but $2t < c_1 s\log s\log\log s$ clearly implies that for $s > s_0 = s_0(c_1)$, $k_r \leqslant s$ (since $a_s = p_s^2 - p_s > c_1 s\log s\log\log s$). Thus

$$B(s)+2t = \sum_{i=2}^{s+1} p_i + \sum_{i=1}^{r} a_{k_i}$$

gives a representation of $B(s)+2t$ as the sum of s distinct primes or squares of primes where p and p^2 are not both used.

Assume next $n = B(s)+2t+1$. Then $n = A(s)+2t_1$, $2t_1 < cs\log s \times \log\log s$. Thus the same proof again gives that n is the sum of s distinct primes of squares of primes where p and p^2 are not both used. Thus (12) and hence our Theorem is proved (the cases $s \leqslant s_0$ can be ignored because of Lemma 1).

Finally we remark that $f_3(s) \geqslant B(s)-2$ since $B(s)-2$ can not be the sum of s distinct integers >1 which are pairwise relatively prime. To see this we only have to observe that by considerations of parity no even number can occur in such a representation.

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Further developments in the comparative prime-number theory V

(The use of "two-sided" theorems)

by

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1. This paper means in this series a methodical digression; its aim is at the same time modest and pretentious. It is modest since we are going to prove a theorem which we proved in stronger form in a previous paper (see Knapowski-Turán [1]). It is still pretentious for the following reason. The second of us observed some years ago that several problems in the analytical number-theory can be reduced to the following "two-sided" theorem.

If m is a positive number, further

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n| \tag{1.1}$$

and

$$B \overset{\text{def}}{=} \min_{\lambda} \Big|\sum_{j=1}^{\lambda} b_j\Big| > 0, \tag{1.2}$$

then there is an integer ν satisfying

$$m \leqslant \nu \leqslant m+n \tag{1.3}$$

such that

$$\Big|\sum_{j=1}^{n} b_j z_j^{\nu}\Big| \geqslant \left(\frac{n}{8e(m+n)}\right)^n \frac{B}{2n}. \tag{1.4}$$

He had in mind further applications too, a typical one being the explicit numerical determination of an X such that for a suitable $2 \leqslant x_0 \leqslant X$ the difference $\pi(x) - \mathrm{Li}\,x$ would change sign at $x = x_0$ (Littlewood's problem). But he came soon to a conclusion that such an application can be expected only after having instead of the "two-sided" theorem (1.1)-(1.4) a "one-sided" one, assuring the existence of integers ν_1 and ν_2 in an interval of type (1.3), for which we should have a positive lower bound for

$$\mathrm{Re}\sum_{j=1}^{n} b_j z_j^{\nu_1}$$

and a negative upper bound for

$$\mathrm{Re}\sum_{j=1}^{n} b_j z_j^{\nu_2},$$

both independent of the configuration of the z_j's, apart from an "argument-restriction". He has found such a theorem since (see Turán [1]) and this was indeed a starting point for a mass of researches, among others for a solution of Littlewood's problem by the first of us (see Knapowski [1]) and for several in the comparative prime-number theory. However, the verification of the argument-restriction presents difficulties to be overwhelmed in each case by individual ideas, whereas the two-sided theorem does not contain such restrictions. This fact makes it desirable to be able to use for these aims the two-sided theorem. Recently we observed that a slight modification of an ingenious idea found by G. Kreisel (see Kreisel [1]) for the solution of Littlewood's problem gives a possibility for such proofs. Applied to Littlewood's problem it would give a very short solution. However, we shall illustrate this new turn of our methods not on this but on another example, where previously we applied the one-sided theorem (see Knapowski-Turán [1]), in order to get some comparison of both forms. This analysis leads to the conclusion that where both methods work, the one-sided theorem leads generally to stronger theorems but then the new form works in some cases when the one-sided theorem fails. So illustrating this thesis on the comparison of

$$\sum_{\substack{n\leqslant x\\ n\equiv 1(k)}} \Lambda(n) \quad\text{and}\quad \sum_{\substack{n\leqslant x\\ n\equiv l(k)}} \Lambda(n),$$

this new turn leads to the following

THEOREM. *If for a* $0<\delta<\frac{1}{10}$ *and for* (1)

