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On the Sign Changes of $(\pi(x) - \operatorname{li} x)$. II.

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Dedicated to Prof. Dr. E. Hlawka on the occasion of his 60th birthday

Abstract

If $\pi(x)$ stands for the number of primes not exceeding x then the existence of two effectively calculable constants c_1 and c_2 is proved so that for $T > c_1$, the number $V(T)$ of sign changes of $(\pi(x) - \int_2^x dv/\log v)$ in $(0, T)$ is greater than $c_2 \log\log\log T$.

1. If $\pi(x)$ stands for the number of primes $\leqq x$ and

$$\operatorname{li}^* x = \int_0^x \frac{dr}{\log r}$$

(taken as principal value at $r = 1$) then Riemann [1] made in his famous paper the assertion that for $x > 2$ the inequality

$$\pi(x) < \operatorname{li}^* x \tag{1.1}$$

holds. Though D. N. Lehmer [2] showed its validity for $x < 10^7$, Littlewood [3] found in the same year that (1.1) is not true. He even showed that replacing the function $\operatorname{li}^* x$ by the more handy function

$$\operatorname{li} x = \int_2^x \frac{dr}{\log r} \tag{1.2}$$

there are sequences

$$\begin{aligned} x_1' < x_2' < \ldots \to \infty \\ x_1'' < x_2'' < \ldots \to \infty \end{aligned} \tag{1.3}$$

for which the inequalities

$$\begin{aligned} \pi(x_\nu') - \operatorname{li} x_\nu' &> c \frac{\sqrt{x_\nu'}}{\log x_\nu} \log\log\log x_\nu' \\ \pi(x_\nu'') - \operatorname{li} x_\nu'' &< -c \frac{\sqrt{x_\nu''}}{\log x_\nu''} \log\log\log x_\nu'' \end{aligned} \tag{1.4}$$

hold. LANDAU wrote in his "Vorlesungen über Zahlentheorie" [4]: "Der Hardy-Littlewoodsche Beweis[1] (of 1.4) gehört zu den schönsten Früchten der analytischen Zahlentheorie".

2. The inequalities (1.4), however beautiful, are not quite satisfactory from two reasons. The first one—noticed by LITTLEWOOD himself—refers to the numerical result of D. N. LEHMER. This indicates that the first sign-changing place of

$$\pi(x) - \operatorname{li} x \stackrel{\text{def}}{=} \Delta_1(x) \tag{2.1}$$

must be "very large" and it would be desirable to find an *effective* numerical upper bound for it. Curiously enough the original proof of (1.4) cannot furnish such a bound. LITTLEWOOD himself as well as others returned repeatedly to this question which was solved not earlier than 1955 by SKEWES [6] with the first effective bound $e_4(7{,}705)$ where $e_\nu(x)$ stands for ν-times iterated exponentials. The second one refers to the fact that the inequality (1.4) does not give any indication on the oscillatory nature of the error-term $\Delta_1(x)$. Such work started with a paper of PÓLYA [7] but the first essential result on $\Delta_1(x)$ was due to INGHAM [8]. He proved the existence of a constant c_0 so that for $Y > 1$ the interval

$$[Y, c_0 Y] \tag{2.2}$$

contains a sign-change of $\Delta_1(x)$, *under an unproved condition however*. This condition asserts the existence of a Θ with $\frac{1}{2} \leqq \Theta < 1$ so that

$$\zeta(\sigma) \neq 0 \quad \text{for} \quad \sigma > \Theta \tag{2.3}$$

but for a suitable real t_0

$$\zeta(\Theta + i t_0) = 0. \tag{2.4}$$

This is satisfied e. g. if Riemann's conjecture is true. In our paper [9] we found the unconditional theorem that for $Y > Y_0$ the interval

$$[Y, Y \exp\{\log^{\frac{3}{4}} Y (\log\log Y)^4\}] \tag{2.5}$$

[1] See [5].

contains a sign-change of $\Delta_1(x)$. However, this theorem is ineffective since our proof does not furnish an effective Y_0.

