ON THE REMAINDER TERM OF THE PRIME NUMBER FORMULA IV. SIGN CHANGES OF $\pi(x) - \operatorname{li} x$

by

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1. Riemann asserted (without proof) in his memoir [12] that for every $x > 2$

$$\pi(x) < \operatorname{li} x. \tag{1.1}$$

His assertion was disproved by LITTLEWOOD [8] in 1914, who proved that the difference $\pi(x) - \operatorname{li} x$ changes sign infinitely many times. However, his proof was ineffective, so it was impossible to give any upper bound for the first sign change. After many attempts this problem was solved in 1955 by SKEWES [13] who proved the first explicit bound $e_4(7{,}705)$[1]. Another interesting problem was — for which Littlewood's original work had no answer — how often has $\pi(x) - \operatorname{li} x$ sign changes. Let us denote the number of sign changes of $\pi(x) - \operatorname{li} x$ in $[2, Y]$ by $V_1(Y)$. So the problem would be to give lower estimate for $V_1(Y)$. Such a result was first achieved by INGHAM [2], under an unproved and very deep condition however. Let us denote by θ the least upper bound of the real parts of the ζ-zeros. Then Ingham's theorem says, that if there exists a zero on the line $\sigma = \theta$, then $\pi(x) - \operatorname{li} x$ has a sign change in every interval

$$[Y, c_0 Y] \tag{1.2}$$

where c_0 is an absolute, however, not effectively computable constant. From this one gets easily that

$$V_1(Y) > c_1 \log Y \quad \text{for} \quad Y > Y_1 \tag{1.3}$$

with ineffective absolute constants c_1[2] and Y_1.

The first unconditional lower bound for $V_1(Y)$ was proved in 1961—62 by S. KNAPOWSKI [3], [4]. He proved

$$V_1(Y) > c_2 \log_2 Y \quad \text{for} \quad Y > Y_2 \tag{1.4}$$

with ineffective Y_2, further the weaker effective inequality

$$V_1(Y) > c_3 \log_4 Y \quad \text{for} \quad Y > Y_3 \tag{1.5}$$

where both c_3 and Y_3 are explicitly calculable.

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[1] We use the notations $e_1(x) = \exp(x) = e^x$, $e_{\nu+1}(x) = \exp(e_\nu(x))$ and analogously with $\log x$.

[2] All the constants c_i are positive.

In 1974—1976 S. KNAPOWSKI and P. TURÁN [5], [6] showed the improvements of (1.4)—(1.5), namely

$$V_1(Y) > c_4 \frac{(\log Y)^{1/4}}{(\log_2 Y)^4} \quad \text{for} \quad Y > Y_4 \tag{1.6}$$

with ineffective Y_4 and

$$V_1(Y) > c_5 \log_3 Y \quad \text{for} \quad Y > Y_5 \tag{1.7}$$

with effective absolute constants c_5 and Y_5.

(1.7) was improved by the author [9] in 1976 to

$$V_1(Y) > c_6 (\log_2 Y)^{c_7} \quad \text{for} \quad Y > Y_6 \tag{1.8}$$

where c_6, c_7 and Y_6 are effective. In part III of this series [10] the effective lower bound

$$V_1(Y) > c_8 \frac{\sqrt{\log Y}}{\log_2 Y} \quad \text{for} \quad Y > Y_7 \tag{1.9}$$

was proved, which is better than (1.8) and the former best ineffective lower estimate (1.6).

Finally we mention that LEVINSON [7] showed in 1975

$$\overline{\lim_{Y\to\infty}} \frac{V_1(Y)}{\log Y} > 0 \tag{1.10}$$

which is the best result in this direction.

Let

$$\begin{aligned}
\Delta_1(x) &\overset{\text{def}}{=} \pi(x) - \operatorname{li} x \overset{\text{def}}{=} \sum_{p \leq x} 1 - \int_0^x \frac{dt}{\log t},\\
\Delta_2(x) &\overset{\text{def}}{=} \Pi(x) - \operatorname{li} x \overset{\text{def}}{=} \sum_{p^m \leq x} \frac{1}{m} - \operatorname{li} x,\\
\Delta_3(x) &\overset{\text{def}}{=} \Theta(x) - x \overset{\text{def}}{=} \sum_{p \leq x} \log p - x,\\
\Delta_4(x) &\overset{\text{def}}{=} \psi(x) - x \overset{\text{def}}{=} \sum_{p^m \leq x} \log p - x,
\end{aligned} \tag{1.11}$$

and let $V_i(Y)$ denote (for $1 \leq i \leq 4$) the number of sign changes of $\Delta_i(x)$ in the interval $[2, Y]$.

With these notations we shall prove the partially ineffective

THEOREM. *There are absolute constants Y_i $(1 \leq i \leq 4)$ such that for $Y > Y_i$ the inequality*

$$V_i(Y) > \frac{1}{10^{11}} \cdot \frac{\log Y}{(\log_2 Y)^3} \qquad (1 \leq i \leq 4) \tag{1.12}$$

holds, where the constants Y_i are effectively computable for $i = 2, 4$ and they are ineffective for $i = 1, 3$.

This is already very near to Ingham's conditional (and also ineffective) lower bound (1.3), however, here we cannot give a corresponding localization of a sign change. The inequality (1.12) is also not far from Levinson's result (1.10).

