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ON INVESTIGATIONS IN THE COMPARATIVE PRIME NUMBER THEORY

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(*Presented by* P. TURÁN)

1. Introduction. In the analytical number theory an important direction of the investigations is the omega-estimations of number-theoretical functions. The first, significant steg was the famous Landau's theorem concerning the Dirichlet-series with positive coefficients (E. LANDAU [1]). This theorem asserts that if the Dirichlet-series $F(s)=\sum \frac{a_n}{n^s}$ has nonnegative coefficients, further its convergence-abscissa is α, then α is a singular point of $F(s)$. From this theorem easily follows that e.g. $\overline{\lim}\, M(x)\cdot x^{-\frac{1}{2}+\varepsilon}>0$, $\underline{\lim}\, M(x)\cdot x^{-\frac{1}{2}+\varepsilon}<0$, $M(x)=\sum_{n\leqq x}\mu(n)$. In other words there exists a sequence $0<x_1'<x_1''<\ldots<x_n'<x_n''<\ldots$, $x_n'\to\infty$ such that $M(x_\nu')>x_\nu'^{\frac{1}{2}-\varepsilon}$, $M(x_\nu'')<-x_\nu''^{\frac{1}{2}-\varepsilon}$. But the question concerning the density of the values is not answered by this theorem.

Recently S. KNAPOWSKI and P. TURÁN [1], [2], [3] elaborated a method which gives omega-estimations of many number theoretic functions, in more effective form. Their proofs were based on the very deep results of Prof. TURÁN in the theory of diophantine approximation. S. KNAPOWSKI has dealt with similar questions in the papers [1], [2], [3], and W. STAS in [1], [2], [3].

In this paper we shall deal with similar questions. The author in [1] obtained some new effective results for number-theoretic functions without any conjectures. In the proofs of these theorems an idea of RODOSSKY [1] is important. These results were published partially in author's paper [4]. The aim of this paper is to obtain a more general theorem. Our earlier theorems will be direct consequences of this theorem.

Throughout this paper $c, c_1, c_2, \ldots$ will denote explicitely calculable numerical constants not necessarily the same at every occurence. Further we use the notation $e_1(x)=e^x$.

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2. A theorem on Dirichlet integrals. Let

$$f(s)=\int_1^\infty \frac{dA(x)}{x^s} \qquad (s=\sigma+it) \tag{2.1}$$

and let the integral be absolutely convergent on the halfplane $\sigma>\sigma_1$. Let us suppose that $A(x)$ is real and that

$$|A(x)|\leqq c_1 x^{\theta_1} \quad \text{if} \quad x\geqq 1, \tag{2.2}$$

where c_1 is a constant. Let, further $f(s)$ be analitically continuable on the halfplane $\sigma > \theta_2 - \delta_1$, where $\delta_1 > 0$ is a suitable constant, $0 < \theta_2 < \theta_1$. Let

(2. 3) $$\varrho = \theta_2 + i\gamma \qquad (\gamma \neq 0)$$

a pole of multiplicity k of $f(s)$.

Let the Laurent-expansion over $s = \varrho$ of $f(s)$ be of the form

(2. 4) $$f(s) = P((s-\varrho)^{-1}) + g(s),$$

where

(2. 5) $$P(x) = b_1 x + b_2 x^2 + \ldots + b_k x^k \qquad (b_k \neq 0),$$

is a polynomial of degree k, and $g(s)$ is a function, which is regular in the neighbourhood of $s = \varrho$.

Let $D(\varepsilon)$ denote the set of $s = \sigma + it$, which satisfy the inequalities $\sigma^2 - t^2 > (\theta_2 - \varepsilon)^2$, $\theta_2 - \varepsilon \leqq \sigma \leqq \sigma_1 + \varepsilon$ i.e. in notation

(2. 6) $$D(\varepsilon) = \{s;\ \sigma^2 - t^2 > (\theta_2 - \varepsilon)^2,\ \theta_2 - \varepsilon \leqq \sigma \leqq \sigma_1 + \varepsilon\}.$$

Now we define the curves $C_1(\varepsilon)$, $C_2(\varepsilon)$ as

(2. 7) $$C_1(\varepsilon) = \left\{s;\ \sigma^2 - t^2 = \left(\theta_2 - \frac{\varepsilon}{2}\right)^2, \theta_2 - \frac{\varepsilon}{2} \leqq \sigma \leqq \sigma_1 + \varepsilon\right\},$$

(2. 8) $$C_2(\varepsilon) = \left\{s;\ \sigma = \sigma_1 + \varepsilon,\ t^2 > (\sigma_1 + \varepsilon)^2 - \left(\theta_2 - \frac{\varepsilon}{2}\right)^2\right\}.$$

Let us suppose that for a suitable $\varepsilon > 0$ $f(s)$ is regular in the domain $s \in D(\varepsilon)$, and $f(s + i\gamma)$ is also regular in $s \in D(\varepsilon)$, except for the point $s = \theta_2$.

Let us fix this value of ε.