3. So what we really want to do, would be *unconditional and effective* theorems of (2.2)- or (2.5)-type. No such results are known up to now. But if we are more modest and content ourselves with a lower bound for $V_1(Y)$, the number of sign-changes of $\Delta_1(x)$ with

$$2 < x \leqq Y, \tag{3.1}$$

the situation changes. As was found by the first of us (see [10]) the inequality

$$V_1(Y) > e^{-35} \log\log\log\log Y \tag{3.2}$$

holds unconditionally for

$$Y > e_5(35). \tag{3.3}$$

The essential new tool in the proof of (3.2)—(3.3) was a "onesided" powersum theorem (see [11]). (3.2)—(3.3) includes (though with a worse constant) SKEWES' result.

In the recent book of W. J. ELLISON and M. MENDÈS FRANCE (see [12]), after announcing the theorem (3.2)—(3.3) on p. 224 the authors write the following: "Une amélioration de ce résultat, ne serait-ce que la suppression d'un Log, semble être un problème très difficile". The aim of the present note is to prove that such a suppression is possible. More exactly

Theorem. *There are effectively computable constants c_1 and c_2 so that for $Y > c_1$ the inequality*

$$V_1(Y) > c_2 \log\log\log Y$$

holds.

4. In the proof of our theorem we shall use the following

Lemma I. *There is an effectively calculable constant c_3 with the following property: If for a $Z > c_3$ the function $\zeta(s)$ has a zero*

$$\varrho^* = \beta^* + i\gamma^* \tag{4.1}$$

with

$$\beta^* \geqq \tfrac{1}{2} + \frac{2(\log\log Z)^5}{\sqrt[4]{\log Z}}, \quad |\gamma^*| \leqq \sqrt[4]{\log Z}, \tag{4.2}$$

then with the notation

$$[Z, Z \exp \{\log^{\frac{3}{4}} Z (\log\log Z)^4\}] \stackrel{\text{def}}{=} I(Z) \tag{4.3}$$

the inequalities

$$\begin{aligned} \max_{x \in I(Z)} \Delta_1(x) &\geqq \tfrac{1}{2} Z^{\beta^*} \exp(-\sqrt{\log Z}) \\ \min_{x \in I(Z)} \Delta_1(x) &\leqq -\tfrac{1}{2} Z^{\beta^*} \exp(-\sqrt{\log Z}) \end{aligned} \tag{4.4}$$

hold.

This was theorem III in our paper [9]. Hence $\Delta_1(x)$ has a sign change in $I(Z)$.

5. Let the sequence a_ν be defined recursively by $a_0 > e$ and

$$a_{\nu+1} = a_\nu \exp\{\log^{\frac{3}{4}+\varepsilon} a_\nu\}, \ \nu = 0, 1, \ldots \tag{5.1}$$

where

$$0 < \varepsilon < \tfrac{1}{70}. \tag{5.2}$$

Then we shall need the easy

Lemma II. *For* $\nu \geqq 1$ *the inequality*

$$a_\nu \leqq \exp\{2 \log a_0 \cdot \nu^{4+17\varepsilon}\} \tag{5.3}$$

holds.

For the proof we have with $\log a_\nu = b_\nu$ the recursion

$$b_{\nu+1} = b_\nu + b_\nu^{\frac{3}{4}+\varepsilon}, \quad \nu \geqq 0. \tag{5.4}$$

It is enough to show

$$b_\nu \leqq 2 b_0 \nu^{4+17\varepsilon}, \quad \nu \geqq 1. \tag{5.5}$$

This is true for $\nu = 1$. Further we remark that for $\nu \geqq 1$

$$1 + \nu^{-1+(51/4-13)\varepsilon+17\varepsilon^2} = 1 + \nu^{-1-\varepsilon/4+17\varepsilon^2} < 1 + 1/\nu <$$

$$< 1 + (4 + 17\varepsilon)/\nu < (1 + 1/\nu)^{4+17\varepsilon}$$

i. e.

$$\nu^{4+17\varepsilon} + \nu^{3+(51/4+4)\varepsilon+17\varepsilon^2} < (\nu + 1)^{4+17\varepsilon}. \tag{5.6}$$

