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ON THE REMAINDER TERM OF THE PRIME NUMBER FORMULA III

Sign changes of $\pi(x)-\operatorname{li} x$

by

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1. Riemann [14] asserted in his famous paper in 1859

$$\pi(x) < \operatorname{li} x = \int_0^x \frac{dt}{\log t} \qquad (x > 2). \tag{1.1}$$

About 50 years later E. SCHMIDT [15] proved that dealing with the function

$$\Pi(x) = \sum_{p^m \leqq x} \frac{1}{m} = \sum_{\nu \geqq 1} \frac{1}{\nu}\pi\left(x^{\frac{1}{\nu}}\right) \tag{1.2}$$

the analogous difference $\Pi(x)-\operatorname{li} x$ changes sign infinitely often as $x\to\infty$. This naturally could not decide Riemann's problem, since

$$\Pi(x)-\pi(x) \sim \frac{\sqrt{x}}{\log x}. \tag{1.3}$$

On the other hand he observed that if (1.1) is true then the famous Riemann conjecture concerning the zeros of $\zeta(s)$ is also true. This remark made naturally even more interesting the question whether (1.1) is true or not.

The truth of (1.1) seemed to be supported by the calculations of D. N. LEHMER [9], who showed in 1914 that (1.1) is valid for all $x<10^7$. But LITTLEWOOD [10] in the same year disproved Riemann's assertion (1.1) proving that $\pi(x)-\operatorname{li} x$ has infinitely many sign changes.

2. However, after Littlewood's work the curious problem arose that it was known that $\pi(x)-\operatorname{li} x$ changes sign infinitely many times but no explicit bound X could be given that (1.1) would be false for any $x<X$. Namely, Littlewood's proof was completely ineffective and so it could not give any explicit upper bound for the first sign change of $\pi(x)-\operatorname{li} x$. The problem of the effectivization turned out to be a very deep one. Only 14 years later in 1955 could SKEWES [16] give the first explicite, however, incredibly large upper bound

$$e_4(7{,}705) \tag{2.1}$$

for the first sign change, where $e_4(x)$ means the four times iterated exponential function. In 1966 this was improved by SHERMAN LEHMAN [8] to

$$1{,}65\cdot 10^{1165}. \tag{2.2}$$

3. Littlewood's proof also did not give any information about the problem how often $\pi(x)-\mathrm{li}\, x$ changes his sign. So the more deep and general problem was: what can one say from the oscillatorical behaviour of various forms of the remainder term of the prime number formula.

Let

$$\begin{aligned} \Delta_1(x) &\overset{\mathrm{def}}{=} \pi(x)-\mathrm{li}\, x, \\ \Delta_2(x) &\overset{\mathrm{def}}{=} \Pi(x)-\mathrm{li}\, x, \\ \Delta_3(x) &\overset{\mathrm{def}}{=} \vartheta(x)-x = \sum_{p\leqq x} \log p - x, \\ \Delta_4(x) &\overset{\mathrm{def}}{=} \psi(x)-x = \sum_{p^m\leqq x} \log p - x, \end{aligned} \tag{3.1}$$

and let $V_i(Y)$ denote the number of sign changes of $\Delta_i(x)$ in $[2, Y]$.

The first result in this direction is due to PÓLYA [12] who showed

$$\varlimsup_{Y\to\infty} \frac{V_4(Y)}{\log Y} > 0 \tag{3.2}$$

but he could not prove anything for the more difficult case $i=1$. For $V_1(Y)$ the first and very important though conditional result was proved by INGHAM [3] in 1936. Let Θ denote the least upper bound of the real parts of the zeros of $\zeta(s)$. Ingham showed that if there is a zero on the line $\sigma=\Theta$ then for $Y>Y_0$ $\Delta_1(x)$ has a sign change in every interval of the form

$$[Y, c_0 Y] \tag{3.3}$$

with a constant c_0. Ingham's result trivially implies that if the condition is satisfied then

$$\varliminf_{Y\to\infty} \frac{V_1(Y)}{\log Y} > 0. \tag{3.4}$$

We must note, however, that Ingham's condition is very deep. It is satisfied naturally if, e.g., the Riemann hypothesis is true, but it implies $\Theta<1$, i.e., the so called quasi Riemann hypothesis. A second disadvantage of this beautiful theorem is that the constant c_0 and thus also the lower bound for the left side of (3.4) cannot be computed effectively even if we suppose the Riemann hypothesis.

4. The first unconditional lower bound for $V_1(Y)$ was given by S. KNAPOWSKI [4], [5] in 1961 and 1962 using Turán's method. He proved the completely effective lower estimate

$$V_1(Y) > e^{-35} \log_4 Y \quad \text{for} \quad Y > e_5(35) \tag{4.1}$$

where $\log_4 Y$ denotes the four times iterated logarithm function, i.e., $\log_4 Y = \log\log\log\log Y$. He also showed the sharper but ineffective inequality

$$\varliminf_{Y\to\infty} \frac{V_1(Y)}{\log_2 Y} > 0. \tag{4.2}$$

These results were improved in 1974—76 by S. KNAPOWSKI and P. TURÁN [6], [7]. In the first work [6] they proved that $\pi(x)-\mathrm{li}\, x$ changes sign in every interval of the form

$$[Y, Y \exp\{\log^{3/4} Y (\log_2 Y)^4\}] \tag{4.3}$$

if $Y>Y_0$ ineffective constant. This implies for $V_1(Y)$ the ineffective inequality

$$\varliminf_{Y\to\infty} \frac{V_1(Y)}{\log^{1/4} Y (\log_2 Y)^{-4}} > 0. \tag{4.4}$$

In the second work [7] they showed the existence of effectively computable positive constants c_1 and c_2 such that for $Y>c_1$ the inequality

$$V_1(Y) > c_2 \log_3 Y \tag{4.5}$$

holds.

This result was recently improved by the author [11] to

$$V_1(Y) > C_1 (\log_2 Y)^{C_2} \quad \text{for} \quad Y > C_3 \tag{4.6}$$

where C_1, C_2 and C_3 are positive effective constants.

5. In this paper we shall prove using Turán's method

THEOREM 1. *For $Y>Y_i$ $(1\leqq i\leqq 4)$ the interval*

$$[Y, Y \exp\{63 \sqrt{\log Y} \log_2 Y\}] \tag{5.1}$$

contains a sign-change of $\Delta_i(x)$ $(1\leqq i\leqq 4)$, where Y_1 and Y_3 are ineffective constants, Y_2 and Y_4 are effective ones.

This theorem already implies the partially ineffective inequality (see **10**)

$$\varliminf_{Y\to\infty} \frac{V_i(Y)}{\sqrt{\log Y} (\log_2 Y)^{-1}} > 0. \tag{5.2}$$

However, using a more explicit form of Theorem 1 it will be possible to deduce from it the inequality (5.2) in an effective form for all $i\leqq 4$, namely we state further

THEOREM 2. *There exist effectively computable constants c_3 and c_4 such that for $Y>c_3$ the inequality*

$$V_i(Y) > c_4 \frac{\sqrt{\log Y}}{\log_2 Y} \qquad (1\leqq i\leqq 4) \tag{5.3}$$

holds.

The reason why we stated the seemingly weaker (because for $i=1, 3$ ineffective) Theorem 1 as a separate theorem is that it contains a localization for the sign changes of $\Delta_i(x)$, whereas Theorem 2 gives only a lower bound for the total number of sign changes without any localization (or more precisely with a very weak but effective localization, which one can read out from the proof, namely the interval $[\exp(c \sqrt{\log Y} (\log_2 Y)^{-1}); Y]$.

We mention that part IV of this series will be devoted to the proof of the partially ineffective improvement of (5.2), namely, we shall prove

$$V_i(Y) > 10^{-11}\frac{\log Y}{(\log_2 Y)^3} \quad \text{for} \quad Y > Y_i \tag{5.4}$$

where the constants Y_i are effective for $i=2$ and 4 and ineffective for $i=1$ and 3.