The function $f(s)$ and $f(s + i\gamma)$ are bounded on $C_1(\varepsilon) \cup C_2(\varepsilon)$. The boundedness on $C_1(\varepsilon)$ follows from the finiteness of the length of $C_1(\varepsilon)$, and from $C_1(\varepsilon) \subset D(\varepsilon)$. The boundedness on $C_2(\varepsilon)$ follows from the absolute convergence of (2. 1) on $\sigma > \sigma_1$.

Let

(2. 9)—(2. 10) $$M_1 = \max_{s \in C_1 \cup C_2} |f(s)|, \quad M_2 = \max_{s \in C_1 \cup C_2} |f(s + i\gamma)|.$$

From these conditions we obtain the following

THEOREM. *For $T > c_2$ we have*

(2. 11) $$\max_{T \leqq x \leqq T^{\varkappa}} \frac{A(x)}{x^{\theta_2} (\log x)^{k-1}} > \delta,$$

(2. 12) $$\min_{T \leqq x \leqq T^{\varkappa}} \frac{A(x)}{x^{\theta_2} (\log x)^{k-1}} < -\delta,$$

where

(2. 13) $$\varkappa = \left(\frac{\theta_1}{\theta_2} + \sqrt{\left(\frac{\theta_1}{\theta_2}\right)^2 - 1}\right)^2,$$

and $\delta>0$, $c_2>0$ *are suitable constants. They are numerically calculable functions of* $\theta_1, \theta_2, \varepsilon, M_1, M_2, \gamma$ *and of the coefficients of* $P(x)$.

3. Lemmas. For the proof of Theorem we need the following lemmas.

LEMMA 1. *Let* τ *be a real number and*

$$I(\tau)=\int_1^{\infty} e_1\left(-i\tau\log x-\frac{\log^2 x}{4u}\right)dA(x). \tag{3.1}$$

Then the relation

$$I(\tau)=-\frac{i\sqrt{u}}{\sqrt{\pi}}\int_{\sigma_1+\varepsilon-i\infty}^{\sigma_1+\varepsilon+i\infty} f(w+i\tau)e_1(w^2u)\,dw \tag{3.2}$$

holds by means of (2. 1).

For the proof see K. PRACHAR [1], p. 381. In the PRACHAR's book is proved only the case $A(x)=\sum_{n\le x}a_n$, but it is easily seen, that in the general case it is also true.

LEMMA 2. *If* $0<\beta\le 1$ *and* $u\to\infty$, *then*

$$\frac{1}{2u}\int_1^{\infty} x^{\beta-1}\log x\cdot e_1\left(-\frac{\log^2 x}{4u}\right)dx=2\sqrt{\pi u}\,\beta e_1(\beta^2u)+O(1). \tag{3.3}$$

LEMMA 3. *If* $0<\alpha<\theta$, $\log y=2u(\theta-\sqrt{\theta^2-\alpha^2})$, $\log z=2u(\theta+\sqrt{\theta^2-\alpha^2})$, *then we get*

$$\frac{1}{2u}\int_1^{y} x^{\theta-1}\log x\cdot e_1\left(-\frac{\log^2 x}{4u}\right)dx<\theta(\theta^2-\alpha^2)^{-\frac{1}{2}}e_1(\alpha^2u), \tag{3.4}$$

$$\frac{1}{2u}\int_z^{\infty} x^{\theta-1}\log x\cdot e_1\left(-\frac{\log^2 x}{4u}\right)dx<2\theta(\theta^2-\alpha^2)^{-\frac{1}{2}}e_1(\alpha^2u), \tag{3.5}$$

$$\frac{1}{2u}\int_1^{y} x^{\theta-1}e_1\left(-\frac{\log^2 x}{4u}\right)dx<(\theta^2-\alpha^2)^{-\frac{1}{2}}e_1(\alpha^2u). \tag{3.6}$$

For the proofs of Lemmas 2 and 3 see K. A. RODOSSKY [1].

4. Proof of the Theorem. First we prove that

$$I(0)=O\left(M_1\sqrt{u}\,e_1\left(\left(\theta_2-\frac{\varepsilon}{2}\right)^2u\right)\right), \tag{4.1}$$

$$I(\gamma)=2\sqrt{u\pi}\,\operatorname*{Res}_{w=\theta_2}f(w+i\gamma_0)e_1(w^2u)+O\left(M_2\sqrt{u}\,e_1\left(\left(\theta_2-\frac{\varepsilon}{2}\right)^2u\right)\right). \tag{4.2}$$

Applying the representation (3. 2) at $\tau=0$, and transforming the way of integration to $C_1\cup C_2$ (the value of integral is not changing), so

$$|I(0)|\le\sqrt{\frac{u}{\pi}}M_1\int_{C_1}e_1((\sigma^2-t^2)u)\,|dw|+2\sqrt{\frac{u}{\pi}}M_1\int_{C_2}e_1((\sigma^2-t^2)u)\,dt.$$