Hence if (5.5) is true for $\nu = \nu_1$, then from (5.4) and (5.6) follows indeed

$$b_{\nu_1+1} \leqq 2 b_0 \cdot \nu_1^{4+17\varepsilon} + (2 b_0 \nu_1^{4+17\varepsilon})^{\frac{3}{4}+\varepsilon} \leqq 2 b_0 (\nu_1^{4+17\varepsilon} + \nu_1^{(4+17\varepsilon)(\frac{3}{4}+\varepsilon)}) =$$

$$= 2 b_0 (\nu_1^{4+17\varepsilon} + \nu_1^{3+(51/4+4)\varepsilon+17\varepsilon^2}) < 2 b_0 (\nu_1 + 1)^{4+17\varepsilon}.$$

6. Let $0 < \varepsilon < \frac{1}{70}$ be fixed; let c_4 be so large that for $x > c_4$ the inequalities

$$(\log\log x)^{12} \leqq \log^{\varepsilon} x$$
$$\log\log\log x \leqq \tfrac{1}{10} \log\log x \tag{6.1}$$
$$\log^2 x \leqq \sqrt{x}$$

hold. With c_3 of lemma I we fix an arbitrary Y with

$$Y > \max\{\exp(\tfrac{6}{5}\exp 20), \exp(\log^{6/5}\max(c_3, c_4))\}. \tag{6.2}$$

In the proof of our theorem we distinguish two cases.

Case I. There is a zero $\varrho_0 = \beta_0 + i\gamma_0$ of $\zeta(s)$ with

$$\beta_0 \geqq \tfrac{1}{2} + 2\log^{-\frac{1}{8}} Y, \quad 0 < \gamma_0 \leqq \log^{\frac{1}{8}} Y. \tag{6.3}$$

Then we can apply lemma I with

$$Z = \exp\log^{5/6} Y, \quad \varrho^* = \varrho_0;$$

this assures the existence of a sign-change of $\Delta_1(x)$ in $I(Z)$ and consequently in (a_0, a_1) with

$$a_0 = \exp\log^{5/6} Y, \quad a_1 = a_0 \exp\log^{3/4+\varepsilon} a_0.$$

But lemma I is also applicable with

$$Z = a_1, \quad \varrho^* = \varrho_0;$$

this gives again a sign-change of $\Delta_1(x)$ in (a_1, a_2) with

$$a_2 = a_1 \exp\log^{3/4+\varepsilon} a_1.$$

And so on. Since we want a lower bound for $V_1(Y)$, we have

$$V_1(Y) \geqq m \tag{6.4}$$

where

$$a_0 = \exp\log^{5/6} Y, \quad a_{\nu+1} = a_\nu \exp\{\log^{3/4+\varepsilon} a_\nu\}$$
$$a_m \leqq Y < a_{m+1}. \tag{6.5}$$

By lemma II we have

$$Y < \exp\{2\log^{5/6} Y \cdot (m+1)^{4+17\varepsilon}\}, \tag{6.6}$$

i. e.

$$V_1(Y) > \tfrac{1}{2}\left(\frac{\log Y}{2\log^{5/6} Y}\right)^{\frac{1}{5}} > \tfrac{1}{4}\log^{1/30} Y \tag{6.7}$$

and this is much stronger than required by our theorem.

7. Now we turn to the second case which can be written owing to the functional equation as

Case II. All zeros $\varrho = \beta + i\gamma$ of $\zeta(s)$ with

$$|\gamma| \leqq \log^{1/5} Y \tag{7.1}$$

are in the narrow strip

$$|\beta - \tfrac{1}{2}| \leqq 2 \log^{-1/5} Y. \tag{7.2}$$

The treatment of this case uses ideas of LITTLEWOOD, INGHAM and SKEWES too.

Let—with the usual notations—

$$\Delta_3(r) = \psi(r) - r = \sum_{n \leqq r} \Lambda(n) - r, \tag{7.3}$$

$$\Delta_2(r) = \Pi(r) - \mathrm{li}\, r = \pi(r) + \tfrac{1}{2}\pi(\sqrt{r}) + \ldots - \mathrm{li}\, r, \tag{7.4}$$

$$\Delta_4(r) = \int_1^r \Delta_3(\vartheta)\, d\vartheta. \tag{7.5}$$

We shall need some well-known lemmata which we shall explicitly state for the reader's convenience.

