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Quadratic Residues and the Distribution of Prime Numbers

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Abstract. D. SHANKS [11] has given a heuristical argument for the fact that there are "more" primes in the non-quadratic residue classes mod q than in the quadratic ones. In this paper we confirm SHANKS' conjecture in all cases $q < 25$ in the following sense. If l_1 is a quadratic residue, l_2 a non-residue mod q, $\varepsilon(n, q, l_1, l_2)$ takes the values $+1$ or -1 according to $n \equiv l_1$ or $l_2 \bmod q$, then

$$\lim_{x\to\infty} \sum_p \varepsilon(p, q, l_1, l_2) \log p\, p^{-\alpha} \exp(-(\log p)^2/x) = -\infty$$

for $0 \leqslant \alpha < 1/2$. In the general case the same holds, if all zeros $\varrho = \beta + i\gamma$ of all $L(s, \chi \bmod q)$, q fix, satisfy the inequality $\beta^2 - \gamma^2 < 1/4$.

1. CHEBYSHEV asserted in 1853 that

$$\lim_{x\to\infty} \sum_{p>2} (-1)^{(p-1)/2} e^{-p/x} = -\infty \tag{1}$$

but never gave a proof [5]. His statement is indeed very deep, for it is equivalent to an analogue of the famous Riemann-Hypothesis:

$$L(s, \chi_1) \neq 0, \ \mathrm{Re}(s) > \tfrac{1}{2}, \tag{2}$$

with nonprincipal χ_1 mod 4. This was shown by HARDY, LITTLEWOOD and LANDAU in 1917/18 [6], [9].

If correct, (1) would imply a certain predominance of the primes $\equiv 3 \bmod 4$ over those $\equiv 1 \bmod 4$. Numerical calculations show that

$$\Delta(X) := \pi_1(X) - \pi_3(X) \tag{3}$$

is predominantly negative. Although sign changes of Δ seem to be very rare (the first one occurs at $X = 26\,861$ [10]) we know from theoretical investigations [6] that

$$\lim_{X\to\infty} \left\{ {\sup \atop \inf} \Delta(X) \right\} = \pm\infty. \tag{4}$$

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So, the "naive" stament that there are more primes $\equiv 3 \bmod 4$ than $\equiv 1 \bmod 4$ is certainly wrong and one has to look for another interpretation of the numerical phenomena.

KNAPOWSKI and TURÁN created a new path in the theory of prime numbers, the so called "Comparative Prime Number Theory". Their series of papers during the years 1960–70 [7], [8], contain a lot of interesting problems and results. In the Chebyshevian case they found that

$$\lim_{x\to\infty} \sum_{p>2} (-1)^{(p-1)/2} \log p \, e^{-\log^2(p/x)} = -\infty \tag{5}$$

is as well equivalent to (2).

In 1978 the first of us[1] gave an unconditional proof of the predominant behaviour of the primes in the class 3 mod 4, namely

$$\lim_{x\to\infty} \sum_{p>2} (-1)^{(p-1)/2} \log p \, p^{-\alpha} e^{-(\log p)^2/x} = -\infty \tag{6}$$

for $0 \leqslant \alpha \leqslant \frac{1}{2}$ [3].

The proof is based on explicit formulas. A more direct way, which is very simple in the case $\alpha = 0$, can be found in our paper [1].

It turned out that the weight-function $\exp(-(\log^2 p)/x)$ is easier to handle than $\exp(-\log^2(p/x))$ (KNAPOWSKI, TURÁN) or $\exp(-p/x)$ (CHEBYSHEV). Therefore it is naturally to investigate some generalisations of the "Chebyshev-problem", using the first of these functions.

Under some hypothesis (weaker than the Riemann—Piltz hypothesis and fulfilled for every character mod q with $q < 25$) we have shown in another paper [2] that the following statement holds for $0 \leqslant \alpha < \frac{1}{2}$:

$$\lim_{x\to\infty} \sum_{p} \chi_1(p) \log p \, p^{-\alpha} e^{-(\log^2 p)/x} = -\infty, \tag{7}$$

where $\chi_1(n)$ is a real non-principal character.