2. In the course of proof we shall use the following "kernel function":

$$\mathscr{I}_{k,\mu}(u) = \frac{1}{2\pi i} \int_{(2)} \left(\frac{e^s - e^{-s}}{2s}\right)^k e^{us + \frac{s^2}{\mu}} ds \tag{2.1}$$

where $k \geq 1$ integer, $\mu \geq 1$ real, and u is real.

First we state some properties of the $\mathscr{I}_{k,\mu}(u)$ functions as

LEMMA 2.1. *For the $\mathscr{I}_{k,\mu}(u)$ function defined by (2.1) we have*

$$\mathscr{I}_{k,\mu}(u) = \mathscr{I}_{k,\mu}(-u); \tag{2.2}$$

$$\mathscr{I}_{k,\mu}(u) \geq 0; \tag{2.3}$$

$$\textit{if} \quad |u| \geq k+2 \quad \textit{then} \quad |\mathscr{I}_{k,\mu}(u)| \leq \frac{1}{e^{(|u|-k-1)\mu}}. \tag{2.4}$$

For the proof of (2.2) we note that shifting the line of integration to $\sigma = 0$ we get

$$\mathscr{I}_{k,\mu}(u) = \frac{1}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^k e^{-\frac{t^2}{\mu}} \cos(ut)\, dt \tag{2.5}$$

from which (2.2) follows.

Now it is enough to prove (2.4) for $u \geq k+2$. Then shifting the line of integration to $\sigma = -\mu$ we get

$$\begin{aligned}
|\mathscr{I}_{k,\mu}(u)| &= \left| \frac{1}{2\pi i} \int_{(-\mu)} \left(\frac{e^s - e^{-s}}{2s}\right)^k e^{us + \frac{s^2}{\mu}} ds \right| = \\
&= \left| \frac{1}{2^k} \sum_{d=0}^{k} (-1)^d \binom{k}{d} \cdot \frac{1}{2\pi i} \int_{(-\mu)} \frac{\exp\left\{(k-2d+u)s + \frac{s^2}{\mu}\right\}}{s^k} ds \right| \leq \\
&\leq \max_{0 \leq d \leq k} \left| \frac{1}{2\pi i} \int_{(-\mu)} \frac{\exp\left\{(k-2d+u)s + \frac{s^2}{\mu}\right\}}{s^k} ds \right| \leq \\
&\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp\left\{(k-2k+u)(-\mu) + \frac{\mu^2 - t^2}{\mu}\right\}}{\mu^k} dt \leq \\
&\leq \frac{\exp\{-\mu(u-k-1)\}}{\mu^k} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{t^2}{\mu}} dt < \frac{1}{e^{\mu(u-k-1)}}.
\end{aligned} \tag{2.6}$$

Thus we have also

$$\lim_{u\to\infty} \mathscr{I}_{k,\mu}(u) = 0. \tag{2.7}$$

So (2.3) will be proved considering (2.7) and (2.2) if we show that $\mathcal{I}_{k,\mu}(u)$ is monotonically decreasing for $u \geqq 0$.

This will be proved by induction with respect to k. Using the well-known formula

$$(2.8) \qquad \frac{1}{2\pi i}\int_{(2)} e^{As^2+Bs}\,ds = \frac{1}{2\sqrt{\pi A}}\exp\left(-\frac{B^2}{4A}\right)$$

valid for real positive A and arbitrary complex B, we get for $k=1$

$$\frac{d\mathcal{I}_{1,\mu}(u)}{du} = \frac{1}{2\pi i}\int_{(2)} \frac{e^s-e^{-s}}{2s}\cdot e^{\frac{s^2}{\mu}}\cdot s\cdot e^{us}\,ds =$$

$$(2.9) \qquad = \frac{1}{2}\left\{\frac{1}{2\pi i}\int_{(2)} e^{\frac{s^2}{\mu}+(u+1)s}\,ds - \frac{1}{2\pi i}\int_{(2)} e^{\frac{s^2}{\mu}+(u-1)s}\,ds\right\} =$$

$$= \frac{\sqrt{\mu}}{4\sqrt{\pi}}\left\{\exp\left(-\frac{\mu}{4}(u+1)^2\right)-\exp\left(-\frac{\mu}{4}(u-1)^2\right)\right\} \leqq 0.$$

If it is proved already for $k=k_0-1$ then we have

$$\frac{d\mathcal{I}_{k_0,\mu}(u)}{du} = \frac{1}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^{k_0}(-t\sin(ut))e^{-\frac{t^2}{\mu}}\,dt =$$

$$(2.10) \qquad = \frac{1}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^{k_0-1}\cdot\frac{1}{2}\{\cos((u+1)t)-\cos((u-1)t)\}e^{-\frac{t^2}{\mu}}\,dt =$$

$$= \frac{1}{2}(\mathcal{I}_{k_0-1,\mu}(u+1)-\mathcal{I}_{k_0-1,\mu}(u-1)) \leqq 0$$

or alternately

$$(2.11) \qquad = \frac{1}{2}(\mathcal{I}_{k_0-1,\mu}(1+u)-\mathcal{I}_{k_0-1,\mu}(1-u)) \leqq 0.$$

Thus (2.10) proves the assertion in case of $u \geqq 1$ and (2.11) in case of $0 \leqq u \leqq 1$, and so Lemma 1 is completely proved.