Lemma III. *For $r \to \infty$ we have*

$$\frac{\Delta_1(r)}{\left(\frac{\sqrt{r}}{\log r}\right)} - \frac{\Delta_3(r)}{\sqrt{r}} + o(1) + 1 =$$

$$= \frac{\Delta_4(r)}{r^{3/2} \log r} + \frac{\log r}{\sqrt{r}} \int_2^r \Delta_4(u) \frac{\log u + 2}{u^2 \log^3 u}\, du.$$

For a proof see e. g. [14], p. 103—104.

Lemma IV. *For $n \to \infty$ we have*

$$\Delta_4(u) = -\sum_{\varrho} \frac{u^{\varrho+1}}{\varrho(\varrho+1)} + O(u).$$

For a proof see e. g. [14], p. 73.

8. So far (7.1)—(7.2) was not used. We shall need the simple

Lemma V. *Under the restrictions* (7.1)—(7.2) *we have for*

$$u \leqq r \leqq \log^{1/5} Y \tag{8.1}$$

the relation (o-sign refers to r)

$$\frac{\Delta_1(r)}{\left(\frac{\sqrt{r}}{\log r}\right)} - \frac{\Delta_3(r)}{\sqrt{r}} = -1 + o(1).$$

For the proof of this lemma we have owing to lemma III to investigate $\Delta_4(u)$, especially the sum $\sum_{\varrho} \frac{u^{\varrho+1}}{\varrho(\varrho+1)}$. This is absolutely

$$< \sum_{|\gamma| \leq \log^{1/5} Y} \frac{|u|^{3/2+2\log^{-1/5} Y}}{|\varrho(\varrho+1)|} + |u|^2 \sum_{|\gamma| > \log^{1/5} Y} \frac{1}{\gamma^2} <$$
$$< \left(|u|^{3/2} + |u|^2 \frac{\log\log Y}{\log^{1/5} Y}\right) O(1), \tag{8.2}$$

and hence by (8.1)

$$\frac{|\Delta_4(r)|}{r^{3/2}\log r} = o(1). \tag{8.3}$$

Further we get by (8.2) and (8.1)

$$\frac{\log r}{\sqrt{r}} \int_2^r |\Delta_4(u)| \frac{\log u + 2}{u^2 \log^3 u} du < O(1) \frac{\log r}{\sqrt{r}} \int_2^r \left(\frac{1}{\sqrt{u}\log^2 u} + \right.$$
$$\left. + \frac{\log\log Y}{\log^{1/5} Y} \cdot \frac{1}{\log^2 u}\right) du = o(1),$$

which proves the lemma.

9. Due to lemma V $\Delta_1(r)$ has certainly a sign change in a subinterval J of (8.1) if we can show that

$$\max_{r \in J} \frac{\Delta_3(r)}{\sqrt{r}} > \frac{5}{4}, \quad \min_{r \in J} \frac{\Delta_3(r)}{\sqrt{r}} < 0 \tag{9.1}$$

say. The advantage of it lies in the fact that owing to the von Mangoldt's form of Riemann's "exact" primenumber formula

$\Delta_3(r)$ (see [15]) can be more directly dealt with than $\Delta_1(r)$. This can be written—conveniently to our purposes—

$$G(v) \overset{\text{def}}{=} \Delta_3(e^v)\, e^{-v/2} = - \sum_{|\gamma| \leqq \log^{1/5} Y} \frac{e^{(\varrho - \frac{1}{2})v}}{\varrho} + \tag{9.2}$$
$$+ O(1)\, e^{v/2} \log^{-1/5} Y \cdot \{v^2 + (\log\log Y)^2\} + O(v\, e^{-v/2}).$$

The restriction (8.1) is satisfied requiring[2]

$$\log\log\log Y \leqq v \leqq \tfrac{1}{5} \log\log Y; \tag{9.3}$$

hence (9.2) takes the form

$$G(v) = - \sum_{|\gamma| \leqq \log^{1/5} Y} \frac{e^{(\varrho - \frac{1}{2})v}}{\varrho} + o(1). \tag{9.4}$$