6. We shall give only the proof for the most interesting and deep case $i=1$. Our proof implicitly contains the case $i=2$, too, with ineffective Y_2. In **24** we shall mention the slight changes in the course of proof of Theorem 1 which make possible to get for Y_2 an explicitly calculable value. The cases $i=3$ and 4 are very similar to the cases $i=1$ and 2, resp., they are even easier, so we do not work them out.

Theorem 1 will be the immediate consequence of the following

LEMMA 1. *If for a* $Z > c_5$ *(effective constant) the function* $\zeta(s)$ *has a zero* $\varrho^* = \beta^* + i\gamma^*$ *with*

$$\beta^* \geqq \frac{1}{2} + \frac{10\log\gamma^*}{4\sqrt{\log Z}\,(\log_2 Z)^{-1}}, \tag{6.1}$$

$$0 < \gamma^* \leqq \exp\left(\frac{1}{10}\sqrt{\log Z}\,(\log_2 Z)^{-1}\right)$$

then the interval

$$I(Z) = \left(Z\exp\left(-31\sqrt{\log Z}\log_2 Z\right),\ Z\exp\left(31\sqrt{\log Z}\log_2 Z\right)\right) \tag{6.2}$$

contains a sign-change of $\Delta_1(x)$.

Namely, it is easy to see that if the Riemann conjecture is not true, than any zero $\varrho^* = \beta^* + i\gamma^*$ with $\beta^* > \frac{1}{2}$ satisfies (6.1) if $Z > Z_0(\varrho^*)$, and thus in this case Theorem 1 follows from Lemma 1 indeed.

On the other hand if the Riemann conjecture is true then Ingham's quoted theorem (see 3.3) shows the validity of Theorem 1 even in a stronger form. (For the sake of completeness we note that Ingham's theorem with essentially unchanged proof is true for the cases $i=2, 3, 4$, too.)

With the proof of our lemma we shall use the following notations:

$$L \stackrel{\text{def}}{=} \log Z, \tag{6.3}$$

$$M \stackrel{\text{def}}{=} 100(\log_2 Z)^2 = 100\log^2 L, \tag{6.4}$$

$$\lambda \stackrel{\text{def}}{=} \frac{\sqrt{\log Z}}{10\log_2 Z} = \sqrt{\frac{L}{M}}. \tag{6.5}$$

Let k be a real number to be determined later, which will be restricted at present only by

$$M \leqq k \leqq M\left(1+\frac{1}{L}\right). \tag{6.6}$$

Let further

$$\mu \stackrel{\text{def}}{=} k\lambda^2, \tag{6.7}$$

$$A \stackrel{\text{def}}{=} \exp(\mu - 3k\lambda), \tag{6.8}$$

$$B \stackrel{\text{def}}{=} \exp(\mu + 3k\lambda), \tag{6.9}$$

$$f(x) \stackrel{\text{def}}{=} \Pi(x) - \lg x \pm \sqrt{x} \stackrel{\text{def}}{=} \Pi(x) - \sum_{2\leqq n\leqq x}\frac{1}{\log n} \pm \sqrt{x}, \tag{6.10}$$

$$H(s) \stackrel{\text{def}}{=} \frac{\zeta'}{\zeta}(s) + \zeta(s) - 1 \mp \frac{1}{2\left(s-\frac{1}{2}\right)^2}, \tag{6.11}$$

where both in (6.10) and (6.11) the upper or in both the lower signs are meant.

Let us assume that $f(x)$ has no sign change in the interval

$$I' \stackrel{\text{def}}{=} [A, B] \subset I(Z). \tag{6.12}$$

We shall show that this assumption leads to contradiction, and thus proves the existence of

$$x', x'' \in I' \subset I(Z) \tag{6.13}$$

for which

$$\Pi(x) - \lg x' > \sqrt{x'} \tag{6.14}$$

and

$$\Pi(x'') - \lg x'' < -\sqrt{x''} \tag{6.15}$$

and so the inequalities

$$\pi(x') - \mathrm{li}\, x' > \frac{1}{2}\sqrt{x'} \tag{6.16}$$

and

$$\pi(x'') - \mathrm{li}\, x'' < -\frac{1}{2}\sqrt{x''} \tag{6.17}$$

hold owing to the trivial estimate

$$\Pi(x) - \pi(x) = O\left(\frac{\sqrt{x}}{\log x}\right) \tag{6.18}$$

and

$$\lg x = \mathrm{li}\, x + O(1). \tag{6.19}$$

7. We shall distinguish the following two cases.

Case A. There exists a zero $\varrho_0=\beta_0+i\gamma_0$ with

$$\beta_0 \geqq \frac{1}{2}+\frac{\log\gamma_0}{4\lambda}, \quad 0<\gamma_0\leqq\lambda^5. \tag{7.1}$$

Then let $\varrho_1'=\beta_1'+i\gamma_1'$ be the zero with the maximal real part β_1' among those satisfying (7.1). Let successively ϱ_{n+1}' be the zero with the maximal real part β_{n+1}' satisfying

$$\gamma_n'\leqq\gamma_{n+1}'\leqq\gamma_n'+2\lambda, \quad \beta_{n+1}'\geqq\beta_n'+\frac{1}{\lambda} \tag{7.2}$$

if such a zero exists. Thus we get after at most $\left[\frac{\lambda}{2}\right]$ steps the existence of a zero $\varrho_N'=\beta_N'+i\gamma_N'\overset{\text{def}}{=}\varrho_1=\beta_1+i\gamma_1$ with

$$\beta_1\geqq\frac{1}{2}+\frac{1}{\lambda}, \quad 0<\gamma_1\leqq 2\lambda^5 \tag{7.3}$$

for which the domains

$$|t|\leqq\lambda^5, \quad \sigma>\beta_1 \tag{7.4}$$

and

$$|t-\gamma_1|\leqq 2\lambda, \quad \sigma>\beta_1+\frac{1}{\lambda} \tag{7.5}$$

are zero-free.

Case B. There is no zero satisfying (7.1).

Then let $\varrho_1=\beta_1+i\gamma_1$ be any zero satisfying (6.1); i.e., with our new notations in this case we have

$$\beta_1\geqq\frac{1}{2}+\frac{\log\gamma_1}{4\lambda}, \quad \lambda^5<\gamma_1\leqq e^{\lambda}. \tag{7.6}$$

8. Then in both cases our starting formula is:

$$\int_1^{\infty} f(x)\frac{d}{dx}(x^{-s}\log x)\,dx = H(s) \tag{8.1}$$

which can be proved easily by partial summation for $\sigma>1$.

Further we shall use the formula ($A>0$, B arbitrary complex)

$$\begin{aligned}\frac{1}{2\pi i}\int_{(2)} e^{As^2+Bs}\,ds &= \exp\left(-\frac{B^2}{4A}\right)\cdot\frac{1}{2\pi i}\int_{(2)} e^{\left(\sqrt{A}s+\frac{B}{2\sqrt{A}}\right)^2}ds = \\ &= \exp\left(-\frac{B^2}{4A}\right)\cdot\frac{1}{\sqrt{A}}\cdot\frac{1}{2\pi i}\int_{(0)} e^{z^2}dz=\frac{1}{2\sqrt{\pi A}}\exp\left(-\frac{B^2}{4A}\right).\end{aligned} \tag{8.2}$$

Using (8.1) with $s+i\gamma_1$ instead of s, multiplying both sides by $e^{ks^2+\mu s}$ and integrating along $\sigma=2$ we get our basic identity as follows:

$$\begin{aligned}U &= \frac{1}{2\pi i}\int_{(2)} H(s+i\gamma_1)e^{ks^2+\mu s}ds = \\ &= \frac{1}{2\pi i}\int_{(2)}\int_1^{\infty} f(x)\frac{d}{dx}\left(x^{-s-i\gamma_1}\log x\cdot e^{ks^2+\mu s}\right)dx\,ds = \\ &= \int_1^{\infty} f(x)\frac{d}{dx}\left\{x^{-i\gamma_1}\log x\cdot\frac{1}{2\pi i}\int_{(2)} e^{ks^2+(\mu-\log x)s}ds\right\}dx = \\ &= \frac{1}{2\sqrt{\pi k}}\int_1^{\infty} f(x)\frac{d}{dx}\left\{x^{-i\gamma_1}\log x\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)\right\}dx = \\ &= \frac{1}{2\sqrt{\pi k}}\int_1^{\infty}\frac{f(x)}{x}x^{-i\gamma_1}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)\left\{-i\gamma_1\log x+1+\log x\,\frac{\mu-\log x}{2k}\right\}dx.\end{aligned} \tag{8.3}$$

9. The main idea of the proof is that supposing that $f(x)$ does not change its sign in I' one can deduce an upper bound for the absolute value of the right side of (8.3); on the other hand one can give a lower estimate for the absolute value of the left side of (8.3) choosing k suitably in the interval (6.6) and these two estimations will contradict.