D. SHANKS [11] has given a heuristical argument for the fact that there are "more" prime numbers in the "non-quadratic" residue-classes mod q than in the "quadratic" ones. As (6) shows, this is true

[1] Up to now this author has published under the name of BESENFELDER.

in the case $q = 4$. In addition, just the same can be obtained for some other moduli, e.g. $q = 3{,}8$ and in some weaker form for $q = 5$, see [4].

Here, we will fully affirm SHANKS' conjecture in all cases $q < 25$ (theorem 2 and 4 below). In general (theorem 1 and 3), our result tends in the same direction but depends on some hypothesis concerning the non-trivial zeros of $L(s, \chi)$ "near" to the real axis.

2. We formulate the following hypothesis for Dirichlet's L-functions:

$\mathbf{R}_2$: *The domain*

$$\sigma > \tfrac{1}{2}, \ |t| \leqslant 1 \tag{8}$$

is zerofree and there is no zero at $s = \frac{1}{2}$.

With other words the Riemann—Piltz hypothesis (**RP**) is assumed to be valid up to the height $t = 1$, and there is no real zero. This hypothesis can be considered as plausible. In contrary to **RP** it has a further advantage, for it can be verified with computations for any concrete L-function. In fact, $\mathbf{R}_2$ was verified (even for an extended region) for every L-function $L(s, \chi \bmod q)$, $q < 25$, by R. SPÍRA [12]. To prove the above mentioned phenomena for these L-functions it is even enough to assume a weaker form of $\mathbf{R}_2$, namely:

$\mathbf{H}_2$: *All zeros* $\varrho = \beta + i\gamma$ *satisfy the inequality*

$$\beta^2 - \gamma^2 < \tfrac{1}{4}. \tag{9}$$

Further we introduce the notation

$$\varepsilon(n, q, l_1, l_2) = \begin{cases} 1 \text{ if } n \equiv l_1 \bmod q \\ -1 \text{ if } n \equiv l_2 \bmod q. \\ 0 \text{ otherwise} \end{cases} \tag{10}$$

Theorem 1. *If* l_1 *is a quadratic residue,* l_2 *a non-residue* $\bmod q$ *and the hypothesis* $\mathbf{R}_2$ *or even* $\mathbf{H}_2$ *is valid for all L-functions* $\bmod q$, *then*[2]

$$\lim_{x \to \infty} \sum_p \varepsilon(p, q, l_1, l_2) \log p \exp(-(\log^2 p)/x) = -\infty. \tag{11}$$

[2] We always mention both hypothesis $\mathbf{R}_2$ and $\mathbf{H}_2$ together, although $\mathbf{H}_2$ (as the weaker one) would suffice to prove the theorems, but it is easier to imagine the 'region' which is denoted by $\mathbf{R}_2$, and, in addition, some hypothesis of type $\mathbf{R}_2$ are used in related contexts.

Taking in account the cited calculations of SPIRA, this implies immediately

Theorem 2. *If l_1 is a quadratic residue, l_2 a non-residue* mod q, *then*

$$\lim_{x\to\infty} \sum_p \varepsilon(p, q, l_1, l_2) \log p \exp(-(\log^2 p)/x) = -\infty \tag{12}$$

holds for all $q < 25$.

For brevity we take

$$F(s) := -\frac{1}{\varphi(q)} \sum_\chi (\bar\chi(l_1) - \bar\chi(l_2)) \frac{L'}{L}(s, \chi), \tag{13}$$

where χ runs through all characters mod q. Then using the simple integration formula

$$\frac{1}{2\pi i} \int_{(2)} e^{As^2+Bs} ds = \frac{1}{2\sqrt{\pi A}} \exp\left(-\frac{B^2}{4A}\right), \tag{14}$$

which is valid for real $A > 0$ and arbitrary complex B, we get

$$I = \frac{1}{2\sqrt{\pi y}} \sum_n \varepsilon(n, q, l_1, l_2) \Lambda(n) \exp\left(-\frac{\log^2 n}{4y}\right) = \tag{15}$$

$$= \frac{1}{2\sqrt{\pi y}} \cdot \frac{1}{\varphi(q)} \sum_\chi (\bar\chi(l_1) - \bar\chi(l_2)) \sum_n \chi(n) \Lambda(n) \exp\left(-\frac{\log^2 n}{4y}\right) =$$