10. In order to simplify the sum on the right we may use (7.1)—(7.2). Using also (9.3) we have

$$\left| \frac{e^{(\varrho - \frac{1}{2})v}}{\varrho} - \frac{e^{i\gamma v}}{i\gamma} \right| \leqq \left| \frac{e^{(\varrho - \frac{1}{2})v} - e^{i\gamma v}}{\varrho} \right| + \left| \frac{1}{\varrho} - \frac{1}{i\gamma} \right| =$$
$$= O(1) \left(\frac{1}{|\varrho|} \cdot \frac{\log\log Y}{\log^{1/5} Y} + \frac{1}{\gamma^2} \right);$$

Thus we get from (9.4)

$$\Big| G(v) + \sum_{|\gamma| \leqq \log^{1/5} Y} e^{i\gamma v}/i\gamma \Big| \leqq c_5. \tag{10.1}$$

11. The required inequalities of (9.1)-type will be obtained following the idea of BOHR—LITTLEWOOD by using appropriately Dirichlet's theorem. But using it directly the number of terms in the sum would be too large; to avoid this we shall use the "term-shortening" idea of INGHAM which boils down to an appropriate use of the Fejér-kernel

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin(y/2)}{y/2} \right)^2 e^{iry}\, dy = \begin{cases} 1 - |r| & \text{for } -1 \leqq r \leqq +1 \\ 0 & \text{for } |r| \geqq 1. \end{cases} \tag{11.1}$$

For this sake let $A > e^4$ and B be sufficiently large numerical effective constants, B an integer $\geqq 8$ to be determined later and ω so large that

$$\log\log\log Y < \tfrac{3}{4}\omega < \tfrac{5}{4}\omega \leqq \tfrac{1}{5} \log\log Y. \tag{11.2}$$

[2] Owing to (6.1)—(6.2) this makes sense.

Multiplying in (10.1) by $A\left(\frac{\sin(A(v-\omega)/2)}{A(v-\omega)/2}\right)^2$ we can integrate over $[\frac{3}{4}\omega, \frac{5}{4}\omega]$; introducing the new variable y by $v=\omega+y/A$ we get from (10.1)

$$\left|\int_{-A\omega/4}^{A\omega/4}\left(\frac{\sin(y/2)}{y/2}\right)^2 G(\omega+y/A)\,dy+\right.$$

$$\left.+\sum_{|\gamma|\leqq\log^{1/5}Y}\frac{e^{i\gamma\omega}}{i\gamma}\int_{-A\omega/4}^{A\omega/4}\left(\frac{\sin(y/2)}{y/2}\right)^2 e^{i(\gamma/A)y}\,dy\right|< \tag{11.3}$$

$$<c_5\int_{-A\omega/4}^{A\omega/4}\left(\frac{\sin(y/2)}{y/2}\right)^2 dy<c_6\,.^{3}$$

In order to use (11.1) we have to complete the integrals in the sum of (11.3); doing this the error is absolutely

$$<4\sum_{0<\gamma\leqq\log^{1/5}Y}\frac{1}{|\gamma|}\left|\int_{A\omega/4}^{\infty}\left(\frac{\sin(y/2)}{y/2}\right)^2 e^{i(\gamma/A)y}\,dy\right|\overset{\mathrm{def}}{=}H\,. \tag{11.4}$$

The contribution of the terms with $|\gamma|\leqq A$ is evidently

$$\leqq c_7\sum_{0<\gamma\leqq A}\frac{1}{\gamma}\,\frac{1}{A\omega}<\frac{c_8}{\omega}\cdot\frac{\log^2 A}{A}<c_9\,. \tag{11.5}$$

The contribution of the remaining terms in H is after partial integration

$$<c_{10}\sum_{A\leqq|\gamma|\leqq\log^{1/5}Y}\left(\frac{A}{\gamma^2}\cdot\frac{1}{(A\omega)^2}+\frac{A}{\gamma^2}\cdot\frac{1}{A\omega}\right)<c_{11}+\frac{c_{10}}{\omega}\sum_{\gamma\geqq A}\frac{1}{\gamma^2}<c_{12}\,.$$

Now by (11.1) we get from (11.3)

$$\left|\int_{-A\omega/4}^{A\omega/4}\left(\frac{\sin(y/2)}{y/2}\right)^2 G\left(\omega+\frac{y}{A}\right)dy+\right.$$

$$\left.+4\pi\sum_{0<\gamma\leqq A}\left(1-\frac{\gamma}{A}\right)\frac{\sin\gamma\omega}{\gamma}\right|\leqq c_6+c_9+c_{12}=c_{13}\,. \tag{11.6}$$

This is the "short" formula of Ingham, adapted to our situation.