By (2. 7) and (2. 8) the first term is $O\left(M_1\sqrt{u}e_1\left(\left(\theta_2-\frac{\varepsilon}{2}\right)^2 u\right)\right)$, and the second is $O\left(\frac{M_1}{\sqrt{u}}e_1\left(\left(\theta_2-\frac{\varepsilon}{2}\right)^2 u\right)\right)$. From this (4. 1) follows. The proof of (4. 2) is similar. Transforming the way of integral in the relation (3. 2) at $\tau=\gamma$ to $C_1\cup C_2$, we go through a pole $w=\theta_2$. The module of the integral (3. 2) on $C_1\cup C_2$ is $O\left(M_2\sqrt{u}e_1\left(\left(\theta_2-\frac{\varepsilon}{2}\right)^2 u\right)\right)$, and the residue at the point $w=\theta_2$ is the first term of right hand side of (4. 2).

We take into consideration the residue. By easy calculation

$$e_1(w^2u) = e_1(\theta_2^2 u)\sum_{n=0}^{\infty} c_n(u)(w-\theta_2)^n,$$

where

$$c_n(u) = \sum_{\substack{2j+m=n\\ j\geq 0}} \frac{(2\theta_2)^m u^{(m+n)/2}}{m!\,j!}$$

is a polynomial in u of degree n.

Considering (2. 4), (2. 5) we have

$$2\sqrt{u\pi}\operatorname*{Res}_{w=\theta_2} e_1(w^2u)f(w+i\gamma) = 2\sqrt{u\pi}e_1(\theta^2u)\sum_{j=1}^{k} b_j c_{j-1}(u) = \sqrt{u}e_1(\theta_2^2u)Q_{k-1}(u),$$

where $Q_{k-1}(u)$ is a polynomial of degree $(k-1)$, with main coefficient $2\sqrt{\pi}\frac{(2\theta_2)^{k-1}}{(k-1)!}$. So

$$|I(\gamma)| > c_3 u^{k-\frac{1}{2}} e_1(\theta_2^2 u), \tag{4. 3}$$

if $u>c_4$. The choice of c_3 depends only on $\theta_1, \theta_2, \varepsilon, c_1, M_2$.

5. After these preliminaries we begin the proof. Let us introduce the following notations:

$$\log y = 2u\left(\theta_1-\sqrt{\theta_1^2-\theta_2^2}\right); \tag{5. 1}$$

$$\log z = 2u\left(\theta_1+\sqrt{\theta_1^2-\theta_2^2}\right). \tag{5. 2}$$

Let u so large that $\log y>1$. Let $\delta>0$ be a constant, which we choose later, and let

$$B_{\pm}(x) = A(x)\pm\delta(\log x)^{k-1}\cdot x^{\theta_2}. \tag{5. 3}$$

Suppose now that at least one of the following inequalities

$$\min_{y\leqq x\leqq z} B_+(x)\geqq 0, \quad \max_{y\leqq x\leqq z} B_-(x)\leqq 0 \tag{5. 4)—(5. 5}$$

is satisfied. We will show that this assumption contradicts (4. 1) and (4. 4). Hence our theorem follows immediately.

From (3. 1) by partial integration

$$(5.6)\qquad I(\tau)=\int_1^\infty \frac{A(x)}{x}\left(i\tau+\frac{\log x}{2u}\right)e_1\left(-i\tau\log x-\frac{\log^2 x}{4u}\right)dx.$$

Let $K_1(\tau)$, $K_2(\tau)$, $K_3(\tau)$ the parts of (5. 6) on the intervals $[1, y]$, $[y, z]$, $[z, \infty)$.

Applying (5. 4) or (5. 5) we have

$$(5.7)\qquad |K_2(\tau)|\leqq\left|\int_y^z \frac{B_\pm(x)}{x}\left|i\tau+\frac{\log x}{2u}\right|e_1\left(-\frac{\log^2 x}{4u}\right)dx\right|+$$

$$+\delta\int_y^z x^{\theta_2-1}(\log x)^{k-1}\left|i\tau+\frac{\log x}{2u}\right|e_1\left(-\frac{\log^2 x}{4u}\right)dx=R_1+R_2.$$

From (4. 3) it follows that

$$(5.8)\qquad R_1\leqq\left|\int_y^z \frac{A(x)}{x}\left|i\tau+\frac{\log x}{2u}\right|e_1\left(-\frac{\log^2 x}{4u}\right)dx\right|+R_2.$$