$$= \frac{1}{\varphi(q)} \sum_\chi (\bar\chi(l_1) - \bar\chi(l_2)) \sum_n \chi(n) \Lambda(n) \frac{1}{2\pi i} \int_{(2)} \frac{e^{ys^2}}{n^s} ds =$$

$$= \frac{1}{2\pi i} \int_{(2)} F(s) e^{ys^2} ds =$$

$$= \frac{1}{\varphi(q)} \sum_\chi (\bar\chi(l_1) - \bar\chi(l_2)) \sum_{\varrho=\varrho_\chi} e^{y\varrho^2} + \frac{1}{2\pi i} \int_{(-1/4)} F(s) e^{ys^2} ds.$$

Now we use

$$\frac{L'}{L}(-\tfrac{1}{4} + it, \chi) = O(\log(|t| + 2)). \tag{16}$$

The constants in the O-Symbols and the positive constants c_1, c_2, c_3 below, may depend on q. (16) gives

$$\frac{1}{2\pi i} \int_{(-1/4)} F(s)\, e^{ys^2} ds = O\Big(\int_{-\infty}^{\infty} \log(|t|+2)\, e^{y(1/16-t^2)} dt \Big) = O(e^{y/16}). \tag{17}$$

Furthermore we know

$$\sum_{\substack{n \leqslant |\operatorname{Im} \varrho| \leqslant n+1 \\ \varrho = \varrho_\chi}} 1 = O(\log(n+2)), \tag{18}$$

and this gives for $y \to \infty$

$$\begin{aligned} \Big| \sum_{\varrho = \varrho_\chi} e^{y\varrho^2} \Big| &\leqslant \sum_{|\gamma| \leqslant 1} e^{y(\beta^2 - \gamma^2)} + O\Big(\sum_{n=1}^{\infty} e^{y(1-n^2)} \log(n+2) \Big) = \\ &= O\big(\exp\big(y \cdot \max_{|\gamma| \leqslant 1} (\beta^2 - \gamma^2)\big)\big) + O(1) = O(e^{y(1/4 - \delta_1)}), \end{aligned} \tag{19}$$

with $\delta_1 > 0$, depending on q only.

(15), (16), (19) together give

$$I = O(e^{y(1/4 - \delta_2)}), \tag{20}$$

with $\delta_2 = \delta_2(q) > 0$. Elementary estimations imply

$$\begin{aligned} S(x) := \sum_{n \leqslant x,\ n \text{ not a prime}} \varepsilon(n, q, l_1, l_2)\, \Lambda(n) = \\ = \sum_{\substack{p^2 \equiv l_1 (q) \\ p^2 \leqslant x}} \log p + O(\sqrt[3]{x}) > c_1 \sqrt{x}, \end{aligned} \tag{21}$$

for $x > c_2 (\geqslant 3)$. So we get for $y \to \infty$

$$\begin{aligned} &\sum_{n \neq p} \varepsilon(n, q, l_1, l_2)\, \Lambda(n) \exp(-\log^2 n / 4y) = \\ &= -\int_3^{\infty} S(x) \exp(-\log^2 x / 4y)\, (-2 \log x / 4y)\, (dx/x) \geqslant \\ &\geqslant \int_{c_2}^{\infty} c_1 \sqrt{x} \exp(-\log^2 x / 4y)\, (\log x / 2y)\, (dx/x) + O(1/y) \geqslant \\ &\geqslant (c_1/2y) \int_{c_2}^{e^y} \sqrt{x} \exp(-\log x / 4)\, (dx/x) + O(1/y) = \\ &= (2c_1/y)\, e^{y/4} + O(1/y). \end{aligned} \tag{22}$$

Now (15), (20) and (22) prove theorem 1.

