TWO OSCILLATION THEOREMS*

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Ingham established in [7] a method of estimating from below the oscillation of a real-valued function in terms of the singularities of its Laplace transform. This technique and a clever searching procedure with an electronic computer have led to the disproof of various conjectures in number theory (cf. Haselgrove [6], Lehman [9]). In this note we shall give some theorems which are analogous to Ingham's, but are established in a rather different way.

Let f be a real-valued measurable function on $[0,\infty)$ and let $F(s) = \int_0^\infty e^{-su} f(u)\,du$. We assume that the integral converges for $\operatorname{Re} s > 0$ and that F can be continued as a meromorphic function to a region of the complex plane which includes the imaginary axis. Assume further that there are poles of F on the imaginary axis and that they are all simple. Let

$$T = \{t_1, t_2, \ldots\} = \{t > 0: ti \text{ is a pole of } F\},$$

a finite or countable set. Let a_n be the residue of F at it_n and let a_o be the residue of F at 0 ($a_o = 0$ if F is regular at 0).

We say that a finite subset

$$T_1 = \{t_{i_1}, t_{i_2}, \ldots, t_{i_J}\} \subset T$$

is a <u>weakly independent subset of</u> T <u>of order</u> N provided that $\sum_{j=1}^{J} n_j t_{i_j} \in T$ for some integers n_j with $|n_j| \le N$ implies that $\sum_{j=1}^{J} |n_j| = 1$. That is, the only way that a sum $\sum n_j t_{i_j}$ with $|n_j| \le N$ can represent an element of T is that one $n_j = 1$ and all others be zero.

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We formulate our oscillation theorems in terms of the notion of weak independence. Note that if T is linearly independent over the integers, then any finite subset satisfies a weak independence condition of any order. It is not hard to establish the equivalence of our two theorems. We shall prove the second theorem and indicate how one deduces the first.

Theorem 1. *Suppose there exist a finite collection of indices* $\mathcal{J}$ *and a positive integer* N *such that* $\{t_j \in T\colon j \in \mathcal{J}\}$ *is a weakly independent subset of* T *of order* N. *Then*

$$\lim_{X\to\infty} \operatorname*{ess\,sup}_{u \ge X} f(u) \ge a_o + \frac{2N}{N+1} \sum_{j \in \mathcal{J}} |a_j|,$$

$$\lim_{X\to\infty} \operatorname*{ess\,inf}_{u \ge X} f(u) \le a_o - \frac{2N}{N+1} \sum_{j \in \mathcal{J}} |a_j|.$$

Theorem 2. *In addition to the hypotheses of Theorem 1, assume that* $\frac{2N}{N+1} \sum_{j \in \mathcal{J}} |a_j| > |a_o|$. *Then there exists no real number* X *such that* f *(or some* L^1 *equivalent) is of one sign on* $[X,\infty)$.

Proof of Theorem 2. The case in which $a_o = 0$, i.e., F regular at 0, follows immediately from Landau's Theorem on Mellin transforms of nonnegative functions [8; Satz 454]. (It could also be handled directly.) We henceforth assume for the proof of Theorem 2 that $a_o > 0$. Let K_N be the N-th Fejér kernel

$$K_N(x) = \sum_{-N}^{N} \left(1 - \frac{|n|}{N+1}\right)e^{inx} = \frac{1}{N+1} \left|\sum_{0}^{N} e^{inx}\right|^2 \ge 0.$$

Let J be the cardinality of $\mathcal{J}$ and let $x_1, x_2, \ldots, x_J$ be a sequence of real numbers. Form the product

$$0 \le \prod_{j=1}^{J} K_N(x_j) = \sum_{n_1=-N}^{N} \cdots \sum_{n_j=-N}^{N} \prod_{j=1}^{J} \left\{\left(1 - \frac{|n_j|}{N+1}\right)e^{in_jx_j}\right\}$$

$$= 1 + 2\left(1 - \frac{1}{N+1}\right) \sum_{j=1}^{J} \operatorname{Re} e^{ix_j} + S,$$

where S is a real-valued trigonometric polynomial each of whose terms involves at least two x_j's (possibly the same x_j repeated). For notational simplicity we assume that $\mathcal{J} = \{1, 2, \ldots, J\}$. Let $c_1, \ldots, c_J$ be real numbers such that $\arg a_j e^{ic_j} \equiv \pi \pmod{2\pi}$, and set $x_j = -ut_j + c_j$, where $t_j \in T$ and u is an integration variable to be specified.