Using the inequality

$$\left|i\tau+\frac{\log x}{2u}\right|\leqq|\tau|+\frac{\log x}{2u}\leqq\left(|\tau|\cdot\frac{2u}{\log y}+1\right)\frac{\log x}{2u}<c_6(|\tau|+1),$$

we obtain from (5. 8) that

$$(5.9)\qquad R_1\leqq c_6(|\tau|+1)K_2(0)+R_2.$$

Further by (2. 2) and (3. 4), (3. 6) it follows

$$(5.10)\qquad R_2\leqq\delta c_6(|\tau|+1)(\log z)^{k-1}\cdot\frac{1}{2u}\int_y^z x^{\theta_2-1}\cdot\log x\cdot e_1\left(-\frac{\log^2 x}{4u}\right)dx<$$

$$<\delta c_7(|\tau|+1)u^{k-\frac{1}{2}}e_1(\theta_2^2u).$$

From (5. 7), (5. 8), (5. 9), (5. 10) we obtain that

$$(5.11)\qquad |K_2(\tau)|\leqq c_6(|\tau|+1)K_2(0)+2\delta c_7(|\tau|+1)u^{k-\frac{1}{2}}e_1(\theta_2^2u).$$

Further, from (2. 2), (3. 4), (3. 6) it follows

$$(5.12)\qquad |K_1(\tau)|\leqq\int_1^y \frac{|A(x)|}{x}\left|i\tau+\frac{\log x}{2u}\right|e_1\left(-\frac{\log^2 x}{4u}\right)dx\leqq$$

$$\leqq c_1(|\tau|+2\theta_1)(\theta_1^2-\theta_2^2)^{-\frac{1}{2}}e_1(\theta_2^2u)\leqq c_8(|\tau|+1)e_1(\theta_2^2u),$$

and similarly, from (2. 2) and (3. 5) we get

$$(5.13)\qquad |K_3(\tau)|\leqq c_9(|\tau|+1)e_1(\theta_2^2u).$$

Now taking into account that $I(\tau)=K_1(\tau)+K_2(\tau)+K_3(\tau)$, from the inequalities (5. 7)—(5. 13) it follows that

$$(5.14)\qquad |I(\tau)|\leqq c_{10}(|\tau|+1)\,\{|I(0)|+\delta u^{k-\frac{1}{2}}e_1(\theta_2^2 u)+e_1(\theta_2^2 u)\}.$$

Now choose $\tau=\gamma$ and compare (5. 14) with (4. 1) and (4. 3). So, from (5. 14) the inequality

$$c_3 u^{k-\frac{1}{2}}e_1(\theta_2^2 u)\leqq c_{10}(|\gamma|+1)\left\{O\left(M_1\sqrt{u}\,e_1\left(\left(\theta_2-\frac{\varepsilon}{2}\right)^2 u\right)+\delta u^{k-\frac{1}{2}}e_1(\theta_2^2 u)+e_1(\theta_2^2 u)\right\}$$

follows for $u>c_4$. But this is not true, if $\delta<c_3$ and $u>c_{11}$. So the inequalities (5. 4), (5. 5) cannot be satisfied if $\delta<c_3$, $u>c_{11}$. Let us now take $y=T$, so $z=T^{\varkappa}$ (see (2. 13)) and the Theorem is proved.

6. Number-theoretical applications. We mention now some corollaries of our theorem.

6. 1. *On the Moebius-function. Let*

$$M(x)=\sum_{n\leqq x}\mu(n),$$

where $\mu(n)$ is the Moebius function. Apply the Theorem with $\theta_1=1$, $\theta_2=\frac{1}{2}$, $f(s)=\frac{1}{\zeta(s)}$, $M(x)=A(x)$, $\varrho=\varrho_0$, $\varrho_0=\frac{1}{2}+i\gamma_0$, $\gamma_0=14, 13\ldots$ where ϱ_0 is a simple root of $\zeta(s)$. It is well known that in the domain $0<\sigma<1$, $0<t<20$ the function $\zeta(\sigma+it)$ has no other root. Thus the conditions of our Theorem are satisfied and the following assertion holds.

THEOREM 1. *If* $T>c_1$, *then*

$$\max_{T\leqq x\leqq T^{\varkappa}} x^{-\frac{1}{2}}M(x)\geqq\delta,\qquad \min_{T\leqq x\leqq T^{\varkappa}} x^{-\frac{1}{2}}M(x)\leqq-\delta,$$

where $\varkappa=(2+\sqrt{3})^2$, *and* $c_1>0$, $\delta>0$ *are explicitely calculable numerical constants.*

This theorem can be found in the author's paper [4], too. With similar questions deals Mr. KNAPOWSKI in his papers [1], [2], [3], and the author in [3], [4].

6. 2. *On the k-free numbers.* Let

$$\varrho_k(n)=\begin{cases}1, & \text{if } n \text{ is } k\text{-free}\\ 0 & \text{otherwise,}\end{cases}$$

$$P_k(x)=R_k(x)-\frac{[x]}{\zeta(k)}=\sum_{n\leqq x}\varrho_k(n)-\frac{[x]}{\zeta(k)}.$$

It is well known that $P_k(x)=O(x^{1/k})$. Applying the Theorem with

$$\theta_1=\frac{1}{k},\quad \theta_2=\frac{1}{2k},\quad f(s)=\frac{\zeta(s)}{\zeta(ks)}-\frac{\zeta(s)}{\zeta(k)}=\sum\frac{\varrho_k(n)-\frac{1}{\zeta(k)}}{n^s},$$

$$A(x)=P_k(x),\quad \varrho=\frac{\varrho_0}{k}$$

we obtain the following

THEOREM 2. *If $T>d_k$, then*

$$\max_{T\leqq x\leqq T^{\varkappa}} P_k(x)\cdot x^{-\frac{1}{2k}} \geqq \delta_k, \quad \min_{T\leqq x\leqq T^{\varkappa}} P_k(x)\cdot x^{-\frac{1}{2k}} \leqq -\delta_k$$

where $\varkappa=(2+\sqrt{3})^2$, and $\delta_k>0$, $d_k>0$ are constants, which are explicitely calculable functions of k.