3. In this section we shall investigate a generalisation of theorem 1. The same phenomenon occurs if we use a more general weight-function, i. e.

$$\frac{\log p}{p^\alpha} \exp\left(-\frac{\log^2 p}{x}\right), \quad 0 \leqslant \alpha < \tfrac{1}{2}. \tag{23}$$

Theorem 3. *If* $\mathbf{R}_2$ *or only* $\mathbf{H}_2$ *is true for all L-functions* mod q, l_1 *is a quadratic residue,* l_2 *a non-residue* mod q, *then for* $0 \leqslant \alpha < \frac{1}{2}$

$$\lim_{x\to\infty} \sum_p \varepsilon(p,q,l_1,l_2) \frac{\log p}{p^\alpha} \exp\left(-\frac{\log^2 p}{x}\right) = -\infty. \tag{24}$$

Since our assumption $\mathbf{H}_2$ or $\mathbf{R}_2$ is valid for $q < 25$ we can state

Theorem 4. *If* l_1 *is a quadratic residue,* l_2 *a non-residue* mod q, $q < 25$, *then for* $0 \leqslant \alpha < \frac{1}{2}$

$$\lim_{x\to\infty} \sum_p \varepsilon(p,q,l_1,l_2) \frac{\log p}{p^\alpha} \exp\left(-\frac{\log^2 p}{x}\right) = -\infty. \tag{25}$$

With some extra effort, one can prove theorem 3 (and 4) for $\alpha = \frac{1}{2}$, but if $\alpha > \frac{1}{2}$ the situation changes.

Analogously to (15) we get from (13) and (14)

$$\begin{aligned} I_\alpha &:= \left(\frac{1}{2\sqrt{\pi y}}\right) \sum_n \varepsilon(n,q,l_1,l_2) \Lambda(n) n^{-\alpha} \exp\left(-\frac{\log^2 n}{4y}\right) = \\ &= \left(\frac{1}{2\sqrt{\pi y}}\right) \cdot \frac{1}{\varphi(q)} \sum_\chi (\bar\chi(l_1) - \bar\chi(l_2)) \sum_n \chi(n)\Lambda(n) n^{-\alpha} \exp\left(-\frac{\log^2 n}{4y}\right) = \\ &= \left(\frac{1}{\varphi(q)}\right) \sum_\chi (\bar\chi(l_1) - \bar\chi(l_2)) \sum_n \chi(n)\Lambda(n) n^{-\alpha} \cdot \frac{1}{2\pi i} \int_{(2)} \frac{e^{ys^2}}{n^s} ds = \\ &= \left(\frac{1}{2\pi i}\right) \int_{(2)} F(s+\alpha) e^{ys^2} ds. \end{aligned} \tag{26}$$

Now we transform the line (2) of integration. The new line l consists of a part l_1 of the hyperbola:

$$(\tfrac{1}{2} - \sigma)^2 - t^2 = \tfrac{1}{4}, \quad -\tfrac{1}{4} \leqslant \sigma \leqslant 0, \tag{27}$$

and of the straight lines l_2, l_3:

$$\sigma = -1, \quad |t| \geqslant \sqrt{2}. \tag{28}$$

With this choice of l we can observe that for all non-trivial zeros ϱ the points $\varrho - \alpha$ lie to the right of l. This is trivial for $|\gamma| \geqslant \sqrt{2}$. If $|\gamma| \leqslant \sqrt{2}$ and $\beta \geqslant \frac{1}{2}$ we have

$$|\tfrac{1}{2} + \alpha - \beta| \leqslant \beta, \tag{29}$$

and hypothesis $\mathbf{H}_2$, i.e. (9), gives

$$[\tfrac{1}{2} - (\beta - \alpha)]^2 - \gamma^2 \leqslant \beta^2 - \gamma^2 < \tfrac{1}{4}. \tag{30}$$

If $|\gamma| \leqslant \sqrt{2}$ and $\beta \leqslant \frac{1}{2}$ by (9) we have again

$$[\tfrac{1}{2} - (\beta - \alpha)]^2 - \gamma^2 \leqslant (1 - \beta)^2 - \gamma^2 < \tfrac{1}{4}, \tag{31}$$

for if $\varrho = \beta + i\gamma$ is a zero of $L(s, \chi)$ then by the functional equation $\varrho' = 1 - \beta - i\gamma$ is a zero of $L(s, \bar{\chi})$.
Thus we get

$$I_\alpha = \frac{1}{2\pi i} \int_{(l)} F(s + \alpha) e^{y s^2} ds - \frac{1}{\varphi(q)} \sum_\chi (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \sum_{\varrho = \varrho_\chi} e^{y(\varrho - \alpha)^2}. \tag{32}$$