In course of the proof of the lower bound for the left side of (8.3) essential role will be played by a powersum theorem of Vera T. Sós—P. Turán, which we state as Theorem A of the Appendix.

10. We split the integral U on the right side of (8.3) into the following three parts:

$$U=U_1+U_2+U_3 \tag{10.1}$$

where

$$U_1=\int_1^A, \quad U_2=\int_A^B, \quad U_3=\int_B^{\infty}. \tag{10.2}$$

Considering our notations (6.7)—(6.12) and (8.3) we have

$$\begin{aligned}|U_2| &\leqq \frac{1}{2\sqrt{\pi k}}\int_A^B\frac{|f(x)|\log x}{x}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)\left(\gamma_1+\frac{1}{\log x}+\frac{|\mu-\log x|}{2k}\right)dx\leqq \\ &\leqq \frac{1}{2\sqrt{\pi k}}\int_A^B\frac{|f(x)|\mu\left(1+\frac{3}{\lambda}\right)}{x}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)\left(\gamma_1+1+\frac{3k\lambda}{2k}\right)dx\leqq \\ &\leqq \frac{2\mu(\gamma_1+\lambda)}{2\sqrt{\pi k}}\int_A^B\frac{|f(x)|}{x}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx = \\ &= \frac{2\mu(\gamma_1+\lambda)}{2\sqrt{\pi k}}\left|\int_A^B\frac{f(x)}{x}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx\right|\end{aligned} \tag{10.3}$$

since $f(x)$ does not change its sign in $[A, B]$.

On the other hand we can trivially estimate

(10.4)
$$|U_4| \overset{\text{def}}{=} \left| \int_B^\infty \frac{f(x)}{x} \exp\left(-\frac{(\log x-\mu)^2}{4k}\right) dx \right| \leqq$$
$$\leqq \int_B^\infty \exp\left(-\frac{(\log x-\mu)^2}{4k}\right) dx =$$
$$= \int_{3k\lambda}^\infty \exp\left(\mu+y-\frac{y^2}{4k}\right) dy \leqq$$
$$\leqq \int_{3k\lambda}^\infty \exp(\mu+y-2y-2k\lambda^2)\, dy \leqq \int_0^\infty e^{-\mu-y} dy = e^{-\mu}$$

and analogously

(10.5)
$$|U_5| \overset{\text{def}}{=} \left| \int_1^A \frac{f(x)}{x} \exp\left(-\frac{(\log x-\mu)^2}{4k}\right) dx \right| \leqq$$
$$\leqq \int_1^{e^{\mu-3k\lambda}} \exp\left(-\frac{9k^2\lambda^2}{4k}\right) dx = e^{\mu-3k\lambda-\frac{9}{4}\mu} \leqq e^{-\mu}.$$

Naturally we have mutatis mutandis

(10.6) $$U_1 \leqq e^{-\mu} \quad \text{and} \quad U_3 \leqq e^{-\mu}.$$

Thus using (10.1)—(10.6) we can change the intervals in the left and right side of (10.3) from $[A, B]$ to $[1, \infty)$ and so with the notation

(10.7) $$K \overset{\text{def}}{=} \frac{1}{2\sqrt{\pi k}} \int_1^\infty \frac{f(x)}{x} \exp\left(-\frac{(\log x-\mu)^2}{4k}\right) dx$$

we get

(10.8) $$|U| = |U_2| + o(1) = 2\mu(\gamma_1+\lambda)|K| + o(1).$$

11. With the upper estimate of $|K|$ our starting formula will be

(11.1)
$$\int_1^\infty \frac{f(x)}{x^{s+1}} dx = \frac{1}{s}\left\{\int_2^s \left(\frac{\zeta'}{\zeta}(z)+\zeta(z)\right) dz + h\right\} \pm \frac{1}{s-\frac{1}{2}} =$$
$$\overset{\text{def}}{=} \varphi(s) \pm \frac{1}{s-\frac{1}{2}}$$

which is valid for $\sigma>1$ with a constant h. This can be proved easily by partial integration.

Now multiplication by $e^{ks^2+\mu s}$ and integration along the line $\sigma=2$ gives:

(11.2)
$$K = \pm \frac{1}{2\pi i} \int_{(2)} \frac{e^{ks^2+\mu s}}{s-\frac{1}{2}} ds + \frac{1}{2\pi i} \int_{(2)} \varphi(s) e^{ks^2+\mu s} ds =$$
$$\overset{\text{def}}{=} \pm K_1 + K_2.$$

Applying Cauchy's theorem for K_1 we get

(11.3) $$K_1 = e^{\frac{k}{4}+\frac{\mu}{2}} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-kt^2+i\mu t}}{it-\frac{1}{2}} dt = e^{\frac{k}{4}+\frac{\mu}{2}} + O(1).$$

12. Now in Case A we transform the way of integration in K_2 on the broken line l defined for $t\geqq 0$ by

(12.1)
$$l_1:\ \sigma = \frac{5}{4} \qquad \text{for} \quad t \geqq \lambda,$$
$$l_2:\ \beta_1+\frac{1}{\mu} \leqq \sigma \leqq \frac{5}{4} \qquad \text{for} \quad t = \lambda,$$
$$l_3:\ \sigma = \beta_1+\frac{1}{\mu} \qquad \text{for} \quad 10 \leqq t \leqq \lambda,$$
$$l_4:\ \frac{1}{4} \leqq \sigma \leqq \beta_1+\frac{1}{\mu} \qquad \text{for} \quad t = 10,$$
$$l_5:\ \sigma = \frac{1}{4} \qquad \text{for} \quad 0 \leqq t \leqq 10,$$

and for $t\leqq 0$ by reflection on the real axis, because by the choice of ϱ_1 (see (7.3)—(7.5)) $\varphi(s)$ is regular right of the broken line l and on l. Thus we have

(12.2) $$K_2 = \frac{1}{2\pi i} \int_{(l)} \varphi(s) e^{ks^2+\mu s} ds.$$

We shall use the fact, that if $\zeta(s)$ has no zero in the domain

(12.3) $$\sigma > \beta\left(\geqq \frac{1}{2}\right), \quad |t| \leqq T+1$$

then for

(12.4) $$\sigma \geqq \beta+\eta, \quad 2 \leqq |t| \leqq T$$

one has

(12.5) $$\left|\frac{\zeta'}{\zeta}(z)\right| = O\left(\frac{\log |t|}{\eta}\right).$$

This follows easily from Satz 4.1 of PRACHAR [13] (p. 225) in case of $k=1$.) Further we use the classical estimate

(12.6) $$|\zeta(z)| = O(\sqrt{|t|}) \quad \text{for} \quad \sigma \geqq \frac{1}{2}, \quad |t| \geqq 10.$$

From (12.5) and (12.6) we get by easy computation for the integrals J_i on the intervals I_i $(1 \leqq i \leqq 5)$ the estimates

$$|J_1| = O\left(\exp\left(k \cdot \frac{25}{16} - k\lambda^2 + \frac{5}{4}\mu\right)\right) \leqq e^{\frac{\mu}{3}},$$

$$|J_2| = O\left(\log\lambda \cdot \mu \exp\left(k \cdot \frac{25}{16} - k\lambda^2 + \frac{5}{4}\mu\right)\right) \leqq e^{\frac{\mu}{3}},$$