We assume that the conclusion of Theorem 2 is false, and there exists an X such that f is of one sign on $[X, \infty)$. For $\sigma > 0$ we form

$$\left\{\int_{u=0}^{X} + \int_{X}^{\infty}\right\}\left\{f(u)e^{-\sigma u} \prod_{j=1}^{J} K_N(-ut_j + c_j)\,du\right\}$$

$$= F(\sigma) + \frac{2N}{N+1} \sum_{j=1}^{J} \operatorname{Re}\,\{F(\sigma+it_j)e^{ic_j}\} + \sum e_n F(\sigma+i\tau_n),$$

with suitable e_n's and τ_n's. By the weak independence condition no τ_n lies in T. Now let $\sigma \to 0+$ and note that $\int_{u=0}^{X} = O(1)$, $\sum e_n F(\sigma+i\tau_n) = O(1)$, $\int_{u=X}^{\infty} \geq 0$, and $F(\sigma+it_j)e^{ic_j} \sim -|a_j|\sigma^{-1}$. We conclude that

$$a_o - \frac{2N}{N+1} \sum_{j=1}^{J} |a_j| \geq 0,$$

in contradiction to one of the hypotheses. Thus Theorem 2 is true.

We sketch the proof of Theorem 1 for the lower bound. Let ϵ be a small positive number and take

$$a_o' = a_o - \frac{2N}{N+1} \sum_{j \in \mathcal{J}} |a_j| + \epsilon.$$

Set $g(u) = f(u) - a_o'$ and apply Theorem 2 to show that there exist arbitrarily large values of X such that $\operatorname{ess\,inf}_{u \geq X} g(u) < 0$. Changing the last relation to one involving f and letting $\epsilon \to 0+$, we obtain the second estimate of Theorem 1.

Remarks. 1. It is clear from the proof that the meromorphy requirement in both theorems is unnecessarily strong and can be replaced by weaker but messier conditions.

2. The idea of using a product of Fejér kernels plus an independence condition to estimate the values of a function dates back at least to Bohr and Jessen [2].

3. Theorems 1 and 2 yield new proofs of some theorems and offer the possibility of improving some known estimates. In particular, we obtain another proof of the "variant form of Ingham's theorem" of Bateman et al. [1] and can improve estimates of Grosswald [4], [5] and Saffari [10], [11], [12].

4. To obtain better numerical estimates in specific cases, one should alter the proof of Theorem 2 as follows. First, in place of the N-th Fejér kernel one should use the nonnegative trigonometric polynomial

$$P_N^*(t) = \left|\sum_{j=0}^{N} \left(\sin\frac{[j+1]\pi}{N+2}\right)e^{ijt}\right|^2 \left(\sum_{j=0}^{N} \sin^2 \frac{[j+1]\pi}{N+2}\right)^{-2}$$

$$= 1 + \left(2\cos\frac{\pi}{N+2}\right)\cos t + \cdots .$$

It was proved by Fejér [3; page 79] that, for each N, this polynomial has the largest value of λ_1 among all nonnegative trigonometric polynomials of the form

$$1 + \lambda_1 \cos t + \cdots + \lambda_N \cos Nt.$$

Second, in place of the product of J nonnegative trigonometric polynomials each of degree N, one should multiply together the polynomials $P_{N_j}^*(x_j)$, where the N_j are judiciously chosen and are generally not all equal.

With these changes Theorem 1 can be stated as follows: Suppose that $N_1, \ldots, N_J$ are nonnegative integers such that

$$\sum_{j=1}^{J} n_j t_j \in T, \quad |n_j| \le N_j \Rightarrow \sum_{j=1}^{J} |n_j| = 1.$$

Then

$$\lim_{X\to\infty} \operatorname*{ess\,sup}_{u \geq X} f(u) \geq a_o + 2\sum_{j=1}^{J} \left(\cos\frac{\pi}{N_j+2}\right)|a_j|$$

and

$$\lim_{X\to\infty} \operatorname*{ess\,inf}_{u \geq X} f(u) \leq a_o - 2\sum_{j=1}^{J} \left(\cos\frac{\pi}{N_j+2}\right)|a_j|.$$

Also, we can replace the hypotheses of Theorem 2 by the above assumption and the inequality

$$2\sum_{j=1}^{J} \left(\cos\frac{\pi}{N_j+2}\right)|a_j| > |a_o|.$$

The conclusion of Theorem 2 then follows.

5. The present theorems appear to be of no use in disproving Mertens' conjecture that $\left|\sum_{n \leq x} \mu(n)\right| \leq \sqrt{x}$, $x \geq 1$, μ = Möbius' function. The number of weak independence relations to be checked is far beyond the capacity of present day computers. We investigated the applicability of the present method to the Pólya conjecture that $\sum_{n \leq x} \lambda(n) \leq 0$, $x \geq 2$, λ = Liouville's function. This was shown to be false by Haselgrove [6] using Ingham's result and computing. Using our method, improved as described in the preceding remark, and a table contained in [6], we showed how one could hope to obtain the required weak independence by checking about 10^{12} sums, each involving at most 12 summands. Unfortunately, it would require several weeks of computer time to carry out such a project. This example suggests that Ingham's theorem and "educated" trials on a computer is more practical that our method for numerical work.

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