^{1/n})/n,$$

and for $k\geqq 1$ and $n\geqq 2$ let

$$A_7^{(k+1)}(x,3)=\frac{1}{x}\sum_{t=1}^{x}A_7^{(k)}(t,3),\ h_{k+1}(x,n)=\frac{1}{x}\sum_{t=1}^{x}h_k(t,n).$$

As a natural generalization of (4. 3), one may investigate for x_0 (in computer range) whether there exists a small $k=k(x_0)$ with

$$(5.\ 15)\qquad |A_7^{(k)}(x,3)-h_k(x,3)|<h_k(x,3)\quad \text{for every } x<x_0.$$

Remarks. The main purpose of this paper is to uncover the common combinatorial principle unifying the quadratic and higher order effects in prime number theory, not to compile numerical information regarding approximations such as (5. 15).

With the help of Carolyn Brauer Hudson, I have discovered that for small x, $A_7^{(1)}(x,3)$ is predominantly positive and oscillates more or less symmetrically about $h_1(x,3)$. Of course, values of $A_7^{(2)}(x,3)$ will vary much less and it would be of interest to have values of the latter to compare with values of $h_2(x,3)$ (and, also, with values of $F_2(x,3)$). Our preliminary computations suggest that values of $A_7^{(2)}(x,3)$ and $h_2(x,3)$ agree well for $x<10^9$; however, our computations consist of gross approximations not meriting publication at this time.

Noting that $V_2-V_1\cong(T_1+T_2+T_4)/3\cong(x^{1/3})/6$, the reader may well wonder why we do not make the additional assumption of approximate equidistribution, $(T_1+T_2+T_4)/3\cong T$ and describe the cubic residue effect as follows.

Let b be any modulus having cubic non-residues, and let

$$(5.16)\qquad A_b^{(1)}(x, c_3, c_3') = \frac{1}{x}\left(\sum_{t=1}^{x} \pi_{b,c_3}(t) - \pi_{b,c_3'}(t)\right).$$

Moreover, let

$$(5.17)\qquad A_b^{(k+1)}(x, c_3, c_3') = A_b^{(k)}(x, c_3, c_3')$$

for $k \geqq 1$, where c_3 is any cubic non-residue and c_3' any cubic residue of b and c_3 and c_3' have the same quadratic character (mod b).

Analogy with the quadratic case, see (6. 16) of [7], suggests an approximation of the type

$$(5.18)\qquad A_b^{(k)}(x, c_3, c_3') \cong \frac{h^{(k)}(x, 3)}{\gamma_3(b)}$$

where $\gamma_3(b)$ is the number of cubic residues of b. However, the surprising finding that (6. 16) of [7] breaks down so utterly if the number of quadratic residues of b exceeds 4 or 5 (see, also, [11], p. 295) that, for example,

$$(5.19)\qquad \frac{1}{x}\sum_{t=1}^{x} (\pi_{19,15}(t) - \pi_{19,4}(t)) < 0$$

for almost all $x < 8 \cdot 10^9$ (in spite of the fact that 15 is clearly the quadratic non-residue of 19) is more than enough reason to be cautious in the cubic case. Consequently, although Carolyn Hudson and I have found that $A_7^{(3)}(x, 2, 1)$ *is* positive for almost all $x < 10^9$, the fact that it is necessary to average three times when the modulus has only two cubic residues, the relative tininess of $\pi(x^{1/3})/3\gamma(b)$, and most of all (5. 19), all suggest that the approximation (5. 18) may break down even for small x if the number of cubic residues is at all large.

Thus, in contrast to (5. 15), (5. 18) is probably devoid of meaning for all large moduli b, even when x is small, unless it has meaning in a wider context than prime number theory — see remarks on quasi-prime number theory at the end of this paper.

Finally, we remark that the expression "for every $x < x_0$" in (5. 15) may appear overly restrictive since, as a consequence, (5. 15) cannot hold for any k if $A_7^{(1)}(x, 3)$ is negative for the first few integers. Since (on the basis of numerical evidence) this appears not to be the case, however, we find it unnecessary to weaken (5. 15).