(12.7) $$|J_3| = O\left(\log\lambda \cdot \mu \exp\left(-98k + \mu\left(\beta_1 + \frac{1}{\mu}\right)\right)\right) \leqq e^{-97k+\mu\beta_1},$$

$$|J_4| = O\left(\exp\left(-98k + \mu\left(\beta_1 + \frac{1}{\mu}\right)\right)\right) \leqq e^{-97k+\mu\beta_1},$$

$$|J_5| = O\left(\exp\left(\frac{k}{16} + \frac{\mu}{4}\right)\right) \leqq e^{\frac{\mu}{3}}.$$

Thus we have considering (11.2)—(11.3) and (12.7) the upper bound

(12.8) $$|K| = O\left(e^{-97M+\mu\beta_1} + e^{\frac{k}{4}+\frac{\mu}{2}}\right).$$

Further, using (7.3), (10.8) and (12.8) we get

(12.9) $$|U| \leqq e^{-96M\mu\beta_1} + e^{\frac{k}{4}+\mu\beta_1-\frac{\mu}{\lambda}+\lambda} \leqq \frac{e^{k\beta_1^2+\mu\beta_1}}{e^{96M}}.$$

13. Now we shall give a lower bound for the absolute value of the integral U on the left of (8.3) by suitable choice of k using Theorem A of the Appendix. Shifting the line of integration to $\sigma = -\frac{1}{2}$ we get

(13.1) $$U = \sum_{\varrho} \exp\{k[(\varrho - i\gamma_1)^2 + \lambda^2(\varrho - i\gamma_1)]\} \mp \mp\frac{1}{2}\frac{d}{ds}\left(e^{ks^2+\mu s}\right)_{s=\frac{1}{2}-i\gamma_1} + + \frac{1}{2\pi i}\int_{(-\frac{1}{2})} H(s+i\gamma_1)\exp(ks^2+\mu s)\,ds.$$

Easy computation shows that the last integral is $O(1)$ and the second residue is absolutely

(13.2) $$\leqq \frac{1}{2}\left(2k\left|\frac{1}{2} - i\gamma_1\right| + \mu\right)e^{\frac{k}{4}+\frac{\mu}{2}-k\gamma_1^2} \leqq \frac{e^{k\beta_1^2+\mu\beta_1}}{e^{99M}}.$$

Further we shall use that the number of zeros of $\zeta(s)$ in $T \leqq t \leqq T+1$ is

(13.3) $$< c\log T \quad \text{where} \quad c = 15 \quad \text{for} \quad T > T_0$$

(see, e.g., W. J. ELLISON—M. MENDÈS FRANCE [1] p. 165).

The number of zeros with

(13.4) $$10 \leqq |\gamma - \gamma_1| \leqq 2\lambda$$

is owing to (13.3) and (7.3)

(13.5) $$\leqq 2 \cdot 2\lambda \cdot c \cdot 6\log\lambda \leqq \mu.$$

Thus for the contribution of such zeros to the infinite powersum we get by (7.5) the upper bound

(13.6) $$\mu\exp\left\{k\left(\beta_1 + \frac{1}{\mu}\right)^2 - 100k + \mu\left(\beta_1 + \frac{1}{\mu}\right)\right\} \leqq \frac{e^{k\beta_1^2+\mu\beta_1}}{e^{99M}}.$$

Using again (13.3) we have for the contribution of zeros with $|\gamma - \gamma_1| > 2\lambda$ the upper estimate

(13.7) $$2\sum_{n=[2\lambda]}^{\infty} c\log(\gamma_1+n)\exp(k - kn^2 + \mu) = O(1).$$

So the number of remaining zeros with

(13.8) $$|\gamma - \gamma_1| < 10$$

is again by (13.3) and (6.5)

(13.9) $$1 \leqq n < 20 \cdot c\log(\gamma_1 + 10) < 300\log(2\lambda^5 + 10) < 900\log L.$$

Now we can apply Theorem A of the Appendix for the numbers

(13.10) $$\alpha_j = (\varrho_j - i\gamma_1)^2 + \lambda^2(\varrho_j - i\gamma_1)$$

with

(13.11) $$|\gamma_j - \gamma_1| < 10$$

and with the choice

(13.12) $$a = M, \quad d = \frac{M}{L}.$$

So we get the existence of a k satisfying (6.6) for which

(13.13) $$|W_1| = \left|\sum_{|\gamma_j-\gamma_1|<10} \exp\{k[(\varrho_j - i\gamma_1)^2 + \lambda^2(\varrho_j - i\gamma_1)]\}\right| \geqq \geqq \frac{e^{k\beta_1^2+k\lambda^2\beta_1}}{(30L)^{900\log L}} \geqq \frac{e^{k\beta_1^2+\mu\beta_1}}{e^{1000\log^2 L}} = \frac{e^{k\beta_1^2+\mu\beta_1}}{e^{10M}}.$$

Thus (13.2), (13.6), (13.7) and (13.13) together imply

(13.14) $$|U| \geqq \frac{e^{k\beta_1^2+\mu\beta_1}}{2e^{10M}}$$

which contradicts to (12.9) and thus proves the lemma in Case A.

14. In Case B we transform the way of integration in K_2 on the broken line l defined for $t \geqq 0$ by

$$(14.1)\qquad \begin{aligned} &l_1\colon \sigma = \frac{5}{4} && \text{for } t \geqq \lambda, \\ &l_2\colon \alpha + \frac{1}{\mu} \leqq \sigma \leqq \frac{5}{4} && \text{for } t = \lambda, \\ &l_3\colon \sigma = \alpha + \frac{1}{\mu} && \text{for } 10 \leqq t \leqq \lambda, \\ &l_4\colon \frac{1}{4} \leqq \sigma \leqq \alpha + \frac{1}{\mu} && \text{for } t = 10, \\ &l_5\colon \sigma = \frac{1}{4} && \text{for } 0 \leqq t \leqq 10, \end{aligned}$$

where

$$(14.2)\qquad \alpha = \frac{1}{2} + \frac{\log \lambda}{2\lambda}$$

and for $t \leqq 0$ by reflection on the real axis, because there is no zero with (7.1) and so $\varphi(s)$ is regular right of l and on l.

Considering that the only change compared to the way in (12.1) is that β_1 is replaced by α, and so we get for K_2 analogously to (12.7) the estimate

$$(14.3)\qquad |K_2| = O\left(e^{-97M+\mu\alpha} + e^{\frac{\mu}{3}}\right) \leqq e^{k\beta_1^2+\mu\alpha}.$$

Taking in account (7.6) and (14.2) we have

$$(14.4)\qquad \beta_1 - \alpha \geqq \frac{\log \gamma_1}{4\lambda} - \frac{\log \lambda}{2\lambda} \geqq \frac{\log \gamma_1}{4\lambda} - \frac{\frac{1}{5}\log \gamma_1}{2\lambda} > \frac{\log \gamma_1}{8\lambda}.$$

Thus estimating here $|U|$ we get from (10.8), (14.3) and (14.4) the inequality

$$(14.5)\qquad |U| \leqq e^{2\lambda} \frac{e^{k\beta_1^2+\beta_1}}{e^{\mu \cdot \frac{\log \gamma_1}{8\lambda}}} \leqq \frac{e^{k\beta_1^2+\mu\beta_1}}{e^{\frac{1}{9} M\lambda \log \gamma_1}}.$$

To get a lower bound for $|U|$ we have to consider that in (13.1) the integral is again $O(1)$; the residue is owing to (7.6) absolutely

$$(14.6)\qquad \leqq \frac{1}{2}\left(2k\left|\frac{1}{2} - i\gamma_1\right| + \mu\right) e^{\frac{k}{4} + \frac{\mu}{2} - k\gamma_1^2} \leqq e^{2\lambda} \cdot e^{\frac{k}{4} + \frac{\mu}{2} - k\lambda^{10}} \leqq e^{\mu - \lambda^8 \mu} \leqq 1.$$

Further the infinite powersum belonging to zeros with $|\gamma - \gamma_1| \geqq 2\lambda$ is, as given by (13.7), $O(1)$.

Thus here again only the behaviour of the finite power sum belonging to the zeros with

$$(14.7)\qquad |\gamma - \gamma_1| < 2\lambda$$

is interesting.