This theorem can be found in author's paper [4].

6. 3. Prof. GELFOND raised the problem to extend our investigations for

$$M(x,k,l) = \sum_{\substack{n\leqq x\\ n\equiv l \pmod k}} \mu(n),$$

and for

$$M(x,\chi,k) = \sum_{n\leqq x} \mu(n)\chi(n),$$

if χ is a real character mod k. It seems to be difficult to answer these questions, because we do not know much on the real root of $L(s,\chi)$ in the critical strip. We hope to return to these questions later.

We are dealing with the case $k=4$ only.

THEOREM 3. *If $T>c_1$ then*

$$\max_{T\leqq x\leqq T^{\varkappa}} x^{-\frac{1}{2}} M(x,4,l) \geqq \delta, \quad \min_{T\leqq x\leqq T^{\varkappa}} x^{-\frac{1}{2}} M(x,4,l) \leqq -\delta, \tag{6. 3. 1}$$

$$\max_{T\leqq x\leqq T^{\varkappa}} x^{-\frac{1}{2}} M(x,\chi) \geqq \delta, \quad \min_{T\leqq x\leqq T^{\varkappa}} x^{-\frac{1}{2}} M(x,\chi) \leqq -\delta, \tag{6. 3. 2}$$

where $\varkappa=(2+\sqrt{3})^2$, $l\equiv 1$ or 3 mod 4, and χ is the non-principal character mod 4, *c_1 and δ are calculable numerical constants.*

For the proof of the inequalities (6. 3. 1) we apply the Theorem with

$$f(s) = \frac{1}{2}\left(\frac{1}{L(s,\chi_0)}+\frac{\chi(l)}{L(s,\chi)}\right), \quad \theta_1 = 1, \quad \theta_2 = \frac{1}{2}, \quad \varrho = \varrho_1 = \frac{1}{2}+i\cdot\gamma_1,$$

$\gamma_1=6{,}020\ldots$ where ϱ_1 is a simple root of $L(s,\chi)$. It is known, that in the domain $0<\sigma<1$, $0\leqq t\leqq 10$ the function has no other root, and in this domain $L(s,\chi_0)$ is non-vanishing. (See KNAPOWSKI—TURÁN [3], p. 254).

For the proof of (6. 3. 2) let

$$f(s) = \frac{1}{L(s,\chi)}, \quad \theta_1 = 1, \quad \theta_2 = \frac{1}{2}, \quad \varrho = \varrho_1.$$

6. 4. An other interesting question is the oscillatorious behavior of

$$M_0(x) = \sum_{n\leqq x} \frac{\mu(n)}{n}.$$

From the prime number theorem it follows $M_0(x)=o(1)$, if $x\to\infty$, and from the quoted theorem of Landau we obtain $M_0(x)=\Omega_{\pm}(x^{-\frac{1}{2}})$. We prove the following

THEOREM 4. *If $T > c_1$ then*

$$\max_{T \leqq x \leqq T^{\varkappa}} M_0(x) \cdot x^{\frac{1}{2}} \geqq \delta, \quad \min_{T \leqq x \leqq T^{\varkappa}} M_0(x) \cdot x^{\frac{1}{2}} \leqq -\delta,$$

where $\varkappa = (2+\sqrt{3})^2$, $c_1 > 0$, $\delta > 0$ are explicitely calculable absolute constants.

PROOF. Let

$$f(s) = \int_1^\infty x^{-s}\, dx M_0(x) = \int_1^\infty \frac{M_0(x)}{x^s}\, dx = \frac{1}{(s-1)} \cdot \frac{1}{\zeta(s)}; \quad A(x) = x M_0(x),$$

$\theta_1 = 1$, $\theta_2 = \frac{1}{2}$, $\varrho = \varrho_0$. The conditions of our Theorem are satisfied.

6. 5. *On a formula of Ramanujan.* Let

$$S(\beta) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e_1\left(-\left(\frac{\beta}{n}\right)^2\right),$$

be the so-called Ramanujan-formula. HARDY and LITTLEWOOD [1] proved that the estimation $S(\beta) = O(\beta^{-\frac{1}{2}+\varepsilon})$ is equivalent with the Riemann-conjecture. W. STAS [1], [2], [3] considered the Ω-properties of this sum assuming the Riemann-hypothesis and other conjectures. The author dealt with similar questions in [2], [6]. We prove

THEOREM 5. *If $T > c_1$ then*

$$\max_{T \leqq \beta \leqq T^{\varkappa}} \beta^{\frac{1}{2}} S(\beta) > \delta, \quad \min_{T \leqq \beta \leqq T^{\varkappa}} \beta^{\frac{1}{2}} S(\beta) < -\delta,$$

where $\varkappa = (2+\sqrt{3})^2$, $\delta > 0$, $c_1 > 0$ are explicitely calculable numerical constants.