Our main tool in the proof will be Turán's method. Here we shall use the so called second main theorem in the special case when all the coefficients are equal to 1.

LEMMA 2 (T. Sós—Turán). *For arbitrary complex numbers* z_j

$$(2.12) \qquad \max_{m<\nu\leqq m+n}\frac{\left|\sum_{j=1}^n z_j^\nu\right|}{|z_1|^\nu} \geqq \left(\frac{1}{8e\left(\frac{m}{n}+1\right)}\right)^n.$$

The proof is contained in VERA T. SÓS—P. TURÁN [14].

If we have only an upper bound N for the number of z_j's then we can define

$$(2.13) \qquad z_j = 0 \quad \text{for} \quad n < j \leqq [N].$$

This implies immediately the modified form of (2.12), namely the inequality

$$(2.14) \qquad \max_{m<\nu\leqq m+N}\frac{\left|\sum_{j=1}^n z_j^\nu\right|}{|z_1|^\nu} \geqq \left(\frac{1}{8e\left(\frac{m}{[N]}+1\right)}\right)^N.$$

We shall use Lemma 2 in this form.

Further we shall use some known properties of the zeta function which we state here.

The number of zeros with imaginary part between T and $T+1$

$$(2.15) \qquad N(T+1)-N(T) < c\log T \quad \text{where} \quad c = 15 \quad \text{for} \quad T > T_0$$

(see W. J. ELLISON—M. MENDÈS FRANCE [1] p. 165).

If $\zeta(s) \neq 0$ in the domain

$$(2.16) \qquad \sigma > \beta, \quad |t| \leqq T+1$$

then for

$$(2.17) \qquad 2 \geqq \sigma \geqq \beta+\eta, \quad 2 \leqq |t| \leqq T$$

one has

$$(2.18) \qquad \left|\frac{\zeta'}{\zeta}(s)\right| = O\left(\frac{\log t}{\eta}\right).$$

(This follows easily from Satz 4.1 of PRACHAR [11], p. 225, in the special case $k=1$.)

Finally we shall use the standard estimate

$$(2.19) \qquad \zeta(s) = O(\sqrt{t}) \quad \text{for} \quad \sigma \geqq \frac{1}{2}, \quad t \geqq 1.$$

3. First we shall treat the (ineffective) case $i=1$. If the Riemann hypothesis is true then the quoted theorem of Ingham (see (1.3)) already settles the problem. For the sake of completeness we mention that Ingham's theorem with essentially unchanged proof is valid for $2 \leqq i \leqq 4$, too.

Thus we shall suppose the Riemann hypothesis to be false.

So let $\varrho_0 = \beta_0 + i\gamma_0$ be a zero with $\beta_0 > \frac{1}{2}$ and with minimal $\gamma_0 > 0$. If there are more such zeros then let $\varrho_1' = \beta_1' + i\gamma_1'$ be the zero among those with maximal real part. If there is only one such zero then let $\varrho_1' = \varrho_0$.

Let denote successively $\varrho_{n+1}' = \beta_{n+1}' + i\gamma_{n+1}'$ the zero with maximal real part among those satisfying

$$(3.1) \qquad \gamma_n' < \gamma \leqq \gamma_n' + 2\log Y, \quad \beta \geqq \beta_n' + \frac{1}{\log Y}$$

if such a zero exists.

Thus we get after at most $\left[\frac{\log Y}{2}\right]$ steps a zero $\varrho_N' = \beta_N' + i\gamma_N' \overset{\text{def}}{=} \varrho_1 = \beta_1 + i\gamma_1$ with

$$\beta_1 > \frac{1}{2}, \quad 0 < \gamma_1 < 2\log^2 Y \tag{3.2}$$

(because $\log^2 Y + \gamma_1' < 2\log^2 Y$ if $Y > Y_0$ ineffective constant), such that the domains

$$0 \leqq t \leqq \gamma_1, \quad \sigma > \beta_1 \tag{3.3}$$

and

$$|t - \gamma_1| \leqq 2\log Y, \quad \sigma \geqq \beta_1 + \frac{1}{\log Y} \tag{3.4}$$

are zero-free.

4. Let us introduce the following notations. Let

$$\mu \overset{\text{def}}{=} \log Y, \quad L \overset{\text{def}}{=} \log_2 Y. \tag{4.1}$$

Let k be any positive integer to be chosen later, for which

$$4000L \leqq k \leqq 4400L. \tag{4.2}$$

Let λ be any real number satisfying

$$\frac{\mu}{10^4 L} \leqq \lambda \leqq \frac{2\mu}{10^4 L}. \tag{4.3}$$

Let further

$$A \overset{\text{def}}{=} \exp\{k(\lambda - 2)\}, \tag{4.4}$$

$$B \overset{\text{def}}{=} \exp\{k(\lambda + 2)\}, \tag{4.5}$$

$$g_{k,D}(u, s) \overset{\text{def}}{=} \left(\frac{e^s - e^{-s}}{2s}\right)^k e^{us + \frac{s^2}{D}}, \tag{4.6}$$

$$f(x) \overset{\text{def}}{=} \Pi(x) - \lg x \pm \sqrt{x} \overset{\text{def}}{=} \Pi(x) - \sum_{2 \leqq n \leqq x} \frac{1}{\log n} \pm \sqrt{x}, \tag{4.7}$$

$$H(s) \overset{\text{def}}{=} \frac{\zeta'}{\zeta}(s) + \zeta(s) - 1 \mp \frac{1}{2\left(s - \frac{1}{2}\right)^2}, \tag{4.8}$$

where both in (4.7) and in (4.8) the upper or in both the lower signs are meant.