The number of terms is here by (13.3)

$$(14.8)\qquad 1 \leqq n < 4\lambda \cdot 15 \cdot \log(\gamma_1 + 2\lambda) < 61\lambda \log \gamma_1$$

and thus proceeding as in (13.10)—(13.13) we get for our finite powersum by appropriate choice of k satisfying (6.6) the lower bound:

$$(14.9)\qquad \begin{aligned} |W_2| &= \left| \sum_{|\gamma_j - \gamma_1| < 2\lambda} \exp\{k[(\varrho_j - i\gamma_1)^2 + \lambda^2(\varrho_j - i\gamma_1)]\}\right| \geqq \\ &\geqq \frac{e^{k\beta_1^2 + k\lambda^2\beta_1}}{(30L)^{61\lambda \log \gamma_1}} \geqq \frac{e^{k\beta_1^2+\mu\beta_1}}{e^{62\lambda \log \gamma_1 \log L}} \geqq \frac{e^{k\beta_1^2+\mu\beta_1}}{e^{\lambda \log \gamma_1 \log^2 L}} = \frac{e^{k\beta_1^2+\mu\beta_1}}{e^{\frac{1}{100}\lambda M \log \gamma_1}}. \end{aligned}$$

This implies essentially the same estimate for $|U|$, namely taking in account the upper estimate of the integral, the residue and the contribution of the zeros with $|\gamma - \gamma_1| > 2\lambda$ we have

$$(14.10)\qquad |U| \geqq \frac{e^{k\beta_1^2+\mu\beta_1}}{2e^{\frac{1}{100}\lambda M \log \gamma_1}}$$

in contradiction to (14.5). Thus the proof of Case B is also finished; Lemma 1 is proved.

15. For the proof of Theorem 2 let first

$$(15.1)\qquad \lambda_0 \overset{\text{def}}{=} \frac{1}{10} \frac{\sqrt{\log \sqrt{Y}}}{\log_2 \sqrt{Y}} \left(> \frac{1}{20} \frac{\log Y}{\log_2 Y} \right).$$

In the course of proof we shall distinguish the following two cases.

Case I. There is a zero $\varrho^* = \beta^* + i\gamma^*$ with

$$(15.2)\qquad \beta^* \geqq \frac{1}{2} + \frac{\log \gamma^*}{4\lambda_0}, \quad 0 < \gamma^* \leqq e^{\lambda_0}.$$

This case is essentially settled already by Lemma 1. Namely, the zero ϱ^* for which (15.2) holds, satisfies the condition of Lemma 1 for any $Z \geqq \sqrt{Y}$ (if $Y > c_6$) and thus for any Z with

$$(15.3)\qquad \sqrt{Y} \leqq Z \leqq Y$$

there is at least one sign change of $\Delta_1(x)$ in the interval

$$(15.4)\qquad I^*(Z) = \left(\exp\left(\log Z - 31\sqrt{\log Y}\log_2 Y\right), \exp\left(\log Z + 31\sqrt{\log Y}\log_2 Y\right)\right)$$

because of (15.3) $I^*(Z)$ contains the interval $I(Z)$ given by (6.2).

Applying this for

$$(15.5)\qquad Z_\nu = \sqrt{Y}\exp\left(62\nu\sqrt{\log Y}\log_2 Y\right), \quad 1 \leqq \nu \leqq \left[\frac{\sqrt{\log Y}}{124 \log_2 Y}\right] - 1$$

we get at least

$$\left[\frac{\sqrt{\log Y}}{124\log_2 Y}\right]-1>\frac{1}{125}\cdot\frac{\sqrt{\log Y}}{\log_2 Y} \tag{15.6}$$

disjoint intervals contained in

$$[\sqrt{Y}, Y] \tag{15.7}$$

such that every interval contains at least one sign change of $\Delta_1(x)$. Thus in Case I the total number of sign changes in $[2, Y]$ is

$$V_1(Y)>\frac{1}{125}\cdot\frac{\sqrt{\log Y}}{\log_2 Y} \tag{15.8}$$

which proves Theorem 2 for the Case I.

16. As the zeros of $\zeta(s)$ lie symmetrically to the line $\sigma=\frac{1}{2}$ we can formulate the other case as

Case II. All zeros $\varrho=\frac{1}{2}+\delta+i\gamma$ of $\zeta(s)$ with

$$|\gamma|\leqq e^{\lambda_0} \tag{16.1}$$

satisfy

$$|\delta|<\frac{\log|\gamma|}{4\lambda_0}. \tag{16.2}$$

In this case we shall prove that there are at least

$$c_7\lambda_0 \tag{16.3}$$

(c_7 positive effectively computable constant) sign changes of $\Delta_1(x)$ in $[2, e^{\frac{\lambda_0}{2}}]$ and thus we get the inequality

$$V_1(Y)\geqq V_1(e^{\frac{\lambda_0}{2}})\geqq c_7\lambda_0>\frac{c_7}{20}\frac{\sqrt{\log Y}}{\log_2 Y} \tag{16.4}$$

which will prove Theorem 2 also in Case II.

In this case we shall use ideas of Littlewood, Ingham and Skewes, too.

17. Now we shall show that under the condition (16.1)—(16.2) the investigation of $\Delta_1(x)$ can be reduced to the investigation of the easier manageable $\Delta_4(x)$ (for the notation see (3.1)).

Introducing the notations

$$\Delta(x)=\int_1^x \Delta_4(\vartheta)\,d\vartheta, \tag{17.1}$$

$$\Delta_1^*(r)=\frac{\Delta_1(r)}{\left(\frac{\sqrt{r}}{\log r}\right)}, \tag{17.2}$$

$$\Delta_4^*(r)=\frac{\Delta_4(r)}{\sqrt{r}} \tag{17.3}$$

we shall use two well-known lemmata. (All the constants as well as those implied by the O and o symbols will be absolute, effective constants.)

LEMMA 2.

$$\Delta(u)=-\sum_{|\gamma|\leqq u^2}\frac{u^{\varrho+1}}{\varrho(\varrho+1)}+O(u). \tag{17.4}$$

The proof follows easily from Theorem 28 (p. 73) of INGHAM [2].

LEMMA 3. *For $r\to\infty$ we have*

$$\Delta_1^*(r)-\Delta_4^*(r)+1+o(1)=\frac{\Delta(r)}{r^{\frac{3}{2}}\log r}+\frac{\log r}{\sqrt{r}}\int_2^r \Delta(u)\frac{\log u+2}{u^2\log^3 u}\,du. \tag{17.5}$$

For the proof see, e.g., INGHAM [2], formula (33) in Theorem 35 (p. 104).

18. Combining this with Lemma 2 we get

LEMMA 4. *Under the condition* (16.1)—(16.2) *we have for*

$$u\leqq e^{\frac{\lambda_0}{2}} \tag{18.1}$$

the inequality

$$|\Delta(u)|\leqq c_8 u^{\frac{3}{2}}. \tag{18.2}$$

For the proof we consider (17.4). This gives using (18.1) and (16.1)—(16.2) the estimate

$$\begin{aligned}|\Delta(u)+O(u)|&\leqq\sum_{|\gamma|\leqq u^2}\frac{u^{\frac{3}{2}+\delta}}{\gamma^2}\leqq\sum_{|\gamma|\leqq u^2}\frac{u^{\frac{3}{2}}e^{\frac{\lambda_0}{2}\delta}}{\gamma^2}\leqq\\ &\leqq u^{\frac{3}{2}}\sum_{|\gamma|\leqq u^2}\frac{e^{\frac{\lambda_0}{2}\cdot\frac{\log|\gamma|}{4\lambda_0}}}{\gamma^2}\leqq u^{\frac{3}{2}}\sum_{\gamma}\frac{|\gamma|^{\frac{1}{8}}}{\gamma^2}=O(u^{\frac{3}{2}})\end{aligned} \tag{18.3}$$

which proves the lemma.

Now using Lemmata 3 and 4 we get

LEMMA 5. *Under the condition* (16.1)—(16.2) *we have for*

$$r\leqq e^{\frac{\lambda_0}{2}} \tag{18.4}$$

the relation

$$\Delta_1^*(r)=\Delta_4^*(r)-1+o(1). \tag{18.5}$$

(By the $o(1)$ symbol we mean that the corresponding quantity is absolutely less than ε if $r>r_0(\varepsilon)$ and r satisfies (18.4).)