(1.5) $$k>\max(c_1, e^{\delta^{-20}})$$

no $L(s,\chi)$*-function with* $\chi(l)\neq 1$*, belonging to the modulus* k*, vanishes for*

(1.6) $$|s-1|\leqslant \tfrac{1}{2}+4\delta,$$

(1) c_1 and later on $c_2, c_3, \ldots$ denote always positive numerical, explicitly calculable constants.

then if

(1.7) $$a>\max(c_2, e^{k\log^3 k})$$

and

(1.8) $$b=e^{\log^2 a(\log\log a)^3},$$

we have for suitable x_1 *and* x_2 *in* $[a,b]$ *the inequalities*

(1.9) $$\sum_{\substack{n\leqslant x_1\\ n\equiv 1(k)}} \Lambda(n)-\sum_{\substack{n\leqslant x_1\\ n\equiv l(k)}} \Lambda(n)\geqslant x_1^{\frac{1}{2}-4\delta}$$

and

(1.10) $$\sum_{\substack{n\leqslant x_2\\ n\equiv 1(k)}} \Lambda(n)-\sum_{\substack{n\leqslant x_2\\ n\equiv l(k)}} \Lambda(n)\leqslant -x_2^{\frac{1}{2}-4\delta}.$$

Here not only the localisation of x_1 and x_2 is worse than in our paper [1], where the interval had the form $[a, a^3]$, but also the one-sided theorem worked there in the case, when instead of (1.6) the weaker Haselgrove condition was fulfilled, which required only the existence of an $\eta>0$ such that no $L(s,\chi)$-function mod k would vanish for

$$\sigma\geqslant\tfrac{1}{2},\quad |t|\leqslant\eta.$$

2. First we expose the (slightly modified) idea of Kreisel. Let with $s=\sigma+it$ be

(2.1) $$f(s)=\int_1^\infty \frac{B(x)}{x^s}\,dx$$

with a real $B(x)$-function such that $f(s)$ be regular in the closed half-plane $\sigma\geqslant 1$ and all integrals

(2.2) $$\int_1^\infty \frac{|B(x)|\log^\nu x}{x}\,dx$$

exist for $\nu=0,1,2,\ldots$ Then for $\sigma\geqslant 1$ we have

(2.3) $$f^{(\nu)}(s)=(-1)^\nu\int_1^\infty \frac{B(x)\log^\nu x}{x^s}\,dx.$$

Suppose, we have for an $(1\leqslant)$ $a<b$, for an appropriate integer $\nu_0\geqslant 3$, $t_0\geqslant 10$ and $1+it_0=s_0$, the inequality

(2.4) $$|f^{(\nu_0)}(s_0)|>|f^{(\nu_0)}(1)|+2\big(\psi_1(\nu_0,a)+\psi_2(\nu_0,b)\big),$$

where ψ_1 and ψ_2 are determined by

$$(2.5)\qquad \int_1^a \frac{|B(x)|\log^\nu x}{x}\,dx \leqslant \psi_1(\nu, a),$$

$$(2.6)\qquad \int_b^\infty \frac{|B(x)|\log^\nu x}{x}\,dx \leqslant \psi_2(\nu, b).$$