PROOF. It is known that

$$\beta S(\beta) = \frac{1}{2\pi i} \int_{(\frac{1}{2}+\varepsilon_1)} \beta^{2s} \frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\, ds \qquad (\varepsilon_1 > 0).$$

From this and by the Mellin-formula we obtain

$$\int_0^\infty \beta^{-s} \frac{S(\sqrt{\beta})}{\sqrt{\beta}}\, d\beta = \frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}.$$

Using the prime-number theorem in the form $\sum \frac{\mu(n)}{n} = 0$, we have

$$S(\beta) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left\{ e_1\left(-\left(\frac{\beta}{n}\right)^2\right) - 1 \right\} = O(\beta^2) \quad \text{if} \quad 0 \leqq \beta \leqq 1.$$

Thus the function $\varphi(s)$ defined by

$$\varphi(s) = \int_0^1 \beta^{-s} \frac{S(\sqrt{\beta})}{\sqrt{\beta}}\, d\beta$$

is regular and $\varphi(s)=O\left(\frac{1}{|\sigma-\frac{3}{2}|}\right)$ on the halfplane $\sigma<\frac{3}{2}$. By partial integration

$$h(s)\overset{\text{def}}{=}\int_1^\infty \beta^{-s}\,d(\beta^{\frac{1}{2}}S(\sqrt{\beta}))=s\left(\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}-\varphi(s)\right)+S(1).$$

Let us now apply our Theorem, choosing $f(s)=h(s)$, $\theta_2=\frac{1}{4}$, $\theta_1=\frac{1}{2}$, $\varrho=\frac{\varrho_0}{2}$. Hence the theorem follows.

6. 6. *On a theorem of M. Riesz.* Let

$$\mathcal{T}(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{(k-1)!\,\zeta(2k)}.$$

M. Riesz proved [1] that the estimation $\mathcal{T}(x)=O(x^{\frac{1}{4}+\varepsilon})$ and the Riemann-conjecture are equivalent.

Using similar arguments as in § 6. 5, we can prove the following

THEOREM 6. *If $T>c_1$ then*

$$\max_{T\leqq x\leqq T^\varkappa}\mathcal{T}(x)\cdot x^{-\frac{1}{4}}>\delta,\qquad \min_{T\leqq x\leqq T^\varkappa}\mathcal{T}(x)\cdot x^{-\frac{1}{4}}<-\delta$$

where $\varkappa=(2+\sqrt{3})^2$, $c_1>0$, $\delta>0$ are explicitely calculable numerical constants.

6. 7 *Ω estimations for the Moebius-function in Abel-sense.* Let

$$m(x)=\sum_{n=1}^{\infty}\mu(n)e_1\left(-\frac{n}{x}\right).$$

The following theorem holds.

THEOREM 7. *If $T>c_1$ then*

$$\max_{T\leqq x\leqq T^\varkappa}m(x)\cdot x^{-\frac{1}{2}}>\delta,\qquad \min_{T\leqq x\leqq T^\varkappa}m(x)\cdot x^{-\frac{1}{2}}<-\delta$$

where $\varkappa=(2+\sqrt{3})^2$, $c_1>0$, $\delta>0$ are calculable absolute constants.

PROOF. Let

$$f(s)=\int_1^\infty x^{-s}\,dm(x).$$

Hence by partial integration

$$f(s)=m(1)+s\int_1^\infty m(x)x^{-s-1}\,dx=m(1)+s\int_0^1 m\left(\frac{1}{y}\right)y^{s-1}\,dy=$$

$$=m(1)+s\int_0^\infty m\left(\frac{1}{y}\right)y^{s-1}\,dy-\varphi(s),$$

where $\varphi(s) = \int_1^\infty m\left(\frac{1}{y}\right) y^{s-1}\, dy$ is an integral function. Using the relation

$$\int_0^\infty y^{s-1} \sum_{n=1}^\infty \mu(n) e_1(-ny)\, dy = \frac{\Gamma(s)}{\zeta(s)}$$

we may apply our Theorem choosing

$$A(x) = m(x), \quad f(s) = s\frac{\Gamma(s)}{\zeta(s)} + m(1) - \varphi(s), \quad \theta_1 = 1, \quad \theta_2 = \frac{1}{2}, \quad \varrho = \varrho_0.$$

6. 8. *On the k-free numbers in Abel-sense.* Let

$$\tau_k(x) = \sum_{n=1}^\infty \left(\varrho_k(n) - \frac{1}{\zeta(k)}\right) e_1\left(-\frac{n}{x}\right).$$

(The definition of $\varrho_k(n)$ see in § 6. 2.)