Owing to Lemma 3 it is enough to prove that the right side of (17.5) is $o(1)$. This is trivially true for the first term by (18.2) but again using (18.2) we have also for the integral on the right side of (17.5) the upper bound

$$\int_2^r c_8 u^{\frac{3}{2}}\frac{\log u+2}{u^2\log^3 u}\,du\leqq c_9\frac{\sqrt{r}}{\log^2 r}=o\left(\frac{\sqrt{r}}{\log r}\right) \tag{18.6}$$

and thus the lemma is proved.

19. Due to Lemma 5 to guarantee a sign change for $\Delta_1(r)$ in an interval

$$J \subset \left[c_{10}, e^{\frac{\lambda_0}{2}}\right] \tag{19.1}$$

it is sufficient to show that

$$\max_{r\in J} \Delta_4^*(r) > \frac{3}{2} \tag{19.2}$$

and

$$\min_{r\in J} \Delta_4^*(r) < -\frac{3}{2}. \tag{19.3}$$

But the shortened form of the Riemann—van Mangoldt exact prime number formula gives for $r \leqq e^{\frac{\lambda_0}{2}}$

$$\Delta_4^*(r) = -\sum_{|\gamma|\leqq e^{\lambda_0}} \frac{r^{\delta+i\gamma}}{\varrho} + o(1) \tag{19.4}$$

(see, e.g., INGHAM [2] Theorem 29, (p. 77)) and thus $\Delta_4^*(r)$ can be treated easier than $\Delta_1^*(r)$.

Thus introducing the notation

$$G(v) \overset{\text{def}}{=} \sum_{|\gamma|\leqq e^{\lambda_0}} \frac{e^{(\delta+i\gamma)v}}{\varrho} \tag{19.5}$$

$\Delta_1(r)$ has certainly a sign change in an interval

$$[e^{a_1}, e^{a_2}] \subset \left[c_{11}, e^{\frac{\lambda_0}{2}}\right] \tag{19.6}$$

if we can show that

$$\max_{a_1\leqq v\leqq a_2} G(v) > 2 \tag{19.7}$$

and

$$\min_{a_1\leqq v\leqq a_2} G(v) < -2. \tag{19.8}$$

(We remark that since the zeros of $\zeta(s)$ lie symmetrically to the real axis, $G(v)$ is real.)

20. Now we shall use an idea of INGHAM [3] which makes use of the Fejér-kernel

$$\int_{-\infty}^{\infty} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 e^{iuy}\,dy = \begin{cases} 2\pi(1-|u|) & \text{for} \quad |u| \leqq 1, \\ 0 & \text{for} \quad |u| \geqq 1 \end{cases} \tag{20.1}$$

and which makes possible to reduce the number of terms in $G(v)$ and so the effective application of Theorem B of the Appendix.

Let $A>20$ and $B>8$ be sufficiently large effective constants, B an integer, to be determined later, further ω any real number, satisfying

$$\log c_{11} + 1 \leqq \omega \leqq \frac{\lambda_0}{2} - 1.$$

We note that the further constants c_ν's with $12 \leqq \nu \leqq 22$ will be absolute effective positive constants whose values do not depend on A, B either.

Using the notation (19.5) we define the integral

$$I_1(\omega) \overset{\text{def}}{=} \int_{-\frac{A}{4}}^{\frac{A}{4}} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 G\left(\omega+\frac{y}{A}\right) dy = \tag{20.2}$$

$$= \sum_{|\gamma|\leqq e^{\lambda_0}} \frac{e^{(\delta+i\gamma)\omega}}{\varrho} \int_{-\frac{A}{4}}^{\frac{A}{4}} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 e^{iy\frac{\gamma}{A}} \cdot e^{\frac{\delta y}{A}}\,dy.$$

Further we define the integrals

$$I_2(\omega) \overset{\text{def}}{=} \sum_{|\gamma|\leqq e^{\lambda_0}} \frac{e^{(\delta+i\gamma)\omega}}{\varrho} \int_{-\frac{A}{4}}^{\frac{A}{4}} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 e^{iy\frac{\gamma}{A}}\,dy \tag{20.3}$$

and

$$I_3(\omega) \overset{\text{def}}{=} \sum_{|\gamma|\leqq e^{\lambda_0}} \frac{e^{(\delta+i\gamma)\omega}}{\varrho} \int_{-\infty}^{\infty} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 e^{iy\frac{\gamma}{A}}\,dy. \tag{20.4}$$

We shall prove that

$$|I_1(\omega) - I_3(\omega)| \leqq c_{12} \tag{20.5}$$

and so with the use of Fejér's kernel we can show that in the investigation of the average of $G(v)$ in the interval $\left[\omega-\frac{1}{4}, \omega+\frac{1}{4}\right]$ in (20.2) only the contribution of the low zeros, i.e., those with $|\gamma|<A$ is essential.

To prove (20.5) we get with easy computation by partial integration

$$\left|\int_{-\frac{A}{4}}^{\frac{A}{4}} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 \left(e^{\frac{\delta y}{A}}-1\right) e^{iy\frac{\gamma}{A}}\,dy\right| \leqq$$

$$\left|\left[\frac{A}{i\gamma} e^{iy\frac{\gamma}{A}} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 \left(e^{\frac{\delta y}{A}}-1\right)\right]_{-\frac{A}{4}}^{\frac{A}{4}}\right| + \left|\int_{-\frac{A}{4}}^{\frac{A}{4}} \frac{A}{i\gamma} e^{iy\frac{\gamma}{A}} \cdot \frac{d}{dy}\left\{\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 \left(e^{\frac{\delta y}{A}}-1\right)\right\} dy\right| \leqq \tag{20.6}$$

$$2\cdot\frac{A}{|\gamma|}\cdot\frac{1}{\left(\frac{A}{8}\right)^2}\left(e^{\frac{|\delta|}{4}}-1\right) + \frac{A}{|\gamma|}\left(\frac{|\delta|}{A} e^{\frac{|\delta|}{4}}\cdot 2\pi + c_{13}\left(e^{\frac{|\delta|}{4}}-1\right)\right) \leqq c_{14}\frac{A|\delta|}{|\gamma|}.$$

Thus we get

$$|I_1(\omega)-I_2(\omega)| \leqq \sum_{|\gamma|\leqq e^{\lambda_0}} \left|\frac{e^{(\delta+i\gamma)\omega}}{\varrho}\right| \left| \int_{-\frac{A}{4}}^{\frac{A}{4}} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 e^{i\gamma\frac{y}{A}}\left(e^{\frac{\delta y}{A}}-1\right) dy\right| \leqq$$

(20.7)
$$\leqq \sum_{|\gamma|\leqq e^{\lambda_0}} \frac{e^{\delta\omega}}{|\gamma|}\cdot c_{14}\frac{A|\delta|}{|\gamma|} \leqq c_{14}A \sum_{|\gamma|\leqq e^{\lambda_0}} \frac{e^{\frac{\log|\gamma|}{4\lambda_0}\cdot\frac{\lambda_0}{2}}}{|\gamma|}\cdot\frac{\log|\gamma|}{4\lambda_0}\cdot\frac{1}{|\gamma|} \leqq$$

$$\leqq \frac{c_{14}A}{4\lambda_0}\sum_{\gamma}\frac{\log|\gamma|\cdot|\gamma|^{\frac18}}{|\gamma|^2} = \frac{c_{15}A}{4\lambda_0} \leqq c_{15}.$$

Further again by partial integration we have

$$\left|\int_{\frac{A}{4}}^{\infty} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 e^{i\gamma\frac{y}{A}}\, dy\right| \leqq$$

(20.8)
$$\left|\left[\frac{A}{i\gamma}e^{i\gamma\frac{y}{A}}\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2\right]_{\frac{A}{4}}^{\infty}\right| + \left|\frac{A}{i\gamma}\int_{\frac{A}{4}}^{\infty} e^{i\gamma\frac{y}{A}}\frac{d}{dy}\left\{\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2\right\}dy\right| \leqq$$

$$\frac{A}{|\gamma|}\cdot\frac{1}{\left(\frac{A}{8}\right)^2}+\frac{A}{|\gamma|}\cdot\frac{c_{16}}{A} < \frac{c_{17}}{|\gamma|},$$

and the same holds for the corresponding integral in $\left[-\infty, \frac{A}{4}\right]$. From this we have analogously to (20.7)

(20.9)
$$|I_3(\omega)-I_2(\omega)| \leqq \sum_{|\gamma|\leqq e^{\lambda_0}} \frac{|\gamma|^{\frac18}}{|\gamma|}\cdot\frac{2c_{17}}{|\gamma|} < c_{18}.$$

So using (20.7) and (20.9) we get (20.5) to be valid with the choice $c_{12}=c_{15}+c_{18}$.