Then we assert that $B(x)$ necessarily changes its sign for $a < x < b$. For if not, then we have on the one hand

$$(2.7)\qquad U_1 \overset{\text{def}}{=} \left|\int_a^b \frac{B(x)\log^{\nu_0}x}{x^{s_0}}\,dx\right| \leqslant \int_a^b \frac{|B(x)|\log^{\nu_0}x}{x}\,dx \overset{\text{def}}{=} U_2$$

and on the other

$$(2.8)\qquad U_1 \geqslant |f^{(\nu_0)}(s_0)| - \int_1^a - \int_b^\infty \frac{|B(x)|\log^{\nu_0}x}{x}\,dx \geqslant |f^{(\nu_0)}(s_0)| - \psi_1(\nu_0, a) - \psi_2(\nu_0, b)$$

and

$$(2.9)\qquad U_2 = \left|(-1)^{\nu_0}\int_a^b \frac{B(x)\log^{\nu_0}x}{x}\,dx\right| = \left|f^{(\nu_0)}(1) - (-1)^{\nu_0}\int_1^a - (-1)^{\nu_0}\int_b^\infty\right|$$
$$\leqslant |f^{(\nu_0)}(1)| + \psi_1(\nu_0, a) + \psi_2(\nu_0, b).$$

Putting (2.8) and (2.9) into (2.7), we get *a fortiori*

$$|f^{(\nu_0)}(s_0)| \leqslant |f^{(\nu_0)}(1)| + 2\big(\psi_1(\nu_0, a) + \psi_2(\nu_0, b)\big),$$

which contradicts to (2.4). Hence (2.4) will imply indeed that $B(x)$ changes sign for $a < x < b$. This is the form of Kreisel's idea on which we shall base our further considerations.

3. It will be enough to deal with (1.9); the proof of (1.10) goes analogously. We choose for our $B(x)$

$$(3.1)\qquad B_0(x) = \frac{1}{x}\Big(\sum_{\substack{n\leqslant x\\ n\equiv 1(k)}} \Lambda(n) - \sum_{\substack{n\leqslant x\\ n\equiv l(k)}} \Lambda(n) - x^{\frac12-4\delta}\Big)$$

(in the proof of (1.10) we had only to replace $-x^{\frac12-4\delta}$ by $x^{\frac12-4\delta}$). Then we have for $\sigma > 1$

$$(3.2)\qquad f(s) = \int_1^\infty \frac{B_0(x)}{x^s}\,dx = -\frac{1}{s\varphi(k)}\sum_\chi \big(1-\bar\chi(l)\big)\frac{L'}{L}(s, \chi) - \frac{1}{s-\frac12+4\delta}.$$

Hence $f(s)$ is regular also for $\sigma \geqslant 1$; since for $x \geqslant e^k$, say, the inequality

$$(3.3)\qquad \Big|\sum_{\substack{n\leqslant x\\ n\equiv l(k)}} \Lambda(n) - \frac{x}{\varphi(k)}\Big| \leqslant xe^{-2c_3\sqrt{\log x}},$$

as is well-known, holds also (2.2) and (2.3) are in our case verified. Restricting at this moment ν_0 only by

$$(3.4)\qquad 10\log a \leqslant \nu_0 \leqslant 10\log a + k\log^{5/4}k,$$

we remark first that owing ot (1.7), choosing c_1 in (1.5) sufficiently large, we have

$$(3.5)\qquad 10\log a \leqslant \nu_0 \leqslant 11\log a.$$

Then we have

$$\int_1^a \frac{|B_0(x)|\log^{\nu_0}x}{x}\,dx \leqslant 2\int_1^a \frac{\log^{\nu_0}x}{x}\,dx < 2\log^{\nu_0+1}a < \frac12\nu_0!$$

if c_2 is sufficiently large; thus we can choose

$$(3.6)\qquad \psi_1(\nu_0, a) = \tfrac12\nu_0!.$$

In order to determine $\psi_2(\nu_0, b)$, we use (3.3). This gives for sufficiently large c_1

$$|B_0(x)| \leqslant e^{-c_3\sqrt{\log x}},$$

and hence

$$(3.7)\qquad \int_b^\infty \frac{|B_0(x)|\log^{\nu_0}x}{x}\,dx \leqslant \int_b^\infty \frac{\log^{\nu_0}x\, e^{-c_3\sqrt{\log x}}}{x}\,dx = 2\int_{\sqrt{\log b}}^\infty r^{2\nu_0+1}e^{-c_3 r}\,dr.$$

