Since $0 < s < r$, from (5.8) $k \geq 1$ and $k = 1$ only if $s \geq 2$; in this case however $r \geq 3$ and then (5.9) is not satisfied. Hence $k \geq 2$ and so from (5.9) $z_2 < 2(r+1)$, i.e. $r < 1$. Thus $r = 1$ and $s = 0$. From (5.8) $k \geq 3$ and from (5.9) $k < 4$ so $k = 3$. Now when $r = 1$, $s = 0$, $t = 3$ we have $(r+1)(s+t)+s+1 > 2(s+1)+2 - \frac{t}{r+1}$ so the only case for which $\frac{u}{m}$ is a better bound is in the range $m+2 < n < 2m$ is $r = 1$, $s = 0$, $t = 3$, $\vartheta = 6$, i.e. $n = 9$, $m = 6$.

References

[1] L. J. Mordell, On the inequality $\sum_{p \leq x} \frac{np}{p-1} \geq \frac{x^2}{2}$ and some others, Abh. Math. Sem. Univ. Hamburg, 23 (1958), pp. 239-241.


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Further developments in the comparative prime-number theory IV

(Accumulation theorems for residue-classes representing quadratic non-residues $\mod k$)

by

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1. In the second and third papers of this series we introduced a new approach instead of that of Chebyshev, in order to find a sense in which there are more primes $\equiv 1 \mod k$ than $\equiv 1 \mod k$ if and only if $l_1$ is a quadratic non-residue, $l_2$ quadratic residue $\mod k$. We succeeded in obtaining results in this direction when the Haselgrove-condition is satisfied for $k$, i.e. when there is an $E = E(k) > 0$ such that no $L(s, \chi)$ belonging to the modulus $k$ vanishes for $1^s$(1)

$$\sigma \geq 1, \quad |t| \leq E(k) \quad (s = \sigma + it).$$

For the sake of brevity we shall call such $k$-values "good" $k$-values. We made a comparison in the second paper for the residue-classes

$$\equiv 1 \mod k \quad \text{and} \quad \equiv l \mod k$$

($l$ quadratic non-residue $\mod k$) in the third one for the residue-classes

$$\equiv 1 \mod k \quad \text{and} \quad \equiv l \mod k$$

($l$ quadratic residue $\mod k$).

In this paper we shall pass to the more general case, when we compare the residue-classes

$$\equiv l_1 \mod k \quad \text{and} \quad \equiv l_2 \mod k$$

($l_1$, $l_2$ both quadratic non-residues).

(1) Though no $k$-value is known for which this would be false, it is desirable to prove its truth at least for an infinity of $k$-values.
This time we succeeded only for \( k \) satisfying a condition more stringent than (1.1). We shall suppose not only (1.1) but also with an \( \eta \), satisfying with a suitably small \( c_1 \) the condition
\[
0 < \eta < \min \left( c_1 \left( \frac{B(k)}{6k} \right)^{1/2} \right)
\]
the nonvanishing of all \( L(s, \chi) \) functions belonging to mod \( k \) for
\[(1.4) \quad \sigma > \frac{1}{3}, \quad |\ell| < 2/\eta.\]
On \( B(k) \) we may suppose without loss of generality that
\[(1.5) \quad B(k) \leq \frac{1}{k^{1/2}}.\]
Then we shall prove the

**Theorem I.** If for \( k > c_1 \) with sufficiently large \( c_1 \) the condition (1.1), (1.3), (1.4) and (1.5) is satisfied, then for
\[(1.6) \quad T > \max \left( c_1, c_{1/2, \eta} \right)
\]
and for quadratic non-residue \( \ell_1 \) and \( \ell_2 \) there are \( x_j \) and \( n_j \) (\( j = 1, 2 \)) with
\[(1.7) \quad T^{-1/\sqrt{\eta}} \leq x_j, n_j \leq T^{1+1/\sqrt{\eta}}\]
and
\[(1.8) \quad 2\eta \log T \leq n_j, x_j \leq 2\eta \log T + 3\sqrt{\log T}\]
so that
\[
\sum_{\ell = 1 \mod k} \log \rho e^{-1/\log^2 \frac{T}{\sqrt{\eta}}} - \sum_{\ell = 1 \mod k} \log \rho e^{-1/\log^2 \frac{T}{\sqrt{\eta}}} > T^{1 - 4/\sqrt{\eta}}.
\]