21. Thus we must investigate now the integral given by (20.4) which can be written according to (20.1) in the form

(21.1)
$$I_3(\omega)=\sum_{|\gamma|<A}\frac{e^{(\delta+i\gamma)\omega}}{\frac12+\delta+i\gamma}\cdot 2\pi\left(1-\frac{|\gamma|}{A}\right).$$

Using the fact that the zeros of $\zeta(s)$ lie symmetrically to the real axis easy computation shows that

(21.2)
$$I_3(\omega) = 2\pi\sum_{0<\gamma<A}\frac{e^{\delta\omega}\{(1+2\delta)\cos(\gamma\omega)+2\gamma\sin(\gamma\omega)\}}{\left(\frac12+\delta\right)^2+\gamma^2}\left(1-\frac{\gamma}{A}\right).$$

Now if we restrict ω beyond the previous $\log c_{11}+1\leqq\omega\leqq\frac{\lambda_0}{2}-1$ by

(21.3)
$$\log c_{11}+1\leqq\omega\leqq\frac{\lambda_0}{2\log A}$$

then we have for the zeros with $0<\gamma<A$

(21.4)
$$|\delta|\omega \leqq \frac{\log A}{4\lambda_0}\cdot\frac{\lambda_0}{2\log A}=\frac18$$

and so

(21.5)
$$0.8<e^{\delta\omega}<1.2.$$

Let us introduce the notation:

(21.6)
$$J_\omega(\eta) = 2\pi\sum_{0<\gamma<A}\frac{e^{\delta\omega}\{(1+2\delta)\cos(\gamma\eta)+2\gamma\sin(\gamma\eta)\}}{\left(\frac12+\delta\right)^2+\gamma^2}\left(1-\frac{\gamma}{A}\right).$$

Then obviously we have

(21.7)
$$I_3(\omega)=J_\omega(\omega).$$

First we note that choosing in $J_\omega(\eta)$

(21.8)
$$\eta=\frac1A \quad\text{and}\quad \eta=-\frac1A, \text{ resp.,}$$

$J_\omega(\eta)$ can be made "big positive" and "big negative", resp., for any ω in (21.3), if we choose A sufficiently large.

Namely, using (21.5) and the well-known fact

(21.9)
$$N(T)\overset{\text{def}}{=}\sum_{0<\gamma<T}1=\frac{T}{2\pi}\log\frac{T}{2\pi e}+O(\log T)>\frac{T}{7}\log T \quad\text{for}\quad T>T_1$$

(see, e.g., INGHAM [2], Theorem 25) further the inequality

(21.10)
$$\sin t\geqq\frac{2}{\pi}\cdot t \quad\text{for}\quad 0\leqq t\leqq\frac{\pi}{2}$$

we get by

(21.11)
$$\gamma>14$$

for any $A>2T_1$ the inequality

$$J_\omega\left(\frac{1}{A}\right) > 2\pi \sum_{0<\gamma<A} \frac{0.8\cdot 2\gamma\cdot\frac{2}{\pi}\cdot\frac{\gamma}{A}}{1.01\gamma^2}\left(1-\frac{\gamma}{A}\right) >$$

(21.12)
$$> \frac{6}{A}\sum_{0<\gamma<A}\left(1-\frac{\gamma}{A}\right) \geqq \frac{6}{A} N\left(\frac{A}{2}\right)\cdot\frac{1}{2} >$$

$$> \frac{1}{2}\cdot\frac{6}{A}\cdot\frac{A}{2\cdot 7}\log\frac{A}{2} > \frac{1}{5}\log\frac{A}{2} > \frac{1}{10}\log A.$$

And analogously we have for $A>2T_1$

(21.13)
$$J_\omega\left(-\frac{1}{A}\right) < 2\pi\left\{\sum_{0<\gamma<A}\frac{1.2\cdot 2}{\gamma^2} - \sum_{0<\gamma<A}\frac{0.8\cdot 2\gamma\cdot\frac{2}{\pi}\cdot\frac{\gamma}{A}}{1.01\gamma^2}\left(1-\frac{\gamma}{A}\right)\right\} <$$

$$< 4.8\pi\sum_{\gamma>0}\frac{1}{\gamma^2} - \frac{6}{A}\sum_{0<\gamma<A}\left(1-\frac{\gamma}{A}\right) < c_{19} - \frac{1}{10}\log A.$$

22. Now we shall apply Theorem B of the Appendix for the numbers

(22.1)
$$\frac{\gamma}{2\pi} \quad \text{with} \quad 0<\gamma<A$$

their total number being

(22.2)
$$N \overset{\text{def}}{=} N(A) < A\log A \quad \text{for} \quad A > c_{20}$$

and owing to $A>20$

(22.3)
$$N(A) \geqq 1.$$

We choose in Theorem B of the Appendix

(22.4)
$$q = B \overset{\text{def}}{=} [\log^2 A]$$

and

(22.5)
$$M = \left[\frac{\frac{\lambda_0}{2\log A} - 1}{B^{N(A)}}\right] \geqq c(A)\lambda_0$$

where $c(A)$ is an effectively computable constant depending only on A.

Thus denoting the distance of a real number x from the nearest integer by $\|x\|$, we get the existence of positive integer n_ν's with

(22.6)
$$1 \leqq n_1 < n_2 < \ldots < n_M \leqq \frac{\lambda_0}{2\log A} - 1$$

for which all the relations

(22.7)
$$\left\|\frac{\gamma_j}{2\pi} n_\nu\right\| \leqq \frac{1}{B} \qquad (1\leqq j\leqq N,\ 1\leqq\nu\leqq M)$$

hold.

This implies

(22.8)
$$\left|\sin\left\{\gamma_j\left(n_\nu\pm\frac{1}{A}\right)\right\} - \sin\left(\pm\frac{\gamma_j}{A}\right)\right| \leqq 2\left|\sin\frac{\gamma_j n_\nu}{2}\right| < \frac{2\pi}{B}$$

and

(22.9)
$$\left|\cos\left\{\gamma_j\left(n_\nu\pm\frac{1}{A}\right)\right\} - \cos\left(\pm\frac{\gamma_j}{A}\right)\right| \leqq 2\left|\sin\frac{\gamma_j n_\nu}{2}\right| < \frac{2\pi}{B}$$

for $1\leqq j\leqq N$, $1\leqq\nu\leqq M$, where in the above formulae always both the upper or both the lower signs are meant.

Choosing the numbers ω_ν' and ω_ν'' as

(22.10)
$$\omega_\nu^{(i)} = n_\nu + \frac{(-1)^i}{A}$$

we get from (22.8)—(22.9) (the inequality (21.3) is satisfied owing to (22.6)) for $A > \log c_{11}+2$

(22.11)
$$\left|J_{\omega_\nu^{(i)}}(\omega_\nu^{(i)}) - J_{\omega_\nu^{(i)}}\left(\frac{(-1)^i}{A}\right)\right| < 2\pi\sum_{0<\gamma<A}\frac{1.2(2+2\gamma)\cdot\frac{2\pi}{B}}{\gamma^2} <$$

$$< \frac{110}{B}\sum_{0<\gamma<A}\frac{1}{\gamma} < \frac{110}{B}\cdot c_{21}\log^2 A < c_{22}$$

considering (22.4) and the relation

(22.12)
$$\sum_{0<\gamma<A}\frac{1}{\gamma} < c_{21}\log^2 A$$

which is an easy consequence of (21.9).