But from (1.8) and (3.5) we get for sufficiently large c_1

$$\sqrt{\log b} = \log a(\log\log a)^{3/2} > \nu_0\log^{5/4}\nu_0 \quad \Big(> \frac{4\nu_0+2}{c_3}\Big),$$

i.e. $r^{2\nu_0+1}e^{-\frac12 c_3 r}$ decreases monotonously in our interval and hence here

$$r^{2\nu_0+1}e^{-\frac12 c_3 r} \leqslant (\nu_0\log^{5/4}\nu_0)^{2\nu_0+1}e^{-\frac{c_3}{2}\nu_0\log^{5/4}\nu_0} \leqslant 1$$

if c_1 is sufficiently large. Hence the last term in (3.7) is

$$< 2\int_{\sqrt{\log b}}^\infty e^{-\frac{c_3}{2}r}\,dr < 1 < \tfrac12\nu_0!$$

and we can choose

$$\psi_2(\nu_0, b) = \tfrac{1}{2}\nu_0!.$$

Hence the relation we have to prove takes the form

(3.8) $$|f^{(\nu_0)}(s_0)| > |f^{(\nu_0)}(1)| + \nu_0!.$$

4. Next we give an upper bound to $|f^{(\nu_0)}(1)|$, using (1.6). Standard use of the theorem of Hadamard-Carathéodory gives the estimate

$$\left|\frac{L'}{L}(s, \chi)\right| \leqslant c_4 \frac{\log k}{\delta^2}$$

for all $L(s, \chi)$ with $\chi \neq \chi_0$, valid in the circle

(4.1) $$|s-1| \leqslant \tfrac{1}{2} + 3\delta.$$

Hence here we get

$$|f(s)| \leqslant (5c_4+1)\frac{\log k}{\delta^2} = c_5 \frac{\log k}{\delta^2}$$

and by Cauchy's inequality

$$|f^{(\nu_0)}(1)| \leqslant c_5 \frac{\log k}{\delta^2}\left(\frac{2}{1+6\delta}\right)^{\nu_0} \nu_0!.$$

Hence (3.8) will be true *a fortiori* if we prove

(4.2) $$\frac{1}{\nu_0!}|f^{(\nu_0)}(s_0)| > (c_5+1)\frac{\log k}{\delta^2}\left(\frac{2}{1+6\delta}\right)^{\nu_0};$$

and having this we can be sure that $B_0(x)$ changes sign in (a, b), which will give exactly (1.9).

In what follows we shall use two known facts on the L-functions.

(a) For a suitable $c_6 \geqslant 10$ the domain

(4.3) $$\sigma \geqslant \tfrac{1}{2}, \quad \tau \leqslant t \leqslant \tau + c_6$$

contains for all real τ's at least one zero of any of $L(s, \chi)$-functions.

(b) With the above c_6 the rectangle

(4.4) $$-14 \leqslant \sigma \leqslant 1, \quad \tau \leqslant t \leqslant \tau + c_6$$

contains for each $L(s, \chi) \bmod k$ at most

(4.5) $$c_7 \log k(2+|\tau|)$$

zeros (counted with multiplicity).

5. Now we shall dispose upon t_0. First we determine a τ_0 with

(5.1) $$\tau_0 \geqslant \max(40, 10c_6)$$

so that with c_7 in (4.5) the inequality

(5.2) $$\frac{1}{2}\cdot\frac{1-\cos(2\pi/\varphi(k))}{\varphi(k)\tau_0} > 4c_7 \frac{\log k(2+\tau_0)}{\tau_0^2}$$

holds. Such a τ_0 obviously exists, even

(5.3) $$\tau_0 < c_8 k^4$$

can be asserted with suitable (large) c_8. Let $\chi_1(n)$ be an arbitrary character mod k with $\chi_1(l) \neq 1$; according to (a) $L(s, \chi_1)$ has a zero $\varrho_1 = \beta^{(1)} + i\gamma^{(1)}$ with