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[3] Here and later the c_ν-s are positive numerical constants whose values do not depend on A, B either!

As a last preparation we need the very elegantly proved lemma of INGHAM in [8].

Lemma VI. *Putting*

$$\frac{\log^2 A}{A} \overset{\text{def}}{=} \alpha_0 \tag{11.7}$$

the inequality

$$\left| 2 \sum_{0<\gamma\leq A} \left(1-\frac{\gamma}{A}\right) \frac{\sin \gamma\alpha_0}{\gamma} - \frac{1}{2}\log\frac{1}{|\alpha_0|} \right| \leq c_{14}$$

holds.

Hence for $\pm\alpha_0$ the sum in (11.6) can be made "big positive", resp. "big negative".

12. Let us denote the number of γ's with $0<\gamma\leq A$ by $N(A)$; it is well-known that

$$N(A) < c_{15} A \log A \tag{12.1}$$

and owing to $A > 15$

$$N(A) \geq 1 . \tag{12.2}$$

We shall construct the disjoint intervals

$$I_\nu = [B^{(2\nu-1)N(A)}, B^{2\nu N(A)}], \quad \nu = 1, 2, \ldots; \tag{12.3}$$

if we succeed in producing sign-changes in all I_ν's satisfying (9.3) we shall be ready. The number of such I_ν's is

$$> \frac{1}{10 N(A) \log B} \cdot \log\log\log Y \tag{12.4}$$

if

$$Y > \psi(A, B) . \tag{12.5}$$

13. In order to produce sign-changes in the I_ν's in (9.3) we shall apply Dirichlet's theorem on simultaneous approximation twice. For each such *fixed* I_ν this gives the existence of an ω_ν' and ω_ν'' with

$$\omega_\nu' - \alpha_0 \in I_\nu , \quad \omega_\nu'' + \alpha_0 \in I_\nu , \tag{13.1}$$

so that for each $0<\gamma\leq A$ we have with suitable rational integers $e_{\gamma,\nu}'$ and $e_{\gamma,\nu}''$ the inequalities

$$\left| \frac{\gamma}{2\pi}(\omega_\nu' - \alpha_0) - e_{\gamma,\nu}' \right| \leq \frac{1}{B} , \quad \left| \frac{\gamma}{2\pi}(\omega_\nu'' + \alpha_0) - e_{\gamma,\nu}'' \right| \leq \frac{1}{B} , \tag{13.2}$$

which hold simultaneously. From this one gets easily

$$|\sin \gamma \omega_\nu' - \sin \gamma \alpha_0| \leqq \frac{c_{16}}{B}, \quad |\sin \gamma \omega_\nu'' + \sin \gamma \alpha_0| \leqq \frac{c_{16}}{B} \tag{13.3}$$

and hence using lemma VI

$$2 \sum_{0<\gamma\leqq A} \left(1 - \frac{\gamma}{A}\right) \frac{\sin \gamma \omega_\nu'}{\gamma} > \frac{1}{2} \log \frac{1}{\alpha_0} - c_{14} - \frac{c_{16}}{B} \sum_{0<\gamma\leqq A} \frac{1}{\gamma}.$$

By (12.1) it follows consequently that

$$\sum_{0<\gamma\leqq A} \frac{1}{\gamma} < c_{17} \log^2 A,$$

i. e. choosing

$$B = [\log^2 A] \, (\geqq 8) \tag{13.4}$$

we get from (11.7)

$$\begin{aligned} 2 \sum_{0<\gamma\leqq A} \left(1 - \frac{\gamma}{A}\right) \frac{\sin \gamma \omega_\nu'}{\gamma} &> \frac{1}{2} \log \frac{A}{\log^2 A} - c_{14} - c_{18} = \\ &= \frac{1}{2} \log \frac{A}{\log^2 A} - c_{19}. \end{aligned} \tag{13.5}$$