Thus we have using (22.12)—(22.13)

(22.13)
$$J_{\omega_\nu'}(\omega_\nu') = I_3(\omega_\nu') < -\frac{1}{10}\log A + c_{19} + c_{22}$$

and

(22.14)
$$J_{\omega_\nu''}(\omega_\nu'') = I_3(\omega_\nu'') > \frac{1}{10}\log A - c_{22}.$$

Combining this with (20.5) we get already the needed results for the average of $G(v)$ in $\left[\omega-\frac{1}{4},\ \omega+\frac{1}{4}\right]$, namely we have

(22.15)
$$I_1(\omega_\nu') < -\frac{1}{10}\log A + c_{19} + c_{22} + c_{12}$$

and analogously

(22.16)
$$I_1(\omega_\nu'') > \frac{1}{10}\log A - c_{22} - c_{12}.$$

23. Now fixing A as

$$A=\max\{e^{10(c_{22}+c_{19}+c_{12}+4\pi)},\ 2T_1,\ c_{20},\ 20\} \tag{23.1}$$

from (22.15) and (22.16) we get considering the definition of $I_1(\omega)$ in (20.2):

$$\int_{-\frac{A}{4}}^{\frac{A}{4}}\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 G\left(\omega_\nu'+\frac{y}{A}\right)dy<-4\pi \tag{23.2}$$

and

$$\int_{-\frac{A}{4}}^{\frac{A}{4}}\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 G\left(\omega_\nu''+\frac{y}{A}\right)dy>4\pi. \tag{23.3}$$

Since by (20.1) we have

$$\int_{-\infty}^{\infty}\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 dy=2\pi \tag{23.4}$$

(23.2)—(23.4) give immediately

$$\min_{\omega_\nu'-\frac{1}{4}\le v\le\omega_\nu'+\frac{1}{4}} G(v)<-2 \tag{23.5}$$

and

$$\max_{\omega_\nu''-\frac{1}{4}\le v\le\omega_\nu''+\frac{1}{4}} G(v)>2. \tag{23.6}$$

Taking in account $\frac{1}{A}<\frac{1}{20}$ and (22.10) we get

$$\left[\omega_\nu^{(i)}-\frac{1}{4},\ \omega_\nu^{(i)}+\frac{1}{4}\right]\subset\left[n_\nu-\frac{1}{3},\ n_\nu+\frac{1}{3}\right]. \tag{23.7}$$

Thus by (19.6)—(19.8) $\Delta_1(r)$ has at least one sign change in every interval

$$I_\nu\overset{\text{def}}{=}\left[e^{n_\nu-\frac{1}{3}},\ e^{n_\nu+\frac{1}{3}}\right]\subset\left[c_{11},\ e^{\frac{\lambda_0}{2}}\right] \tag{23.8}$$

(if $\nu>\log c_{11}+2$).

As the n_ν-s are positive integers, these intervals are all disjoint, further their total number is by (22.5) at least

$$\begin{aligned} M-(\log c_{11}+2) &\ge c(A)\lambda_0-\log c_{11}-2\ge \\ &\ge c_{23}\lambda_0-\log c_{11}-2\ge c_7\lambda_0 \end{aligned} \tag{23.9}$$

(since here A is already fixed and so $c(A)=c_{23}$ is an effectively computable absolute positive constant).

Thus we have in Case B in the interval $[2, e^{\frac{\lambda_0}{2}}]$ at least $c_7\lambda_0$ sign changes of $\Delta_1(x)$, and thus owing to (16.4) Case B is settled, too, so Theorem 2 is completely proved.

Now we describe what sort of changes are necessary in the course of proof of Theorem 1 to get an effective value for Y_2. Analogously, even simpler one can effectivize the proof in the case $i=4$.

In the formulation of Lemma 1 we assert (6.2) without any condition. In (6.10) in the definition of $f(x)$ we work with $x^{1/4}$ instead of $x^{1/2}$ and analogously in (6.11) we define $H(s)$ in the last term with $\dfrac{1}{4\left(s-\frac{1}{4}\right)^2}$ instead of $\dfrac{1}{2\left(s-\frac{1}{2}\right)^2}$. We do not distinguish Cases A and B, we follow the line of Case A. As to $\varrho_0=\beta_0+i\gamma_0$ we choose the zero with the minimal imaginary part, i.e., $\varrho_0=\frac{1}{2}+i\gamma_0$ ($\gamma_0\approx 14.3$). Then (7.1) is satisfied trivially, and we get after at most $\left[\frac{\lambda}{2}\right]$ steps also the zero ϱ_1 as in (7.3) but we have now only $\beta_1\ge\frac{1}{2}$ instead of $\beta_1\ge\frac{1}{2}+\frac{1}{\lambda}$.

The next change is only in (11.1)—(11.3) where we get for K_1 in (11.3)

$$K_1=e^{\frac{k}{16}+\frac{\mu}{4}}+O(1). \tag{24.1}$$

The estimation of K_2 being unchanged valid we get instead of (12.8) for K

$$|K|=O\left(e^{-97M+\mu\beta_1}+e^{\frac{k}{16}+\frac{\mu}{4}}\right)=O\left(e^{-97M+\mu\beta_1}\right) \tag{24.2}$$

by (6.3)—(6.7).

From this we get immediately

$$|U|\le\frac{e^{k\beta_1^2+\mu\beta_1}}{e^{96M}} \tag{24.3}$$

again by (6.3)—(6.7), (10.8) and (7.3); i.e., the upper estimate (12.9) for U is unchanged valid.

Further we get for the residue R in (13.1) even a better upper bound than in (13.2), i.e., the final estimate in (13.2) remains valid

$$|R|\le\frac{1}{4}\left(2k\left|\frac{1}{4}-i\gamma_1\right|+\mu\right)e^{\frac{k}{16}+\frac{\mu}{4}-k\gamma_1^2}\le\frac{e^{k\beta_1^2+\mu\beta_1}}{e^{99M}} \tag{24.4}$$

and so the final lower estimate for $|U|$ in (13.14) is again valid, and the contradiction proves our modified Lemma 1.

Appendix

The following theorem is a special case of the so called second main theorem of the powersum theory.

THEOREM (T. Sós—Turán). *For arbitrary complex numbers z_j, and for a natural number n*

$$\max_{m<\nu\leqq m+n} \frac{\left|\sum_{j=1}^{n} z_j^{\nu}\right|}{|z_1|^{\nu}} \geqq \left(\frac{1}{8e\left(\frac{m}{n}+1\right)}\right)^n.$$

For the proof see VERA T. SÓS—P. TURÁN [17].

If we choose here $m=a\frac{n}{d}$, $z_j=e^{\alpha_j \frac{a}{m}}=e^{\alpha_j \frac{d}{n}}$ we get from this

$$\max_{\frac{n}{a}d<\nu\leqq(a+d)\frac{n}{d}} \frac{\left|\sum_{j=1}^{n} e^{\alpha_j \frac{d}{n}\nu}\right|}{\left|e^{\alpha_1 \frac{d}{n}\nu}\right|} \geqq \left(\frac{1}{8e\left(\frac{a}{d}+1\right)}\right)^n.$$

The above inequality implies immediately the continuous form of the second main theorem:

THEOREM A (T. Sós—Turán). *For arbitrary complex numbers α_j, and for positive real numbers a and d*

$$\max_{a<t\leqq a+d} \frac{\left|\sum_{j=1}^{n} e^{\alpha_j t}\right|}{|e^{\alpha_1 t}|} \geqq \left(\frac{1}{8e\left(\frac{a}{d}+1\right)}\right)^n.$$

The following theorem is an extension of Dirichlet's classical theorem on simultaneous approximation.

THEOREM B. *If $q\geqq 2$ and M are integers, $\gamma_1, \ldots, \gamma_N$ arbitrary real numbers, then there exist integer n_j's with*

$$1 \leqq n_1 < n_2 < \ldots < n_M \leqq Mq^N$$

such that for $1\leqq\mu\leqq M$, $1\leqq\nu\leqq N$

$$\|n_\mu \gamma_\nu\| \leqq \frac{1}{q}$$

where $\|x\|$ denotes the distance of x from the nearest integer.

For the proof see TITCHMARSH [18], p. 153.

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