2. In the first paper of this new series we proved the first “accumulation” theorem. This states in its simplest form that for a sufficiently large \( c_1 \) for \( T > c_1 \) there are \( U_1, U_2, U_3, U_4 \) with
\[
T^{-1/\log^2 \frac{T}{\eta}} \leq U_1 < U_2 < U_3 \leq T,
\]
so that
\[
\sum_{n = 1 \mod k} A(n) - \sum_{n = 1 \mod k} A(n) > \sqrt{T} e^{-1/\log^2 \frac{T}{\eta}}
\]
and
\[
\sum_{n = 1 \mod k} A(n) - \sum_{n = 1 \mod k} A(n) < -\sqrt{T} e^{-1/\log^2 \frac{T}{\eta}}.
\]

The corresponding problems for primes instead of prime-powers are generally more difficult. In this direction we shall prove the following

**Theorem II.** Under the conditions of Theorem I there are \( \mu_1, \mu_2, \mu_3, \mu_4 \) with
\[
T^{-1/\sqrt{\eta}} \leq \mu_1 < \mu_2 < T^{1+1/\sqrt{\eta}}\]
so that
\[
\sum_{n = 1 \mod k} 1 - \sum_{n = 1 \mod k} 1 > T^{1/4 \sqrt{\eta}}\]
and
\[
\sum_{n = 1 \mod k} 1 - \sum_{n = 1 \mod k} 1 < T^{-1/4 \sqrt{\eta}}.
\]

Since this can be derived from Theorem I following the pattern of our paper [3] of this series, we shall omit the details.

3. We shall need a number of lemmas.

**Lemma I.** If no \( L(s, \chi) \) functions mod \( k \) vanish for
\[
\sigma > \frac{1}{2}, \quad |\ell| \leq \log \varphi(k),
\]
then for all \( (i, k) = 1 \) there exists a prime \( P = p_i \) with \( P = l(k) \) for which \((^*)\), with suitable \( c_1 \) and \( c_2 \),
\[
c_p \varphi(k)^{1/2} \leq P < c_2 \varphi(k)^{1/2}.
\]

Let with a fixed \( \ell \), with \( (i, k) = 1 \),
\[
F(s) = -\frac{1}{\varphi(k)} \sum_x \frac{x}{L'}(s+1, x)
\]
and
\[
\gamma = \frac{1}{10} \log \varphi(k),
\]
so that
\[
A = 10 \log \varphi(k).
\]

\(^*\) A weaker lemma is deduced in our paper [1] from the exact prime-number formula (p. 50). We prefer now to give an independent proof. The conditions could have been much weakened.
With these we consider the integral
\[ J = \frac{1}{2\pi i} \int \left( e^{2\pi i - e^{2\pi i - \sigma} \frac{E}{2\pi} } \right) P(s) ds. \]

Replacing \( P(s) \) by its Dirichlet-series and integrating term by term, we get
\[ J = \sum_{\substack{n \geq 1 \\text{odd} \\lambda \geq 0 \\text{even}}} \frac{A(n)}{n} \frac{1}{2\pi i} \int \left( e^{2\pi i - e^{2\pi i - \sigma} \frac{E}{2\pi} } \right) ds. \]

Since the integral is
\[ \frac{1}{A\pi i} \int \left( \frac{\sin t}{t} \cos \frac{25\pi - \log n}{A} \right) dt = a_n(v), \]

which is positive for \( e^{2\pi i - \sigma} < n < e^{2\pi i - \sigma} \) and 0 otherwise, further from (3.2), (3.3)
\[ e^{2\pi i - \sigma} < e^{2\pi i - \sigma} \quad \text{and} \quad e^{2\pi i - \sigma} < e^{2\pi i - \sigma}, \]

we have
\[ J = \sum_{\substack{n \geq 1 \\text{odd} \\lambda \geq 0 \\text{even}}} \frac{A(n)}{n} a_n(v). \]

Next we replace the line
\[ \sigma = 1 + \frac{1}{2}, \]
by the broken line \( (\alpha \leq \log \varphi(k) \leq \frac{1}{2}) \)
\[ K_1: \quad \sigma = 1 + \frac{1}{2}, \quad t = \log \varphi(k), \]
\[ K_2: \quad -\frac{9}{20} \leq \sigma \leq 1 + \frac{1}{2}, \quad t = -\log \varphi(k), \]
\[ K_3: \quad 1 - \log \varphi(k) < t < 1 + \frac{1}{2}, \quad \sigma = 1 - \frac{1}{2}, \quad t = -\log \varphi(k), \]
\[ K_4: \quad \sigma = 1 + \frac{1}{2}, \quad t > \log \varphi(k). \]