Further we choose

$$\mu' = \frac{\mu \cdot 4400L}{k}. \tag{4.9}$$

Thus by (4.2)

$$\mu \leqq \mu' \leqq \frac{11}{10}\mu. \tag{4.10}$$

We shall prove that to every real λ satisfying (4.3) there exists an integer k satisfying (4.2) such that $f(x)$ has a sign change in $[A, B]$.

Let λ be fixed in (4.3) and let us assume in contrary that with any k in (4.2) $f(x)$ does not change his sign in $[A, B]$.

5. We shall start with the formula (valid for $\sigma > 1$)

$$\int_1^\infty f(x) \frac{d}{dx}(x^{-s}\log x)\, dx = H(s). \tag{5.1}$$

Replacing s by $s + i\gamma_1$ in (5.1), multiplying by $g_{k,\mu'}(k\lambda, s)$ and integrating with respect to s along the line $\sigma = 2$, using (2.10) we get

$$\begin{aligned} U &= \frac{1}{2\pi i}\int_{(2)} H(s + i\gamma_1) g_{k,\mu'}(k\lambda, s)\, ds = \\ &= \frac{1}{2\pi i}\int_{(2)}\int_1^\infty f(x)\frac{d}{dx}\left(x^{-s-i\gamma_1}\log x\, g_{k,\mu'}(k\lambda, s)\right) dx\, ds = \\ &= \int_1^\infty f(x)\frac{d}{dx}\left\{x^{-i\gamma_1}\log x \cdot \frac{1}{2\pi i}\int_{(2)} g_{k,\mu'}(k\lambda - \log x, s)\, ds\right\} dx = \\ &= \int_1^\infty f(x)\frac{d}{dx}\{x^{-i\gamma_1}\log x\, \mathscr{I}_{k,\mu'}(k\lambda - \log x)\}\, dx = \\ &= \int_1^\infty f(x)\left\{-i\gamma_1 \cdot \frac{x^{-i\gamma_1}}{x}\log x \cdot \mathscr{I}_{k,\mu'}(\log x - k\lambda) + \frac{x^{-i\gamma_1}}{x}\mathscr{I}_{k,\mu'}(\log x - k\lambda) + \right. \\ &\qquad \left. + x^{-i\gamma_1}\log x \cdot \mathscr{I}'_{k,\mu'}(\log x - k\lambda) \cdot \frac{1}{x}\right\} dx = \\ &= \int_1^\infty \frac{f(x)\log x \cdot x^{-i\gamma_1}}{x}\left\{\mathscr{I}_{k,\mu'}(\log x - k\lambda)\left(-i\gamma_1 + \frac{1}{\log x}\right) + \right. \\ &\qquad \left. + \frac{1}{2}\mathscr{I}_{k-1,\mu'}(\log x - k\lambda + 1) - \frac{1}{2}\mathscr{I}_{k-1,\mu'}(\log x - k\lambda - 1)\right\} dx. \end{aligned} \tag{5.2}$$

Now we shall give an upper bound for the right side of (5.2) using the proved properties of the kernel-function $\mathscr{I}_{k,\mu}(u)$ and the fact that $f(x)$ does not change its sign in $[A, B]$ (defined by (4.4)—(4.5)). On the other hand we shall show that the left side can be reduced essentially to a finite powersum, for which we can give a non-trivial lower estimate by suitable choice of k within (4.2) using the second main theorem, which contradicts to the upper estimate sketched above.

6. To estimate U from above we shall split the integral U into three parts

$$U = U_1 + U_2 + U_3 \tag{6.1}$$

where

$$U_1 = \int_1^A, \quad U_2 = \int_A^B, \quad U_3 = \int_B^\infty. \tag{6.2}$$

Considering (3.2), (5.2), (2.3), (4.2)—(4.5) and that $f(x)$ does not change i sign in $[A, B]$ we get

$$|U_2| \leqq \int_A^B \frac{|f(x)|\,k(\lambda+2)}{x}\left\{\frac{11}{10}\gamma_1\mathcal{I}_{k,\mu'}(\log x-k\lambda)+\right.$$

$$\left.+\frac{1}{2}\mathcal{I}_{k-1,\mu'}(\log x-k\lambda+1)+\frac{1}{2}\mathcal{I}_{k-1,\mu'}(\log x-k\lambda-1)\right\}dx \leqq$$