(5.4) $$\beta^{(1)} \geqslant \tfrac{1}{2}, \quad \tau_0 + \tfrac{1}{2}c_6 \leqslant \gamma^{(1)} \leqslant \tau_0 + \tfrac{3}{2}c_6.$$

We define the (positive continuous) function $h(t)$ for real t-values by

(5.5) $$h(t) = \min|1+it-\varrho|$$

where the minimum is taken over *all* non-trivial ϱ-zeros of *all* $L(s, \chi)$-functions belonging to mod k with $\chi(l) \neq 1$. Owing to (5.4) we have

(5.6) $$h(\gamma^{(1)}) \leqslant \tfrac{1}{2}.$$

Now we define t_0 as one of t-values in

(5.7) $$I: \quad \tau_0 + \tfrac{1}{2}c_6 \leqslant t \leqslant \tau_0 + \tfrac{3}{2}c_6,$$

for which

(5.8) $$\min_{t \in I} h(t)$$

is attained. (5.6) gives at once the inequality

(5.9) $$h(t_0) \leqslant \tfrac{1}{2}.$$

Hence we have

(5.10) $$40 \leqslant t_0 \leqslant 2c_8 k^4$$

and if

$$\varrho^* = \beta^* + i\gamma^*$$

is the zero of the $L(s, \chi)$-functions belonging to mod k with $\chi(l) \neq 1$, which is the nearest to

$$s_0 = 1 + it_0,$$

we have

(5.11) $$|s_0 - \varrho^*| \leqslant \tfrac{1}{2}, \quad 39 \leqslant \gamma^* \leqslant 3c_8 k^4, \quad \beta^* \geqslant \tfrac{1}{2}.$$

6. Taking in account (5.10), a standard reasoning, based again on Hadamard-Carathéodory's theorem, gives for all $L(s,\chi)$-functions belonging to modulus k in the circle $|s-s_0| \leqslant 4$ the inequality

$$(6.1) \qquad \left|\frac{L'}{L}(s,\chi) - \sum_{\substack{\varrho \\ |s_0-\varrho|\leqslant 7}} \frac{1}{s-\varrho}\right| \leqslant c_9 \log k.$$

Hence from (3.2) we get in the same circle

$$(6.2) \qquad \left|f(s) + \frac{1}{s-\frac{1}{2}+2\delta} + \sum_{\chi} \frac{1-\bar{\chi}(l)}{\varphi(k)} \sum_{\substack{\varrho(\chi) \\ |\varrho-s_0|\leqslant 7}} \frac{1}{s(s-\varrho)}\right| \leqslant 2c_9 \log k.$$

Since

$$\frac{1}{s(s-\varrho)} = \frac{1}{\varrho}\left(\frac{1}{s-\varrho} - \frac{1}{s}\right)$$

and the contribution of the second terms to the left-side of (6.2) is absolutely

$$\leqslant c_{10} \log k,$$

we get from (6.2) for $|s-s_0| \leqslant 4$ the inequality

$$\left|f(s) + \sum_{\chi} \frac{1-\bar{\chi}(l)}{\varphi(k)} \sum_{\substack{\varrho(\chi) \\ |\varrho-s_0|\leqslant 4}} \frac{1}{\varrho}\cdot\frac{1}{s-\varrho}\right| \leqslant c_{11} \log k.$$

But then Cauchy's inequality gives

$$\left|\frac{1}{\nu_0!} f^{(\nu_0)}(s_0) + (-1)^{\nu_0} \sum_{\chi} \frac{1-\bar{\chi}(l)}{\varphi(k)} \sum_{\substack{\varrho(\chi) \\ |\varrho-s_0|\leqslant 4}} \frac{1}{\varrho}\cdot\frac{1}{(s_0-\varrho)^{\nu_0+1}}\right| \leqslant \frac{c_{11}\log k}{4^{\nu_0}}.$$