With similar arguments can be seen the following

THEOREM 8. *If $T > d_k$ then*

$$\max_{T \leqq x \leqq T^\varkappa} \tau_k(x) x^{-\frac{1}{2k}} > \delta_k, \qquad \min_{T \leqq x \leqq T^\varkappa} \tau_k(x) \cdot x^{-\frac{1}{2k}} < -\delta_k$$

where $\varkappa = (2+\sqrt{3})^2$, $\delta_k > 0$, $d_k > 0$ are constants, being explicitely calculable functions of k.

6. 9. *On prime-numbers in different arithmetical progressions.* Let us denote

$$\psi(x, k, l) = \sum_{\substack{n \leqq x \\ n \equiv l \pmod k}} \Lambda(n), \quad \psi(x) = \sum_{n \leqq x} \Lambda(n),$$

$$\pi(x, k, l) = \sum_{\substack{p \leqq x \\ p \equiv l \pmod k}} 1, \qquad \pi(x) = \sum_{p \leqq x} 1,$$

where p denotes prime-numbers, $\Lambda(n)$ the Mangoldt's function. S. KNAPOWSKI and P. TURÁN treated sistematically the oscillation of $\psi(x, k, l_1) - \psi(x, k, l_2)$ and of $\pi(x, k, l_1) - \pi(x, k, l_2)$ in the series of their papers entitled „Comparative prime-number theory".

We can prove the following

THEOREM 9. *For $T > c_1$ and for all pairs l_1, l_2, $l_1 \not\equiv l_2 \pmod 8$, $(l_1 l_2, k) = 1$ we have*

$$\max_{T \leqq x \leqq T^\varkappa} \{\psi(x, 8, l_1) - \psi(x, 8, l_2)\} x^{-\frac{1}{2}} > \delta,$$

and if $l_1 \not\equiv 1$, $l_2 \not\equiv 1$ then

$$\max_{T \leqq x \leqq T^\varkappa} \{\pi(x, 8, l_1) - \pi(x, 8, l_2)\} x^{-\frac{1}{2}} \log x > \delta,$$

where $\varkappa = (2+\sqrt{3})^2$, $\delta > 0$, $c_1 > 0$ are explicitely calculable constants.

The corresponding results concerning this and the following theorem of S. KNAPOWSKI and P. TURÁN are better in almost every respect (see [2], p. 31, (1. 2); [3], p. 253, (1. 9)). Let

$$\sigma(x, k, l) = \sum_{n \equiv l \,(\mathrm{mod}\, k)} \Lambda(n) e_1\left(-\frac{n}{x}\right).$$

THEOREM 10. *For all* $T > c_1$ *and for all pairs* l_1, l_2; $l_1 \not\equiv l_2 \pmod 8$, $(l_1, k) = (l_2, k) = 1$ *we have*

$$\max_{T \leqq x \leqq T^{\varkappa}} \{\sigma(x, 8, l_1) - \sigma(x, 8, l_2)\} x^{-\frac{1}{2}} > \delta,$$

where $\varkappa = (2 + \sqrt{3})^2$, $\delta > 0$, $c_1 > 0$ *are calculable constants.*

7. Applications to ineffective theorems. In what follows we quote some Ω-theorems, which correspond to the analog theorem of the preceding paragraph.

Let θ be the upper limit of the real part of the roots of $\zeta(s)$, i.e. in notation

$$\theta = \sup_{\zeta(\varrho) = 0} {}_{\varrho} \operatorname{Re} \varrho.$$

The following theorem holds.

THEOREM 11. *For arbitrary* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ *we have*

$$\max_{T \leqq x \leqq T^{1+\varepsilon_2}} M(x) \cdot x^{-\theta + \varepsilon_1} > 1, \qquad \min_{T \leqq x \leqq T^{1+\varepsilon_2}} M(x) \cdot x^{-\theta + \varepsilon_2} < -1,$$

where $T > d$. *(*d *is a suitable constant. We cannot calculate* d.*)*

PROOF. a) If $\theta = \frac{1}{2}$, then using the estimation

$$|M(x)| < d_1(\varepsilon_3) x^{\frac{1}{2} + \varepsilon_3}$$

(see E. C. TITCHMARSH [1]) we apply the Theorem with $\varrho = \varrho_0 = \frac{1}{2} + i\gamma_0$, $k = 1$, $\theta_1 = \frac{1}{2} + \varepsilon_3$, $\theta_2 = \frac{1}{2}$. Thus

$$\varkappa = \frac{\theta_1}{\theta_2} + \sqrt{\left(\frac{\theta_1}{\theta_2}\right)^2 - 1} < 1 + 2\varepsilon_3 + \sqrt{2\varepsilon_3 + 4\varepsilon_3^2} < 1 + \varepsilon_2$$

if ε_3 is small enough, and so the theorem is proved.

b) The proof of the case $\theta > \frac{1}{2}$ is similar.