6. The cubic residue effect for prime moduli

Recall that if p is a prime modulus $\equiv 1 \pmod 3$, p has $(p-1)/3$ cubic residues and $2(p-1)/3$ cubic non-residues (mod p).

Let S_3 be defined as before and for each α, $\alpha = 1, \ldots, p-1$, let T_α denote the number of primes in the interval $x^{1/4} \leqq p \leqq x^{1/3}$ which are $\equiv \alpha \pmod p$. For each cubic residue R let V_R denote the number of elements of S_3 which are $\equiv R \pmod p$ and, similarly, for each cubic non-residue N let V_N denote the number of elements of S_3 which are $\equiv N \pmod p$.

For each R let R_1, R_2 and R_3 be the integers with $1 \leqq R_1 \leqq R_2 \leqq R_3 \leqq p-1$ for which $R_1^3 \equiv R_2^3 \equiv R_3^3 \equiv R \pmod{p}$. For each R (or N) and each α, $\alpha = 1, \ldots, p-1$, let α_R (or α_N) be the integer with $1 \leqq \alpha_R \leqq p-1$ (or $1 \leqq \alpha_N \leqq p-1$) for which

$$\alpha^2 \alpha_R \equiv R \pmod{p} \quad \text{or} \quad \alpha^2 \alpha_N \equiv N \pmod{p}.$$

Then, for each cubic residue R we have

$$(6.\,1) \qquad V_R = \sum_{S=1}^{3} \frac{T_{R_S}(T_{R_S}+1)(T_{R_S}+2)}{6} + \sum_{\substack{\alpha=1 \\ \alpha \neq \alpha_R}}^{p-1} \frac{T_\alpha(T_\alpha+1)T_{\alpha_R}}{2} + \sum_{\substack{1 \leq \alpha_1 < \alpha_2 < \alpha_3 \leq p-1 \\ \alpha_1\alpha_2\alpha_3 \equiv R \pmod{p}}} T_{\alpha_1} T_{\alpha_2} T_{\alpha_3},$$

whereas for each cubic non-residue N we have

$$(6.\,2) \qquad V_N = \sum_{\alpha=1}^{p=1} \frac{T_\alpha(T_{\alpha+1})T_{\alpha_N}}{2} + \sum_{\substack{1 \leq \alpha_1 < \alpha_2 < \alpha_3 \leq p-1 \\ \alpha_1\alpha_2\alpha_3 \equiv N \pmod{p}}} T_{\alpha_1} T_{\alpha_2} T_{\alpha_3}.$$

Now the number Q_R (or Q_N) of products $T_{\alpha_1} T_{\alpha_2} T_{\alpha_3}$, $1 \leqq \alpha_1 < \alpha_2 < \alpha_3 \leqq p-1$ for which $\alpha_1 \alpha_2 \alpha_3 \equiv R$ (or N) $\pmod{p}$ is

$$(6.\,3) \qquad \binom{p-1}{3} = \frac{(p-1)!}{(p-4)!\,3!} = \frac{(p-1)(p-2)(p-3)}{6}.$$

If p is $\equiv 2 \pmod{3}$ then $p-1 \,\Big|\, \binom{p-1}{3}$ and the number of distinct products Q_R (there are, of course, only cubic residues if p is $\equiv 2 \pmod{3}$) is precisely

$$(6.\,4) \qquad \frac{Q_R}{p-1} = \frac{(p-2)(p-3)}{6}.$$

It is, therefore, transparent that no cubic residue effect of the type described in section 5 exists if p is an odd prime having only cubic residues.

(6. 5) If p is $\equiv 1 \pmod{3}$, then

$$(6.\,6) \qquad p-1 \,\Big|\, \left(\frac{(p-1)(p-2)(p-3)}{6} - \frac{p-1}{3}\right)$$

and it is straightforward to verify that corresponding to each V_N we have

$$(6.\,7) \qquad Q_N = \frac{(p-2)(p-3)}{6} - \frac{1}{3},$$

whereas for each V_R we have

$$(6.\,8) \qquad Q_R = \frac{(p-2)(p-3)}{6} + \frac{2}{3}$$

terms (since in the counting of each V_R there is the "extra" term $T_{R_1} T_{R_2} T_{R_3}$).