Putting this into (11.6) we get

$$\frac{1}{2\pi} \int_{-A\omega_\nu'/4}^{A\omega_\nu'/4} \left(\frac{\sin(y/2)}{y/2}\right) G\left(\omega_\nu' + \frac{y}{A}\right) dy < -\frac{1}{2} \log \frac{A}{\log^2 A} + c_{13} + c_{19}$$

which is for $A > c_{20}$

$$< -\tfrac{1}{3} \log A, \tag{13.6}$$

and analogously

$$\frac{1}{2\pi} \int_{-A\omega_\nu''/4}^{A\omega_\nu''/4} \left(\frac{\sin(y/2)}{y/2}\right)^2 G\left(\omega_\nu'' + \frac{y}{A}\right) dy > \frac{1}{3} \log A. \tag{13.7}$$

The restriction (12.5) takes the form $Y > \psi_1(A)$.

Since by (11.1) we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin(y/2)}{y/2}\right)^2 dy = 1,$$

(13.6)—(13.7) gives—using (9.2)—

$$\max_{(3/4)\omega_\nu'' \leqq v \leqq (5/4)\omega_\nu''} e^{-v/2} \Delta_3(e^v) > \tfrac{1}{3}\log A$$

and

$$\min_{(3/4)\omega_\nu' \leqq v \leqq (5/4)\omega_\nu'} e^{-v/2} \Delta_3(e^v) < -\tfrac{1}{3}\log A;$$

using lemma V we get

$$\max_{(3/4)\omega_\nu'' \leqq v \leqq (5/4)\omega_\nu''} v\, e^{-v/2} \Delta_1(e^v) > \tfrac{1}{3}\log A - \tfrac{5}{4} > \tfrac{1}{4}\log A \tag{13.8}$$

for $A > c_{21}$ and also

$$\min_{(3/4)\omega_\nu' \leqq v \leqq (5/4)\omega_\nu'} v\, e^{-v/2} \Delta_1(e^v) < -\tfrac{1}{4}\log A. \tag{13.9}$$

14. We wrote in **12.** that "... if we succeed in producing sign-changes in all I_ν's satisfying (9.3), we shall be ready ...". This was written only as an indication of the line of reasoning. In reality we can do this only for the intervals

$$I_\nu^* \overset{\text{def}}{=} [\tfrac{3}{4}\min(\omega_\nu', \omega_\nu''), \tfrac{5}{4}\max(\omega_\nu', \omega_\nu'')]. \tag{14.1}$$

The existence of a sign-change of $\Delta_1(e^v)$ in I_ν^* follows from (13.8)—(13.9); we shall keep only those in $[\log\log\log Y, \frac{1}{5}\log\log Y]$ and we have only to show that they are disjoint. But owing to (13.1) and (11.7) and $B \geqq 8$ we have

$$\tfrac{5}{4}\max(\omega_\nu', \omega_\nu'') \leqq \tfrac{5}{4}(B^{2\nu N(A)} + 1) \leqq \tfrac{5}{2}B^{2\nu N(A)} \leqq (5/2B)\, B^{(2\nu+1)N(A)} \leqq$$

$$\leqq (5/B)B^{(2\nu+1)N(A)} - 1) \leqq (5/B)\min(\omega_{\nu-1}', \omega_{\nu+1}'') < \tfrac{3}{4}\min(\omega_{\nu+1}', \omega_{\nu+1}''),$$

and analogously

$$\tfrac{3}{4}\min(\omega_\nu', \omega_a'') > \tfrac{5}{4}\max(\omega_{\nu-1}, \omega_{\nu-1}),$$

so that the I_ν^*'s are disjoint indeed. Thus observing that all of our previous restrictions on A were lower limitations we can choose $A = c_{22}$ and hence for $Y > c_{23}$

$$V_1(Y) > V_1(\log^{1/5} Y) > c_{24}\log\log\log Y$$

indeed. Thus also case II is settled.

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