Hence taking in account (4.2), chosing c_1 in (1.5) sufficiently large, everything will be proved if ν_0 can be chosen so within the interval (3.4) that the inequality

$$(6.3) \qquad \left|\sum_{\chi} \frac{1-\bar{\chi}(l)}{\varphi(k)} \sum_{\substack{\varrho(\chi) \\ |s_0-\varrho|\leqslant 4}} \frac{1}{\varrho}\cdot\frac{1}{(s_0-\varrho)^{\nu_0+1}}\right| > (c_5+1+c_{11})\frac{\log k}{\delta^2}\left(\frac{2}{1+6\delta}\right)^{\nu_0}$$

holds. Multiplying it by $|s_0-\varrho^*|^{\nu_0+1}$, the inequality (6.3) holds owing to (5.11) *a fortiori* if the inequality

$$\left|\sum_{\chi} \frac{1-\bar{\chi}(l)}{\varphi(k)} \sum_{\substack{\varrho(\chi) \\ |s_0-\varrho|\leqslant 4}} \frac{1}{\varrho}\left(\frac{s_0-\varrho^*}{s_0-\varrho}\right)^{\nu_0+1}\right| > 2(c_5+c_{11}+1)\frac{\log k}{\delta^2}\left(\frac{1}{1+6\delta}\right)^{\nu_0}$$

holds, and all the more if the inequality

$$\left|\sum_{\chi} \frac{1-\bar{\chi}(l)}{\varphi(k)} \sum_{\substack{\varrho(\chi) \\ |s_0-\varrho|\leqslant 4}} \frac{1}{\varrho}\left(\frac{s_0-\varrho^*}{s_0-\varrho}\right)^{\nu_0+1}\right| \geqslant 3(c_5+c_{11}+1)\frac{\log k}{\delta^2}\left(\frac{1}{1+6\delta}\right)^{\nu_0+1}$$

holds. Or we have to find a suitable integer μ with the restriction

$$(6.4) \qquad 1+10\log a \leqslant \mu \leqslant 10\log a + k\log^{5/4} k,$$

for which the inequality

$$(6.5) \qquad \left|\sum_{\chi} \frac{1-\bar{\chi}(l)}{\varphi(k)} \sum_{\substack{\varrho(\chi) \\ |s_0-\varrho|\leqslant 4}} \frac{1}{\varrho}\left(\frac{s_0-\varrho^*}{s_0-\varrho}\right)^{\mu}\right| \geqslant 3(c_5+c_{11}+1)\frac{\log k}{\delta^2}\left(\frac{1}{1+6\delta}\right)^{\mu}$$

holds. This will be done by the two-sided theorem. We shall denote the sum on the left as a function of μ by $Z(\mu)$.

7. Before applying the two-sided theorem, we remark that if we know in it for n only an upper bound N, then on the right of (1.4) we can replace n by N everywhere. The role of the z_j's will be played by the numbers $(s_0-\varrho^*)/(s_0-\varrho)$; the normalisation-condition (1.1) is owing to the definition of ϱ^* fulfilled. The role of the b_j-coefficients is played by the numbers

$$\frac{1-\bar{\chi}(l)}{\varphi(k)}\cdot\frac{1}{\varrho}$$

and we have to give a lower bound for B in (1.2). What is the error in B, replacing $\varrho = \beta+i\gamma$ by $i\gamma$? Using (4.5) with $\tau = t_0-4$, this error is absolutely less than

$$\sum_{\chi} \frac{2}{\varphi(k)} \sum_{\substack{\varrho(\chi) \\ |s_0-\varrho|\leqslant 4}} \frac{1}{|\gamma|^2} < 2c_7 \frac{\log k(2+t_0)}{(t_0-4)^2} < 4c_7 \frac{\log k(2+\tau_0)}{\tau_0^2}$$

using (5.1), which is in turn owing to (5.2)

$$< \frac{1}{2}\cdot\frac{1-\cos\left(2\pi/\varphi(k)\right)}{\varphi(k)\tau_0}.$$