(6.3)
$$\leqq \mu^3\int_A^B\frac{|f(x)|}{x}\{\mathcal{I}_{k,\mu'}(\log x-k\lambda)+\mathcal{I}_{k-1,\mu'}(\log x-k\lambda+1)+$$

$$+\mathcal{I}_{k-1,\mu'}(\log x-k\lambda-1)\}\,dx =$$

$$= \mu^3\left|\int_A^B\frac{f(x)}{x}\{\mathcal{I}_{k,\mu'}(\log x-k\lambda)+\mathcal{I}_{k-1,\mu'}(\log x-k\lambda+1)+\right.$$

$$\left.+\mathcal{I}_{k-1,\mu'}(\log x-k\lambda-1)\}\,dx\right|.$$

Similarly we get owing to $f(x)\log x=O(x)$ introducing the new variab $u=\log x-k\lambda$, using (4.2)—(4.3) and (2.4)

$$|U_3| \leqq \mu^3\int_B^\infty\{\mathcal{I}_{k,\mu'}(\log x-k\lambda)+\mathcal{I}_{k-1,\mu'}(\log x-k\lambda+1)+$$

$$+\mathcal{I}_{k-1,\mu'}(\log x-k\lambda-1)\}\,dx =$$

$$= \mu^3\int_{2k}^\infty\{\mathcal{I}_{k,\mu'}(u)+\mathcal{I}_{k-1,\mu'}(u+1)+\mathcal{I}_{k-1,\mu'}(u-1)\}\,e^{u+k\lambda}\,du \leqq$$

(6.4)
$$\leqq e^{\mu}\int_{2k}^\infty\{e^{-(u-k-1)\mu'}+e^{-(u+1-k-1)\mu'}+e^{-(u-1-k-1)\mu'}\}\,e^u\,du \leqq$$

$$\leqq 3\int_{2k}^\infty e^{-(u-k-2)\mu'+u}\,du = o(1)$$

and analogously

(6.5)
$$|U_1| = o(1).$$

Further, mutatis mutandis, we have

(6.6)
$$U_4 \stackrel{\text{def}}{=} \mu^3\int_B^\infty\frac{f(x)}{x}\{\mathcal{I}_{k,\mu'}(\log x-k\lambda)+\mathcal{I}_{k-1,\mu'}(\log x-k\lambda+1)+$$

$$+\mathcal{I}_{k-1,\mu'}(\log x-k\lambda-1)\}\,dx = o(1)$$

and

(6.7)
$$U_5 \stackrel{\text{def}}{=} \mu^3\int_1^A\frac{f(x)}{x}\{\mathcal{I}_{k,\mu'}(\log x-k\lambda)+\mathcal{I}_{k-1,\mu'}(\log x-k\lambda+1)+$$

$$+\mathcal{I}_{k-1,\mu'}(\log x-k\lambda-1)\}\,dx = o(1).$$

Now (6.3)—(6.8) give immediately with the notation

(6.8)
$$K \stackrel{\text{def}}{=} \int_1^\infty\frac{f(x)}{x}\{\mathcal{I}_{k,\mu'}(\log x-k\lambda)+\mathcal{I}_{k-1,\mu'}(\log x-k\lambda+1)+$$

$$+\mathcal{I}_{k-1,\mu'}(\log x-k\lambda-1)\}\,dx$$

the relation

(6.9)
$$|U| = |U_2|+o(1) \leqq \mu^3|K|+o(1).$$

7. In the following we shall estimate K from above using the formula (valid for $\sigma>1$ with a constant h)

(7.1)
$$\int_1^\infty\frac{f(x)}{x^{s+1}}\,dx = \frac{1}{s}\left\{\int_2^s\left(\frac{\zeta'}{\zeta}(z)+\zeta(z)\right)dz+h\right\}\pm\frac{1}{s-\frac{1}{2}} =$$

$$\stackrel{\text{def}}{=} \varphi(s)\pm\frac{1}{s-\frac{1}{2}},$$

which can be proved easily by partial integration.

Multiplying on both sides by $\frac{1}{2\pi i}G(s)$, where

(7.2)
$$G(s) \stackrel{\text{def}}{=} g_{k,\mu'}(k\lambda, s)+g_{k-1,\mu'}(k\lambda-1, s)+g_{k-1,\mu'}(k\lambda+1, s)$$

and integrating along the line $\sigma=2$ one gets easily by (4.6) and (6.8) the formula

(7.3)
$$K = \pm\frac{1}{2\pi i}\int_{(2)}\frac{G(s)}{s-\frac{1}{2}}\,ds+\frac{1}{2\pi i}\int_{(2)}G(s)\varphi(s)\,ds =$$

$$\stackrel{\text{def}}{=} \pm \quad K_1 \quad + \quad K_2.$$

Shifting the line of integration in K_1 to $\sigma=-\mu'$ we get by easy computation

$$K_1 = G\left(\frac{1}{2}\right)+\frac{1}{2\pi i}\int_{(-\mu')}\frac{G(s)}{s-\frac{1}{2}}\,ds =$$

(7.4)
$$= O\left(e^{k+\frac{k\lambda}{2}}\right)+O\left(\int_{-\infty}^{+\infty}\frac{\exp\left\{\mu'k-(k\lambda-1)\mu'+\mu'-\frac{t^2}{\mu'}\right\}}{\mu'^k}\,dt\right) =$$

$$= O\left(e^{k+\frac{k\lambda}{2}}\right)+O\left(e^{-\mu'(k\lambda-k-2)}\right) = O\left(e^{\frac{1}{2}k\lambda+k}\right).$$

In order to estimate the integral K_2 we transform it on the broken line l defined for $t \geqq 0$ by

$$(7.5)\qquad \begin{aligned} &I_1\colon \sigma = \frac{5}{4} && \text{for} \quad t \geqq 2\mu \\ &I_2\colon \beta_1 + \frac{2}{\mu} \leqq \sigma \leqq \frac{5}{4} && \text{for} \quad t = 2\mu \\ &I_3\colon \sigma = \beta_1 + \frac{2}{\mu} && \text{for} \quad 10 \leqq t \leqq 2\mu \\ &I_4\colon \frac{1}{4} \leqq \sigma \leqq \beta_1 + \frac{2}{\mu} && \text{for} \quad t = 10 \\ &I_5\colon \sigma = \frac{1}{4} && \text{for} \quad 0 \leqq t \leqq 10 \end{aligned}$$

and for $t \leqq 0$ by reflection on the real axis, so that

$$(7.6)\qquad K_2 = \frac{1}{2\pi i} \int_{(l)} G(s)\varphi(s)\, ds$$

because owing to (3.3)—(3.4) $\varphi(s)$ is regular right of l and on l.