Assume, as before, that in the interval $x^{1/4} \leqq p \leqq x^{1/3}$ the primes are approximately equidistributed among the progressions (mod p). Then we have for each R and N,

(6. 9)

$$V_R \cong \frac{3T^3 + 9T^2 + 2(T_{R_1} + T_{R_2} + T_{R_3})}{6} + \frac{(p-4)(T^3+T^2)}{2} + \left(\frac{(p-2)(p-3)}{6} + \frac{2}{3}\right)T^3$$

and

$$V_N \cong \frac{(p-1)T^3 + (p-1)T^2}{2} + \left(\frac{(p-2)(p-3)}{6} - \frac{1}{3}\right)T^3. \tag{6. 10}$$

Thus, with $\left(\frac{c}{p}\right)_3$ denoting the cubic residue symbol, we have

$$\sum_{\substack{\left(\frac{R}{p}\right)_3 = 1 \\ 1 \le R \le p-1}} V_R - \sum_{\substack{\left(\frac{N}{p}\right)_3 \neq 0 \text{ or } 1 \\ 1 \le N \le p-1}} \frac{V_N}{2} \cong \left(\sum_{\alpha=1}^{p-1} T_\alpha\right)/3. \tag{6. 11}$$

On the other hand, we have

$$\left(\sum_{\alpha=1}^{p-1} T_\alpha\right)/3 = \frac{\pi(x^{1/3}) - \pi(x^{1/4})}{3}. \tag{6. 12}$$

Since p is $\equiv 1 \pmod 6$, there are $(p-1)/6$ sixth power residues of p. Consequently, half the cubic residues of p must be quadratic non-residues, and similarly for the cubic non-residues.

Letting $\psi_{p,c}(x, t)$ denote the number of integers >1, relatively prime to the first $t = \pi(x^{1/4})$ primes, which are $\equiv c \pmod p$, $c = 1, 2, \ldots, p-1$, letting p_t denote the t-th prime, and letting q, r and s be primes, we have

$$\pi_{p,c}(x) = \psi_p(x, t) - \sum_{\substack{qr \le x \\ q,r > p_t \\ qr \equiv c \ (\mathrm{mod}\ p)}} 1 - \sum_{\substack{qrs \le x \\ q,r,s > p_t \\ qrs \equiv c \ (\mathrm{mod}\ p)}} 1 + \pi_{p,c}(p_t). \tag{6. 13}$$

Since there is a null quadratic effect we expect that for each odd prime $p \equiv 1 \pmod 3$ and fixed x_0, provided $A_p^{(1)}(x, 3)$ is not negative for the first few integers and x_0 is not too large, there exists a small $k = k(x_0)$ with

$$|A_p^{(k)}(x, 3) - h_k(x, 3)| < h_k(x, 3) \quad \text{for every } x < x_0. \tag{6. 14}$$

Remarks. The case of arbitrary composite modulus b presents some technical (balancing) difficulties which will be treated elsewhere. In particular, if b is a modulus having no primitive root then the ratio of quadratic non-residues to quadratic residues is no longer one-to-one (see section 6 of [7]), and this must be taken into account in the definition of $A_b^{(k)}(x, 3)$ in order that the quadratic effect be null (the net quadratic effect is zero in (6. 11) only because half the cubic residues (mod p) are quadratic residues and half quadratic non-residues (and similarly, of course, for the non-residues).