Hence

$$(7.1) \qquad B \geqslant \frac{1}{2}\cdot\frac{1-\cos\left(2\pi/\varphi(k)\right)}{\varphi(k)\tau_0} > \frac{c_{12}}{k^7}$$

using (5.3). Owing to (4.5) we can use as N the quantity

$$(7.2) \qquad N = c_{13} k \log k$$

with a suitable c_{13} and as m we choose

$$(7.3) \qquad m = 1+10\log a.$$

Then the interval $(m, m+N)$ is contained in the interval (6.4) and thus, choosing for μ the value ν given by (1.4), the requirement (6.4) is not violated. But then we get, using (7.1) too,

$$|Z(\mu)| \geqslant \left(\frac{c_{13}k\log k}{8e(1+\log a+c_{13}k\log k)}\right)^{c_{13}k\log k}\frac{c_{12}}{2k^7}\cdot\frac{1}{c_{13}k\log k}.$$

Taking in account (1.7) and choosing c_1 in (1.5) sufficiently large we get, using also the second half of (1.5) and (7.3),

$$|Z(\mu)| > \left(\frac{c_{14}}{\log a}\right)^{c_{13}k\log k} > \left(\frac{1}{1+6\delta}\right)^{m} 3(c_5+c_{11}+1)\frac{\log k}{\delta^2}$$
$$> 3(c_5+c_{11}+1)\frac{\log k}{\delta^2}\left(\frac{1}{1+6\delta}\right)^{\mu}$$

indeed.

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Rational dependence in finite sets of numbers

by

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J. Mikusiński and A. Schinzel ([1]) proved that, in a finite set of points on the real line so that every distance except the maximal one occurs more than once, all distances are commensurable. This theorem was discussed in the Undergraduate Research Program in Mathematics at UCLA under the author's direction; and the proof developed there leads to a generalization which was conjectured (and proved for the case $m' = 2$ in [2]):

THEOREM. *Let $x_1, x_2, \ldots, x_n$ be real numbers and let m be the dimension of the vector space, V, spanned by $\{x_i-x_j|i,j = l, \ldots, n\}$ over the rationals. Let m' be the dimension of the rational vector space, V', spanned only by those x_i-x_j for which $x_i-x_j \neq x_k-x_l$ whenever $(i,j) \neq (k,l)$. Then $m' = m$.*

Proof. Assume, without loss of generality, that $x_1 = 0$ and let $\eta_1, \eta_2, \ldots, \eta_{m'}$, be a basis for V', and $\eta_1, \ldots \eta_{m'}, \ldots, \eta_m$ be a basis for V.

Decompose the n-tuple $X = (x_1, \ldots, x_n)$ into $\sum_{s=1}^{m} X^{(s)}\eta_s$ where the $X^{(s)}$ are n-tuples of rationals, not all of them 0. By this construction we have $x_i^{(s)} = x_j^{(s)}$ for all $s > m'$ whenever x_i-x_j is attained for a unique pair (i,j). Whenever $x_i-x_j = x_k-x_l$ we obviously have $x_i^{(s)}-x_j^{(s)} = x_k^{(s)}-x_l^{(s)}$ for all $s = 1, \ldots, m$.

Now assume $m > m'$ and let $X(t) = X+tX^{(m)}$, t real. There may be a finite number of choices of t so that $x_i(t) = x_j(t)$ for some $i \neq j$ (at most one choice for each pair (i,j)), we exclude all those t. Obviously

$$x_i-x_j = x_k-x_l \Rightarrow x_i(t)-x_j(t) = x_k(t)-x_l(t)$$

for all t. Since $X^{(m)} \neq 0$ the elements of $X(t)$ become unbounded as $t \to \infty$ while $x_1(t) = 0$ for all t. In particular, for t sufficiently large, $\max_i x_i(t) - \min_i x_i(t) = x_j(t)-x_k(t) > x_n-x_1$. Thus $x_j(t)-x_k(t)$ is a unique difference among the $x_i(t)$ and therefore x_j-x_k is a unique difference among the x_i. But we have

$$x_j(t)-x_k(t) = x_j-x_k+t(x_j^{(m)}-x_k^{(m)}) > x_n-x_1$$