Denoting the respective integrals by \( I_1, I_2, \ldots, I_4 \), we have by standard estimations concerning \( L \)-functions for \( |I_2| \) and \( |I_3| \) the upper bound
\[ c_2 e^{\frac{1}{2\pi i}} \int_{\log \varphi(k) \text{ even}} \frac{\log \varphi(k)}{(At)^\gamma} dt < c_3 e^{\frac{1}{2\pi i}} \frac{\log \varphi(k)}{(10a)^{2\pi i}} \]

for \( |I_2| \) and \( |I_3| \)

\[ e^{-\frac{1}{2\pi i}} \frac{\log \varphi(k)}{(10a)^{2\pi i}} < c_1 \frac{\log \varphi(k)}{(10a)^{2\pi i}}, \]

and finally for \( |I_3| \)
\[ c_2 e^{\frac{1}{2\pi i}} \frac{\log \varphi(k)}{(10a)^{2\pi i}} < c_1 \frac{\log \varphi(k)}{(10a)^{2\pi i}} \]

Now we choose \( \alpha \) so large that
\[ \frac{1}{\sqrt{2\pi i}} \log(10a) = 1, \quad \alpha = 1 + \frac{1}{2}; \]

can be done if \( \log \varphi(k) \leq \log \varphi(k) \), i.e. \( \delta > \delta \).

and the residue at \( s = 0 \) is \( 1/\varphi(k) \), we get from (3.6)
\[ \sum_{\substack{n \geq 1 \\text{odd} \\lambda \geq 0 \\text{even}}} \frac{A(n)}{n} a_n(v) > \frac{1}{\varphi(k)} - \frac{c_4}{\varphi(k)^{1+\gamma}} > \frac{1}{2} \cdot \frac{1}{\varphi(k)} \]

if \( k > \delta \). Since the contribution of the prime-powers \( p^\beta \) (\( \beta \geq 2 \)) to (3.7) is owing to (3.5) at most
\[ c_0 \log \varphi(k) \sum_{p > \text{prime}} \frac{1}{p} < c_0 \frac{\log \varphi(k)}{(10a)^{2\pi i}} < \frac{1}{4\varphi(k)} \]

if \( k > \delta \). Hence for \( k > \delta \) the assertion is proved with \( c_0 = e^{\frac{1}{2}}, c_1 = e \).

From this Lemma I follows easily generally.

Further we need the

**Lemma II.** If \( \alpha, \beta, \) are real, further
\[ |\alpha| > U \quad (\leq 1) \]

further with a \( \gamma > 1 \)
\[ \sum_{1 + |\alpha| < V} \frac{1}{1 + |\alpha|} < V \quad (\leq \infty) \]

and \( \gamma > 1/U \), then each real interval of length \( \gamma \) contains a \( k \) with the property that for each \( v \)-index the inequality
\[ \{a, k + \beta\} > \frac{1}{24V \cdot 1 + |\alpha|} \]

holds (*)

(*) \( |\alpha| \) stands as usual for the distance of \( \alpha \) from the nearest integer.
Also we need the

**Lemma III.** Let \( m \) be positive and \( a_j's \) with

\[
1 = |a_1| \geq |a_2| \geq \ldots \geq |a_n| \geq \ldots \geq |a_n|
\]

such that with a \( 0 < \kappa \leq n/2 \)

\[
\kappa \leq |\text{arcsin} a_j| \leq \pi; \quad j = 1, 2, \ldots, n,
\]

the index \( h \) we define by

\[
(3.3) \quad |a_h| > \frac{4N}{m + N(3 + \pi/|a|)}
\]

where \( N \) is an upper bound for \( n \). Then there are integers \( v_i \) and \( \nu_i \) with

\[
m \leq v_i, \nu_i \leq m + N(3 + \pi/|a|)
\]

so that

\[
\text{Re} \sum_{j=1}^{n} d_j a_j^\nu_i \geq \frac{B}{3N} \left( \frac{N}{24\varepsilon(m + N(3 + \pi/|a|)))} \right)^{\nu_i/2} |\nu_i|^{\nu_i + N(3 + \pi/|a|)}
\]

and

\[
\text{Re} \sum_{j=1}^{n} d_j a_j^\nu_i \leq \frac{B}{3N} \left( \frac{N}{24\varepsilon(m + N(3 + \pi/|a|)))} \right)^{\nu_i/2} |\nu_i|^{\nu_i + N(3 + \pi/|a|)};
\]

here \( B \) stands for

\[
(3.9) \quad B = \min_{v \geq n} \text{Re} \sum_{j=1}^{n} d_j |a_j|.
\]

For the proof see our paper [3].