The last term on the right of the equality in (2. 1) sums to

$$-(\pi(x^{1/2})-\pi(x^{1/3}))/\#\text{ of quadratic residues of } b \tag{6. 15}$$

if c is a quadratic residue of b but sums to zero otherwise. Two years ago, when I announced [4] my belief in the existence of a cubic residue effect for small x it was based on the fact that the generalization of Meissel's formula to progressions, when extended to include products of three primes, is readily seen to contain a term, analogous to the above term, which sums to

$$-(\pi(x^{1/3})-\pi(x^{1/4}))/\#\text{ of cubic residues of } b \tag{6. 16}$$

when c is a cubic residue of b and sums to zero otherwise. Since this announcement I have formulated several elementary explanations of the need to further reduce the term in (6. 15) by the factor 2 and to reduce the term in (6. 16) by the factor 3. One of these is conceptually (and technically) very simple. Namely, consider all sequences consisting of products of three primes (see [7] for the quadratic "matching"). Those which start with the cube of a prime are obviously congruent to one of the cubic residues $(\operatorname{mod} p)$. However, because the cubic residues comprise exactly 1/3 of the reduced residues $(\operatorname{mod} p)$, the probability is 1/3 that such a sequence will end in a product congruent to a cubic residue $(\operatorname{mod} p)$. However, this model (even when all details are spelled out) is not as complete nor, perhaps, as compelling as the model derived above. I am, therefore, indebted to J. Ian Richards for initiating my interest in the above formulae.

Incidentally, it appears to have been overlooked (perhaps because it is somewhat disguised) that Meissel's formula,

$$\pi(x)=\phi(x,m)+m(s+1)+\frac{s(s-1)}{2}-1-\sum_{\sigma=1}^{s}\pi\left(\frac{x}{p_{m+\sigma}}\right), \tag{6. 17}$$

where $m=\pi(x^{1/3})$, $s=\pi(x^{1/2})-\pi(x^{1/3})$, and $\phi(x,m)$ is the number of positive integers $\leqq x$ which are relatively prime to the first m primes, contains the important term $\sum_{p^2\leq x} 1=\pi(x^{1/2})$ since

$$\sum_{\sigma=0}^{s-1}\pi(p_{m+\sigma})=m(s+1)+\frac{s(s-1)}{2}-\pi(p_m) \tag{6. 18}$$

implies that

$$m(s+1)+\frac{s(s-1)}{2}-\sum_{\sigma=1}^{s}\pi\left(\frac{x}{p_{m+\sigma}}\right)=\pi(p_m)-\sum_{\substack{pq\leq x\\ p\neq q}}1-\sum_{p^2\leq x}1 \tag{6. 19}$$

where p and q are primes $>x^{1/3}$.

7. The general combinatorial principle and remarks on quasiprime number theory

The general case can now be described as follows. Let S_n denote the set of positive integers $\leqq x$ which are the product of exactly n prime factors, $n\geqq 2$, say $p_1,p_2,\ldots,p_n$ with $x^{1/n}\leqq p_1\leqq p_2\leqq\cdots\leqq p_n\leqq x^{1/n+1}$. The products $p_1\cdots p_n$ are counted $n!$ times if and only if the p_i's, $i=1,\ldots,n$ are distinct. The other products are counted according to well-known combinatorial formulae (e.g., $ppqq$ is counted $\frac{4!}{2!\,2!}=6$ times, i.e., 1/4 as often as $pqrs$, $p \neq q \neq r \neq s$); in particular, the squares of primes, cubes of primes, fourth powers of primes, etc., are counted only once and this results in an 'excess' of integers in S_n congruent to the n-th power residues (mod b) provided that b is a modulus having n-th power non-residues. If $b=2$, then $\Pi(x)$ plays the role of the "missing" non-residue progression and the connection with $\operatorname{li} x$ in Riemann's formula is arrived at via the inversion formula of Moebius. Now, let $\psi_{b,c}(x, t)$ denote the number of integers >1 which are relatively prime to the first $t=\pi(x^{1/n})$ primes and $\equiv c \pmod b$, $(c, b)=1$, $1 \leqq c < b$. Letting p_t denote the t-th prime, and generalizing (5. 12) in the obvious way, it is clear that the values of $\psi_{b,c}(x, t)$ vary far less than the values of $\pi_{b,c}(x)$ if x is small so that an "excess" of integers in S_n congruent to n-th power residues (mod b) appears as a temporary "deficiency" in the primes congruent to these residues when the products of primes are subtracted from $\psi_{b,c}(x, t)$.