Now using already mentioned well-known properties of $\zeta(s)$ (see (2.16)—(2.19) and (3.2)—(3.4) (further the definitions (4.6), (7.2)) we have the following estimate for the integrals on I_ν $(1 \leqq \nu \leqq 5)$:

$$|\mathscr{I}_1| = O\left(\left(\frac{2}{\mu}\right)^{k-2} \exp\left(\frac{5}{4} k\lambda - \frac{4\mu^2}{\mu'}\right)\right) = O(1),$$

$$|\mathscr{I}_2| = O\left(\mu \log \mu \left(\frac{2}{\mu}\right)^{k-1} \exp\left(\frac{5}{4} k\lambda - \frac{4\mu^2}{\mu'}\right)\right) = O(1),$$

$$(7.7)\qquad |\mathscr{I}_3| = O\left(\mu^2 \log \mu \left(\frac{1}{5}\right)^{k-1} \exp\left(\left(\beta_1 + \frac{2}{\mu}\right) k\lambda\right)\right) = O(e^{\beta_1 k\lambda - k}),$$

$$|\mathscr{I}_4| = O\left(\left(\frac{1}{5}\right)^{k-1} \exp\left(\left(\beta_1 + \frac{2}{\mu}\right) k\lambda\right)\right) = O(e^{\beta_1 k\lambda - k}),$$

$$|\mathscr{I}_5| = O(e^{\frac{1}{4}k\lambda + 2k}).$$

Hence

$$(7.8)\qquad |K_2| = O(e^{(\beta_1\lambda - 1)k}) + O\left(e^{\left(\frac{\lambda}{4} + 2\right)k}\right).$$

Further using (7.3)—(7.4) we get

$$(7.9)\qquad |K| = O(e^{(\beta_1\lambda - 1)k}) + O\left(e^{\left(\frac{\lambda}{2} + 1\right)k}\right).$$

Now with an ineffective absolute constant Y_1, by the definition of $\varrho_0 = \beta_0 + i\gamma_0$ and $\varrho_1 = \beta_1 + i\gamma_1$ (in **3**) for $Y > Y_1$ we have (owing to (4.1) and (4.3))

$$(7.10)\qquad \beta_1 - \frac{1}{2} \geqq \beta_0 - \frac{1}{2} \geqq \frac{2 \cdot 10^4 \log_2 Y}{\log Y} = \frac{2 \cdot 10^4 L}{\mu} \geqq \frac{2}{\lambda}.$$

Thus

$$(7.11)\qquad e^{(\beta_1\lambda - 1)k} \geqq e^{\left(\frac{\lambda}{2} + 1\right)k}$$

and so

$$(7.12)\qquad |K| = O(e^{\beta_1\lambda k - k}).$$

Combining this with (6.9) and (4.1) we get already the required upper estimate for the integral U in (5.2), namely we have

$$(7.13)\qquad |U| \leqq \mu^3 |K| + o(1) = O(e^{\beta_1\lambda k - k + 3L}).$$

8. Now we can start with the lower estimate of the left side of U in (5.2) (with the choice of a suitable k). Shifting the line of integration to $\sigma = -\frac{1}{2}$ we get

$$(8.1)\qquad U = \sum_{\varrho} g_{k,\mu'}(k\lambda, \varrho - i\gamma_1) \mp$$

$$\mp \frac{1}{2} \frac{d}{ds} \left(g_{k,\mu'}(k\lambda, s)\right)_{s = -\frac{1}{2} - i\gamma_1} + \frac{1}{2\pi i} \int_{\left(-\frac{1}{2}\right)} H(s + i\gamma_1)\, g_{k,\mu'}(k\lambda, s)\, ds.$$

Easy computation shows that the last integral is $O(1)$ further the second term is

$$\mp \frac{1}{2} \left(\frac{e^{\frac{1}{2} - i\gamma_1} - e^{-\frac{1}{2} + i\gamma_1}}{1 - 2i\gamma_1}\right)^k e^{k\lambda\left(\frac{1}{2} - i\gamma_1\right) + \left(\frac{1}{2} - i\gamma_1\right)^2 \cdot \frac{1}{\mu}} \cdot$$

$$(8.2)\qquad \cdot \left\{ k \frac{\frac{d}{ds}\left(\frac{e^s - e^{-s}}{2s}\right)_{s = \frac{1}{2} - i\gamma_1}}{\frac{e^{\frac{1}{2} - i\gamma_1} - e^{-\frac{1}{2} + i\gamma_1}}{1 - 2i\gamma_1}} + k\lambda + \frac{1 - 2i\gamma_1}{\mu} \right\} =$$

$$= O(k\lambda e^{\frac{k\lambda}{2}}) = O(e^{\frac{k\lambda}{2} + k}) = O(e^{\beta_1\lambda k - k})$$

if we use the inequality (7.11), too.