We shall use further the

**Lemma IV.** There exists a broken line \( W \) in the vertical strip \( \frac{1}{2} \leq \sigma \leq 1 \)
consisting alternately of horizontal and vertical segments so that each horizontal strip of width \( 1 \) contains at most one horizontal segment and on \( W \) for all \( L \)-functions \( \log \) the inequality

\[
\left| L'(s, \chi) \right| \leq c_2 \varphi(\kappa) \log \log (2 + |\kappa|)
\]

holds.

A proof of this lemma follows mutatis mutandis that of the appendix III of paper of the second of us [1]. We shall also need the following simple consequence of a theorem of Siegel ([1]) which we state as

**Lemma V.** For a suitable \( c_{20} \) all \( L \)-functions (for all \( k \geq 1 \)) have a zero in all parallelograms \( \tau \) real

\[
\frac{1}{2} \leq \sigma \leq 1, \quad \tau \leq t \leq t + c_{20}.
\]

4. Finally we shall need the

**Lemma VI.** In the notation of Lemma I, with \( (l, k) = (l_1, k_1) = 1, \)

\[
l \neq \pm 1 \mod k \quad \text{and}
\]

\[
b_k = 2P_k \log P_k, \quad r_k = \frac{1}{P_k \log P_k},
\]

we have for \( k > c_{20} \) the inequality

\[
\frac{1}{\varphi(k)} \text{Re} \sum_{\chi} \left( \sum_{i \leq k} \chi(i)_{-} - \chi(i)_+ \right) \sum_{m \leq k} e^{2\pi i \chi(m)/k} > c_{20} P_k \log P_k,
\]

where \( \sum \) means that the summation is to be extended only to the nontrivial zeros \( \varphi = \varphi(\chi) \) of \( L(s, \chi) \) right to \( W \).

For the proof we shall define

\[
f(s) = \frac{1}{\varphi(k)} \sum_{\chi} \left( \sum_{i \leq k} \chi(i)_{-} - \chi(i)_+ \right) \frac{L'(s, \chi)}{L(s, \chi)}
\]

and we start with the integral

\[
(4.3) \quad J_1 = \frac{1}{2\pi i} \int_{b}^{c} e^{2\pi i \chi(s)/k} f(s) ds.
\]

Since for positive \( \lambda \)'s we have (see our paper [2])

\[
(4.4) \quad \frac{1}{2\pi i} \int_{b}^{c} e^{2\pi i \chi(s)/k - \lambda} ds = \frac{1}{V \sqrt{\pi a}} e^{\frac{\pi}{2\lambda} \frac{\lambda^2}{a} + \frac{\pi}{2\lambda} \frac{\lambda}{a}}
\]

inserting the Dirichlet-series of \( f(s) \) we get, using the notation

\[
(4.5) \quad a_0(n, l_1, l_1) = \begin{cases} 1 & \text{if } n = l_1(k), \\
-1 & \text{if } n = l_1(k), \\
0 & \text{otherwise},
\end{cases}
\]

the relation

\[
(4.6) \quad J_1 = \frac{e^{2\pi i \chi(s)}}{V \sqrt{\pi a}} \sum_{n \leq k} A(n) a_0(n, l_1, l_1) e^{-\frac{\lambda}{2\lambda} \left( \frac{n \varphi(n)}{k} \right)}.
\]