Let $N(\sigma, T)$ represent the number of zeros $\varrho = \beta + i\gamma$ of the ζ-function that satisfy $\sigma \leqq \beta$ and $0 \leqq \gamma \leqq T$. According to a theorem of A. SELBERG (E. C. TITCHMARSH [1], p. 204) we have

$$N(\sigma, T) = O(T^r \log T)$$

uniformly for $\frac{1}{2} \leqq \sigma \leqq 1$, where $r = 1 - \frac{1}{4}(\sigma - \frac{1}{2})$. Applying this with $\sigma = \theta - \varepsilon_2$ we can guarantee the existence of such a ζ-root $\varrho_1 = \beta_1 + i\gamma_1$, for which in the domain $|t - \gamma_1| < 2$, $\sigma > \beta_1$, the ζ-function does not vanish. Since $\zeta(s) \neq 0$ in $0 < s < 1$, we can guarantee the conditions of the Theorem.

We can deduce similarly the following theorems.

THEOREM 12. *For arbitrary but fixed* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$

$$\max_{T \leqq x \leqq T^{1+\varepsilon_2}} m(x) \cdot x^{-\theta+\varepsilon_1} > 1, \quad \min_{T \leqq x \leqq T^{1+\varepsilon_2}} m(x) \cdot x^{-\theta+\varepsilon_1} < -1,$$

$$\max_{T \leqq \beta \leqq T^{1+\varepsilon_2}} S(\beta) \cdot \beta^{1-\theta+\varepsilon_1} > 1, \quad \min_{T \leqq x \leqq T^{1+\varepsilon_2}} m(x) \cdot x^{-\theta+\varepsilon_1} < -1,$$

$$\max_{T \leqq x \leqq T^{1+\varepsilon_2}} M_0(x) \cdot x^{1-\theta+\varepsilon_1} > 1, \quad \min_{T \leqq x \leqq T^{1+\varepsilon_2}} M_0(x) x^{1-\theta+\varepsilon_1} < -1,$$

$$\max_{T \leqq x \leqq T^{1+\varepsilon_2}} \mathcal{T}(x) > 1, \quad \min_{T \leqq x \leqq T^{1+\varepsilon_2}} \mathcal{T}(x) < -1,$$

if T *is large enough.*

Finally we draw up a conditional result as a consequence of our Theorem. Let $l_1 \not\equiv l_2 \pmod k$; l_1, l_2 be coprime to k, and

$$f(s) = -\frac{1}{\varphi(k)} \sum_{\chi} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \frac{L'}{L}(s, \chi),$$

where the χ-'s denote multiplicative characters mod k.

Let us denote by θ^* the upper limit of real parts of the poles of $f(s)$. We proved in [4], that $\theta^* \geqq \frac{1}{2}$. (This assertion implicitely has been stated and was proved in KNAPOWSKI—TURÁN's paper [3] p. 243 earlier.) Now suppose, that $f(s)$ is regular in $0 < s < 1$. (This condition was quoted by KNAPOWSKI and TURÁN as Haselgrove's-condition. See their paper [2], p. 51.)

THEOREM 13. *If* $f(s)$ *is regular in the interval* $0 < s < 1$ *then for arbitrary but fixed* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$

$$\max_{T \leqq x \leqq T^{1+\varepsilon_2}} (\psi(x, k, l_1) - \psi(x, k, l_2)) x^{-\theta^*+\varepsilon_1} > 1,$$

$$\max_{T \leqq x \leqq T^{1+\varepsilon_2}} (\sigma(x, k, l_1) - \sigma(x, k, l_2)) x^{-\theta^*+\varepsilon_1} > 1,$$

when T *is large enough.*

8. The number of sign-changes. Let us denote by $N(A, T)$ the number of sign-changes of $A(x)$ in the interval $1 \leqq x \leqq T$. From our Theorem follows immediately that for $T > c_1$

$$N(A, T) > \frac{\log \log T - c_2}{\log \varkappa}$$

holds where

$$\varkappa = \frac{\theta_1}{\theta_2} + \sqrt{\left(\frac{\theta_1}{\theta_2}\right)^2 - 1}.$$

Thus the following inequalities hold.

If $T > c_1$ then

$$N(M, T) > \varphi(T), \quad N(M_0, T) > \varphi(T), \quad N(S, T) > \varphi(T), \quad N(\mathcal{T}, T) > \varphi(T)$$

where $\varphi(T) = \dfrac{\log \log T - c_2}{2 \log (2 + \sqrt{3})}$. (See Theorems 1, 4, 5, 6.)

Similarly from the Theorems 11, 12 it follows that

$$\lim_{T\to\infty} \frac{N(M,T)}{\log\log T} = \infty, \quad \lim \frac{N(M_0,T)}{\log\log T} = \infty,$$

$$\lim_{T\to\infty} \frac{N(m,T)}{\log\log T} = \infty, \quad \lim \frac{N(\mathcal{T},T)}{\log\log T} = \infty,$$

$$\lim_{T\to\infty} \frac{N(S,T)}{\log\log T} = \infty.$$

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