It is, unfortunately, not the case that the combinatorial formulae can easily be written down (in full detail) for arbitrary n. We cite but one example with $n>3$ in this paper.

The simplest case for $n>3$ is the quartic case for the modulus 5. Letting T_α denote the number of primes in the interval $x^{1/5} \leqq q \leqq x^{1/4}$ which are $\equiv \alpha \pmod 5$, $\alpha=1, \ldots, 4$, and for each β, $\beta=1, \ldots, 4$ letting V_β denote the number of elements in S_4 which are $\equiv \beta \pmod 5$, we have

$$(7.2) \qquad V_1 = \sum_{\substack{i=1 \\ \alpha_i^4 \equiv 1 \pmod 5}}^{4} \frac{T_{\alpha_i}(T_{\alpha_i}+1)(T_{\alpha_i}+2)(T_{\alpha_i}+3)}{24}$$

$$+ \sum_{\substack{i=1 \\ \alpha_i^2\alpha_j\alpha_k \equiv 1 \pmod 5}}^{4} \frac{T_{\alpha_i}(T_{\alpha_i}+1)T_{\alpha_j}T_{\alpha_k}}{2} + \sum_{\substack{i=1 \\ \alpha_i^2\alpha_j^2 \equiv 1 \pmod 5}}^{2} \frac{T_{\alpha_i}(T_{\alpha_i}+1)T_{\alpha_j}(T_{\alpha_j}+1)}{4},$$

$$(7.3) \qquad V_2 = V_3 = \sum_{\substack{i=1 \\ \alpha_i^3\alpha_j \equiv 2 \text{ or } 3 \pmod 5}}^{4} \frac{T_{\alpha_i}(T_{\alpha_i}+1)(T_{\alpha_i}+2)T_{\alpha_j}}{6} + \sum_{\substack{i=1 \\ \alpha_i^2\alpha_j\alpha_k \equiv 2 \text{ or } 3 \pmod 5}}^{4} \frac{T_{\alpha_i}(T_{\alpha_i}+1)T_{\alpha_j}T_{\alpha_k}}{2},$$

$$(7.4) \qquad V_4 = \sum_{i=1}^{4} \frac{T_{\alpha_i}(T_{\alpha_i}+1)(T_{\alpha_i}+2)T_{\alpha_j}}{6}$$

$$+ \sum_{\substack{i=1 \\ \alpha_i^2\alpha_j^2 \equiv 4 \pmod 5}}^{4} \frac{T_{\alpha_i}(T_{\alpha_i}+1)T_{\alpha_j}(T_{\alpha_j}+1)}{4} + T_1 T_2 T_3 T_4.$$

If we now let $T=(T_1+T_2+T_3+T_4)/4$, then

$$(7.\ 5)\qquad V_1 \cong \frac{4(T^4+6T^3+11T^2+6T)}{24}+\frac{4(T^4+T^3)}{2}+\frac{2(T^4+2T^3+T^2)}{4}$$
$$=(32T^4+48T^3+28T^2+12T)/12,$$

$$(7.\ 6)\qquad V_2 \cong V_3 \cong \frac{4(T^4+3T^3+2T^2)}{6}+\frac{4(T^4+T^3)}{2}=(32T^4+48T^3+16T^2)/12,$$

$$(7.\ 7)\qquad V_4 \cong \frac{4(T^4+3T^3+2T^2)}{6}+\frac{4(T^4+2T^3+T^2)}{4}+T^4=(32T^4+48T^3+28T^2)/12.$$