In the first term (in (8.1)) we can trivially estimate the sum containing the infinitely many zeros with

$$(8.3)\qquad |\gamma - \gamma_1| \geqq 2\mu.$$

Namely, by (2.15), we have for the contribution of these zeros the upper estimate

$$(8.4)\qquad 2 \sum_{n \geqq [2\mu]} c \log(\gamma_1 + n) \left(\frac{2}{n}\right)^k e^{k\lambda + \frac{1 - n^2}{\mu'}} = O(1).$$

Similarly we can easily estimate the sum corresponding to the zeros with

$$(8.5)\qquad 6 \leqq |\gamma - \gamma_1| < 2\mu.$$

The number of these zeros is owing to (2.15) and (3.2) at most

$$(8.6)\qquad 4\mu c \log(\gamma_1+2\mu) \leqq 4c\mu \log(\mu^2+2\mu) = O(\mu^2).$$

Further, by (3.4), we have for the zeros with (8.5)

$$(8.7)\qquad \beta \leqq \beta_1 + \frac{1}{\mu}$$

and so for these zeros

$$(8.8)\qquad |g_{k,\mu'}(k\lambda, \varrho - i\gamma_1)| \leqq \left(\frac{e+1}{2\cdot 6}\right)^k e^{k\lambda\left(\beta_1+\frac{1}{\mu}\right)+\frac{1}{\mu'}} < e^{k\lambda\beta_1-k}.$$

Thus we get for the contributions of zeros with (8.5) to the sum (8.1) the upper bound

$$(8.9)\qquad O(\mu^2 e^{k\lambda\beta_1-k}) = O(e^{k\lambda\beta_1-k+2L}).$$

9. These estimates were naturally independent from the choice of k in (4.2). So the essential part of U is the finite powersum, containing the zeros with

$$(9.1)\qquad |\gamma-\gamma_1| < 6.$$

The number n of such zeros is owing to (2.15) and (3.2)

$$(9.2)\qquad 1 \leqq n \leqq 2\cdot 6\cdot c\log(\gamma_1+6) \leqq 180\log(2\log^2 Y+6) \leqq 400\log_2 Y = 400L.$$

So for the zeros with (9.1) we can use Lemma 2 in the form given in (2.14). Thus choosing

$$(9.3)\qquad m = 4000L$$

we get a positive integer k satisfying (4.2) for which

$$(9.4)\qquad \begin{aligned} |W| &= \Big|\sum_{|\gamma-\gamma_1|<6} g_{k,\mu'}(k\lambda, \varrho-i\gamma_1)\Big| = \\ &= \Big|\sum_{|\gamma-\gamma_1|<6} \left\{\frac{e^{\varrho-i\gamma_1}-e^{-(\varrho-i\gamma_1)}}{2(\varrho-i\gamma_1)}\cdot e^{\lambda(\varrho-i\gamma_1)}+\frac{(\varrho-i\gamma_1)^2}{4400L\mu}\right\}^k\Big| \geqq \\ &\geqq \left(\frac{1}{8e\cdot 12}\right)^{400L}\left(\frac{e^{\beta_1}-e^{-\beta_1}}{2\beta_1}\right)^k e^{k\lambda\beta_1+\frac{\beta_1^2 k}{4400L\mu}} > \\ &> \left(\frac{1}{e^6}\right)^{400L}\cdot e^{k\lambda\beta_1} = e^{k\lambda\beta_1-2400L} \end{aligned}$$

(because for real $x>0$ one has $e^x-e^{-x}>2x$).

Now the estimate $O(1)$ for the integral, the upper bound (8.2) for the residue in (8.1), further the inequalities (8.4) and (8.9) concerning the zeros with $|\gamma-\gamma_1|\geqq 6$ give together with the lower bound (9.4) that owing to (4.2) we have for $|U|$ the lower estimate

$$(9.5)\qquad |U| \geqq \frac{1}{2}e^{k\lambda\beta_1-2400L}$$

which contradicts to (7.13) (again owing to (4.2)). Thus we got that to every λ with (4.3) there exists an integer k with (4.2), such that $f(x)$ in (4.7) and thus also $\Delta_1(x)$ and naturally $\Delta_2(x)$, too, has a sign change in the interval

$$(9.6)\qquad [A, B] = [e^{k\lambda-2k}, e^{k\lambda+2k}].$$

This in itself does not give still the required inequality (1.12), i.e. the assertion of the theorem.