Taking in account that by (27) for $s \in l_1$

$$\operatorname{Re} s^2 = \sigma^2 - t^2 = \sigma \leqslant 0 \tag{33}$$

and $F(s + \alpha)$ is regular on l_1, we get

$$(1/2\pi i) \int_{(l_1)} F(s + \alpha) e^{y s^2} ds = O(1). \tag{34}$$

Further we have for $|t| \geqslant \sqrt{2}$

$$\frac{L'}{L}(-1 + \alpha + it) = O(\log |t|) \tag{35}$$

and by (28)

$$(1/2\pi i) \int_{(l_2)} F(s + \alpha) e^{y s^2} ds = O\left(\int_{\sqrt{2}}^{\infty} \log t \cdot e^{(1 - t^2) y} dt\right) = O(1). \tag{36}$$

Now, for all non-trivial zeros, hypothesis $\mathbf{H}_2$ implies

$$\operatorname{Re}(\varrho - \alpha)^2 = (\beta - \alpha)^2 - \gamma^2 < (\tfrac{1}{2} - \alpha)^2. \tag{37}$$

To prove this, first take $\beta \geqslant \frac{1}{2}$, then by (9):

$$(\beta - \alpha)^2 - \gamma^2 = \beta^2 - \gamma^2 - 2\beta\alpha + \alpha^2 < \tfrac{1}{4} - \alpha + \alpha^2. \tag{38}$$

Secondly, if $\beta < \frac{1}{2}$, we consider the zero $\varrho' = \beta' + i\gamma' = 1 - \beta - i\gamma$

of $L(s,\bar{\chi})$. This zero gives $|\beta' - \alpha| > |\beta - \alpha|$ and with (38)

$$(\beta - \alpha)^2 - \gamma^2 < (\beta' - \alpha)^2 - \gamma'^2 < (\tfrac{1}{2} - \alpha)^2. \tag{39}$$

Thus we get with $\delta_3 = \delta_3(q, \alpha) > 0$

$$\sum_{\substack{\varrho = \varrho_\chi \\ |\gamma| \leqslant 2}} e^{y(\varrho - \alpha)^2} = O\left(e^{y[(1/2-\alpha)^2 - \delta_3]}\right). \tag{40}$$

Using (18) we have

$$\sum_{\substack{\varrho = \varrho_\chi \\ |\gamma| \geqslant 2}} e^{y(\varrho - \alpha)^2} = O\left(\sum_{n=2}^{\infty} \log(n+2)\, e^{y(1-n^2)}\right) = O(1). \tag{41}$$

Now, from (32), (34), (36), (40) and (41) we get

$$I_\alpha = O\left(e^{y[(1/2-\alpha)^2 - \delta_3]}\right). \tag{42}$$

On the other hand analogously to (22) we have from (21)

$$\sum_{n \neq p} \varepsilon(n, q, l_1, l_2)\, \Lambda(n)\, n^{-\alpha} \exp\left(-\frac{\log^2 n}{4y}\right) =$$

$$= -\int_3^{\infty} S(x) \exp\left(-\frac{\log^2 x}{4y}\right) \left\{-\left(\frac{2\log x}{4y}\right) \cdot \left(\frac{1}{x^{1+\alpha}}\right) - \left(\frac{\alpha}{x^{1+\alpha}}\right)\right\} dx \geqslant$$

$$\geqslant \int_{c_2}^{\infty} c_1 \sqrt{x} \exp\left(-\frac{\log^2 x}{4y}\right) \left(\frac{\log x}{2y}\right) \left(\frac{dx}{x^{1+\alpha}}\right) \geqslant \tag{43}$$

$$\geqslant \left(\frac{c_1}{2y}\right) \int_{c_2}^{e^{y(1-2\alpha)}} \exp\left(-\frac{\log x}{4}(1 - 2\alpha)\right) x^{-1/2-\alpha}\, dx =$$

$$= \left(\frac{c_3}{y}\right) e^{y(1/2-\alpha)^2} + O\left(\frac{1}{y}\right).$$

This together with (26) and (42) proves theorem 3.