From (7. 6) it is obvious that

$$(7.\ 8)\qquad V_2 \cong V_3.$$

On the other hand, we have from (7. 5) and from (7. 7),

$$(7.\ 9)\qquad V_1 - V_4 \cong T \cong \pi(x^{1/4})/4.$$

Now, let

$$(7.\ 10)\qquad A_5^{(1)}(x,4)=\frac{1}{x}\sum_{t=1}^{x}\left(\pi_{5,4}(t)-\pi_{5,1}(t)\right),$$

and for $k \geqq 1$ let

$$(7.\ 11)\qquad A_5^{(k+1)}(x,4)=\frac{1}{x}\sum_{t=1}^{x}A_5^{(k)}(t,4).$$

Let $h_k(x,4)$ be given by (5. 14) for $k \geqq 1$. As before, the "excess" of integers in S_4 which are $\equiv 1 \pmod 4$ should, for small x, result in a "deficiency" of approximately $\pi(x^{1/4})/4$ in $\pi_{5,1}(x)$ relative to $\pi_{5,4}(x)$. Thus, one may investigate whether for fixed x_0 (in computer range) there exists a small $k=k(x_0)$ such that

$$(7.\ 13)\qquad |A_5^{(k)}(x,4)-h_k(x,4)|<h_k(x,4) \quad \text{for "almost all" } x<x_0.$$

Concluding remarks. It is not even known whether $A_5^{(k)}(x,2)$ is positive for sufficiently large x and any fixed choice of k (see [16] for a recent result on $\pi_{5,2}(x)-\pi_{5,4}(x)$). Indeed, as long as the generalized Riemann hypothesis remains unresolved, every effect in comparative prime number theory investigated in this paper must be treated as a potentially transitory phenomenon. Of course, this is particularly true when one investigates the meaningfulness of terms as tiny as $\pi(x^{1/n})/n$, $n>3$. It would certainly be useful to have data on inequalities such as (7. 13).

It may be that fluctuations in the primes (or when fluctuations in the quadratic effect) totally destroy the meaning of (7. 13).

Even should this be the case, however, the problem may be rooted less in the fact that the quartic, quintic, and higher order effects are too weak "to be of any importance" than in the failure of prime number theory to provide a sufficiently general context for the interpretation of these subtleties.

As is well-known, Meissel's formula is a special case of Viggo Brun's formula, see remarks of A. L. Whiteman, M. R. 36 # 2548,

$$(7.\,14)\qquad E(a, 2, \ldots, p_{s-1}, \ldots, p_r) = E(a, 2, \ldots, p_{s-1})$$
$$- E\left(\frac{a}{p_s}, 2, \ldots, p_{s-1}\right) - E\left(\frac{a}{p_{s+1}}, 2, \ldots, p_{s-1}\right) - \cdots - E\left(\frac{a}{p_r}, 2, \ldots, p_{s-1}\right)$$
$$+ (r-s)(r-s-1)/2,$$

where a is a positive integer and $E(a, 2, 3, \ldots, p_r)$ is the number of integers uncancelled by the primes $2, \ldots, p_r$ $(p_s \leqq p_r)$. Brun's formula, which has proven to be an extremely valuable number theoretical tool, clearly contains the formula of Meissel as a special case (the case $p_{s-1} = m = \pi(x^{1/3})$, $p_r = n = \pi(x^{1/2})$).

A feature of the theory given in this paper, which may well be more valuable than its independence from hypothetical results on the location of zeros of L-functions, is the ease with which the theory can be extended to settings far more general than the distribution of primes in arithmetic progressions. In particular, the theory naturally generalizes to the distribution (in arithmetic progressions) of integers having no prime factor $> x^y$, $y \leqq 1/2$. This is often called quasi-prime number theory and clearly contains prime number theory as the special case $y = 1/2$.

Of course, the formula (7. 14) is valid only if $p_r \leqq a^{1/2}$ and $p_{s-1} \geqq a^{1/3}$ and, as a result, gives no information regarding the effect of products of 3 primes $> x^{1/4}$ on the number of primes $\leqq x$ (and so not on the cubic and higher order effects). However, the Meissel and (7. 14) can easily be extended to study, e.g., the cubic effect.

I suggest that to fully appreciate the significance of n-th order effects when n is > 2 they should be studied in the context of the distribution of integers in arithmetic progressions having $n-1$ or fewer prime factors.

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