10. However, we can notice that as the total Lebesgue measure of λ's in (4.3) is

$$(10.1)\qquad \frac{\mu}{10^4 L}$$

and k can take at most $[400L]+1$ values, there must exist a fixed k_0 with (4.2) for which there are λ's with Lebesgue measure at least

$$(10.2)\qquad \frac{\mu}{10^4 L}\cdot\frac{1}{401L} = \frac{\mu}{4.01\cdot 10^6 L^2}$$

such that $\Delta_1(x)$ has a sign change in

$$(10.3)\qquad [A_\lambda, B_\lambda] = [e^{k_0\lambda-2k_0}, e^{k_0\lambda+2k_0}] \subset (e^{k_0\lambda-10^4 L}, e^{k_0\lambda+10^4 L}).$$

But as the Lebesgue measure of the λ's belonging to this fixed k_0 is at least the quantity given by (10.2) we can choose among them at least

$$(10.4)\qquad N = \left[\frac{\mu}{4.01\cdot 10^6 L^2}\cdot\frac{1}{2\cdot 10^4 L}\right] > \frac{\mu}{10^{11} L^3}$$

λ_j's $(1\leqq j\leqq N)$, such that the difference of any two of them would be

$$(10.5)\qquad |\lambda_j-\lambda_\nu| \geqq 2\cdot 10^4 L \quad (1\leqq \nu<j\leqq N)$$

and so for the corresponding intervals $[A_{\lambda_j}, B_{\lambda_j}]$ we have by (10.3)

$$(10.6)\qquad [A_{\lambda_j}, B_{\lambda_j}] \cap [A_{\lambda_\nu}, B_{\lambda_\nu}] = \emptyset \quad (1\leqq \nu<j\leqq N).$$

Owing to (4.2)—(4.5) to every k and λ satisfying (4.2) and (4.3) resp. the corresponding interval

$$(10.7)\qquad [A, B] = [e^{k(\lambda-2)}, e^{k(\lambda+2)}] \subset [e^{0,3\mu}, e^{0,9\mu}] = [Y^{0,3}, Y^{0,9}] \subset [2, Y].$$

So, considering (10.4), (10.6) and (10.7), we get at least

$$(10.8)\qquad \frac{\mu}{10^{11}L^3} = \frac{\log Y}{10^{11}(\log_2 Y)^3}$$

disjoint intervals, contained in $[2, Y]$ such that $\Delta_1(x)$ changes its sign in each of these intervals, and thus we finished the proof of Theorem 1.

11. In the case $i=2$ the following slight changes are necessary in the course of proof to get the inequality (1.12) for $Y>Y_2$ effective constant.

Here we do not make any difference whether the Riemann hypothesis is supposed to be true or not, and choose $\varrho_0=\frac{1}{2}+i\gamma_0$ as the zero with the minimal imaginary part ($\gamma_0 \approx 14.13$). Thus we get a zero $\varrho_1 = \beta_1 + i\gamma_1$ with the properties described in (3.2)—(3.4) with the only change that instead of $\beta_1 > \frac{1}{2}$ we have only $\beta_1 \geqq \frac{1}{2}$. In (4.7) we modify the exponent $\frac{1}{2}$ of x to $\frac{1}{4}$ and correspondingly we define $H(s)$ in (4.8) with $\frac{1}{4}\left(s-\frac{1}{4}\right)^{-2}$ in the last term instead of $\frac{1}{2}\left(s-\frac{1}{2}\right)^{-2}$. Thus we get for the K_1 in (7.4) the upper bound

$$|K_1| = O(e^{k+\frac{k\lambda}{4}}) \tag{11.1}$$

and so we get without (7.10) and (7.11) immediately (7.12), i.e.

$$|K| = O(e^{\beta_1 \lambda k - k}). \tag{11.2}$$

Further we get for the residue in (8.1) in the point $s = \frac{1}{4} - i\gamma_1$ the upper estimate $O(e^{\frac{k\lambda}{4}+k}) = O(e^{\beta_1 \lambda k - k})$ and the other parts of the proof remain again valid without any change.

The cases $i=3$ and $i=4$ can be treated similarly to the cases $i=1$ and $i=2$, they are even easier, so we do not work them out.

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ON AN ISOPERIMETRIC PROBLEM

by

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Scatter in the plane a finite number of line segments. The region *enclosed* by the segments is defined as the set of those points which cannot be connected by a Jordan arc with the exterior of the convex hull of the segments, without intersecting at least one segment. How should the segments be arranged so as to maximize the area of the region enclosed by them? Specializing a more general problem of L. FEJES TÓTH [1], G. HAJÓS raised the following question. Prove or disprove the intuitively obvious conjecture that, in an extremal arrangement, the region enclosed by the segments is a simple polygon.

Fejes Tóth [2] claimed to have proved the above conjecture in the special case when only polygonal arrangements are compared, i.e., arrangements in which each endpoint of a segment is the endpoint of exactly one other segment, with the additional condition that no subset of the segments has this property. Soon after the publication of [2], Fejes Tóth observed that his proof was wrong and he called my attention to the problem of finding a correct proof.

Our result is contained in the following

THEOREM 1. *Among all polygons of given side lengths the convex polygon inscribed into a circle has the greatest area.*

Here the area of a polygon is defined as the area enclosed by the sides of the polygon.

The convex hull of the vertices of a polygon P will be denoted by conv P. Our Theorem 1 is an immediate consequence of the following stronger result:

THEOREM 2. *Let P and P_c be two polygons with the same lengths of sides, and suppose that P_c is a convex polygon inscribed into a circle. Then the area of* conv P *is at most that of P_c.*

Equality holds here if and only if P is also a convex polygon inscribed into a circle.

REMARK. For polygons not intersecting themselves the proof of Theorem 2 is easy and well-known (see e.g. [3]).

The proof of Theorem 2 is based on the following simple

LEMMA. *Let Q be a strictly convex polygon. Suppose that the area of the triangle xyz is minimum among the triangles spanned by the vertices of Q. Then two sides of Δxyz lie on the boundary of Q.*

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