4. In addition to theorem 3 we shall now determine how fast the infinite sum, occuring in (24), tends to infinity.

Theorem 5. *Under the conditions of theorem* 3 *we have*

$$\sum_{n} \varepsilon(n, q, l_1, l_2) \frac{\log p}{p^\alpha} \exp\left(-\frac{\log^2 p}{x}\right) \sim \frac{N(q)}{\varphi(q)} \sqrt{\pi x}\, e^{(x/4)(1/2-\alpha)^2}, \tag{44}$$

where $N(q)$ *denotes the number of solutions of* $x^2 \equiv 1 \bmod q$.

This theorem of course implies

Theorem 6. *Under the conditions of theorem* 4 *we have*

$$\sum_{n} \varepsilon(n,q,l_1,l_2) \frac{\log p}{p^\alpha} \exp\left(-\frac{\log^2 p}{x}\right) \sim \frac{N(q)}{\varphi(q)} \sqrt{\pi x}\, e^{(x/4)(1/2-\alpha)^2}. \quad (45)$$

To prove theorem 5 we need the estimate

$$S_\alpha(x) = \sum_{\substack{n \leqslant x \\ n \neq p}} \varepsilon(n,q,l_1,l_2) \frac{\Lambda(n)}{n^\alpha} = \sum_{\substack{p^2 \leqslant x \\ p^2 \equiv l_1 (q)}} \frac{\log p}{p^{2\alpha}} + O\left(\sum_{\substack{p^{m\alpha} \leqslant x \\ m \geqslant 3}} \frac{\Lambda(n)}{p^{m\alpha}}\right) \sim$$
$$\sim \frac{N(q)}{\varphi(q)} \cdot \frac{1}{1-2\alpha} x^{1/2-\alpha}, \quad (46)$$

which can be obtained by partial summation from the prime number theorem for arithmetical progressions, using the simple fact that for any quadratic residue l the congruences $x^2 \equiv 1 \bmod q$, $x^2 \equiv l \bmod q$ have the same number of solutions.

$$\sum_{n \neq p} \varepsilon(n,q,l_1,l_2) \frac{\Lambda(n)}{n^\alpha} \exp\left(-\frac{\log^2 n}{4y}\right) =$$
$$= -\int_1^\infty S_\alpha(x) \exp\left(-\frac{\log^2 x}{4y}\right)\left(-\frac{2\log x}{4y}\right)\frac{dx}{x} \sim$$
$$\sim \int_1^\infty \frac{N(q)}{\varphi(q)} \cdot \frac{1}{1-2\alpha} x^{(1/2)-\alpha} \exp\left(-\frac{\log^2 x}{4y}\right) \frac{\log x}{2y} \frac{dx}{x} =$$
$$= \frac{N(q)}{\varphi(q)} \cdot \frac{1}{1-2\alpha} \cdot \frac{1}{2y} \int_0^\infty e^{u(1/2-\alpha)} \exp\left(-\frac{u^2}{4y}\right) u\, du =$$
$$= \frac{N(q)}{\varphi(q)} \frac{e^{y(1/2-\alpha)^2}}{2y(1-2\alpha)} \int_0^\infty \exp\left(-\left[\frac{u}{2\sqrt{y}} - \sqrt{y}\left(\tfrac{1}{2}-\alpha\right)\right]^2\right) u\, du = \quad (47)$$
$$= \frac{N(q)}{\varphi(q)} \frac{e^{y(1/2-\alpha)^2}}{2y(1-2\alpha)} \cdot \int_{-\sqrt{y}(1/2-\alpha)}^\infty e^{-t^2} (2\sqrt{y}\, t + 2y(1-2\alpha)) 2\sqrt{y}\, dt \sim$$
$$\sim \frac{N(q)}{\varphi(q)} \frac{e^{y(1/2-\alpha)^2}}{\sqrt{y}(1-2\alpha)} \int_{-\infty}^\infty e^{-t^2}(2\sqrt{y}\, t + 2y(1-2\alpha))\, dt =$$
$$= \frac{N(q)}{\varphi(q)} \cdot 2\sqrt{\pi y}\, e^{y(1/2-\alpha)^2}.$$

Theorem 5 is now the immediate consequence of formulae (26), (42) and (47).

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