Hence \( J_1 \) is real. The contribution of \( n = P_k \) to the sum is obviously \( \log P_k \). The contribution of the terms \( n < P_k \) to the sum is

\[
< \sum_{n \leq P_k} \log n e^{-\frac{1}{2\lambda} \left( \frac{n \varphi(n)}{k} \right)}
\]

\[
< P_k \log P_k - \frac{1}{2\lambda} \log P_k \left( \frac{n \varphi(n)}{k} \right)
\]

\[
< P_k \log P_k - \frac{1}{2\lambda} \log P_k < c_{20}.
\]
The contribution of the \( n \)'s with \( n > P_1 \) is
\[
\sum_{\substack{n > P_1 \atop \text{odd}}} \log n \cdot e^{-\frac{1}{4} \lambda n^{1/3}} P_1 = \sum_{P_1 < n < P_1^{1/2}} + \sum_{n > P_1^{1/2}} < 2 \log P_1 \int_0^{P_1} e^{-\frac{1}{4} \lambda x^{1/3}} x dx + c_m < c_m
\]
and hence
\[
J_1 > \frac{c_m^{1/2}}{\sqrt{\pi}} P_1 \log P_1 (\log P_1 - c_m).
\]

Shifting the line of integration to the line \( W \) we get
\[
J_1 = \text{Re} f_1 = \frac{1}{\phi(k)} \sum_\chi \left( \frac{\chi(l_1) - \chi(l_2)}{\phi(k)} \right) \sum_{\sigma = 1}^\infty e^{\sigma x + \sigma y} \frac{1}{2\pi i} \int e^{\sigma x + \sigma y} f(s) ds.
\]

The last integral is, using Lemma IV, absolutely
\[
< c_m \log^2 k e^{\delta y (\sigma + y)} < c_m \log^2 P_1 e^{\delta y} \frac{e^{\delta y}}{\phi(k)} < c_m \frac{e^{\delta y}}{\phi(k)} P_1 \log^2 P_1.
\]

Collecting all these and dividing by \( e^{\delta y} \), the lemma follows if \( k > c_m \).

5. Now we can turn to the proof of Theorem I. First we apply Lemma II for
\[
u = \frac{1}{4\pi}, \quad \beta = \frac{1}{8\pi} \im \varphi^2,
\]
with
\[
y = \frac{1}{\eta}, \quad U = E(k)/\pi \sigma, \quad A = \pi \sigma/E(k);
\]
and one evidently choose
\[
V = c_m k \log k.
\]

Owing to (1.3) we have
\[
A < 2/\sqrt{\eta};
\]
and hence Lemma II gives the existence of a \( b_1 \), with
\[
\frac{1}{\eta} - \frac{1}{V \eta} < b_1 < \frac{1}{\eta}
\]
such that for all \( \varphi^2 \)
\[
\left[ \left( \frac{1}{2\eta} \frac{1}{4} \left[ 2b_1 \im \varphi + \im \varphi^2 \right] \right) > c_{m1} \frac{1}{\log k} \frac{1}{1 + |\varphi|^{1/30}},
\]
i.e.
\[
|\im \varphi | e^{\delta + \delta |b_1| |k|^{1/3}} > c_{m1} \frac{1}{(1 + |\varphi|^{1/30}) \log k}.
\]

Let further be
\[
m = 2\eta \log T
\]
and the integer \( r \) be restricted momentarily only by
\[
r \leq \alpha \leq \frac{2}{3}\eta 
\]

Then we start from the integral
\[
H(r) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\sigma x + \sigma y} \frac{1}{\phi(k)} \left( \frac{\chi(l_1) - \chi(l_2)}{\phi(k)} \right) \sum_{\sigma = 1}^\infty e^{\sigma x + \sigma y} f(s) ds,
\]
r, \( \alpha \) from (4.1), \( f(s) \) being as in (4.2), with \( l = l_1 \) however. Obviously \( f(s) \) can be written in the form
\[
f(s) = \sum \frac{s_0(p, l_1, l_2)}{p^{\sigma - 1}} + f_1(s),
\]
where — using the quadratic non-residuocity of \( l_1 \) and \( l_2 - f_1(s) \) is regular for \( \sigma > 0, 34 \) and here
\[
|f_1(s)| < c_{m4}.
\]

The contribution of the sum in (5.7) to \( H(r) \) is, owing to the integral-formula quoted in (4.4)
\[
\frac{1}{\im \varphi |(\sigma - 1/2\pi i)/(\pi \sigma)|} \int_{-1/2\pi i}^{1/2\pi i} e^{\sigma x + \sigma y} \frac{1}{\phi(k)} \sum_{\sigma = 1}^\infty \sum_{p \leq \sqrt{T}} \log p e \frac{1}{\im \varphi |(\sigma - 1/2\pi i)/(\pi \sigma)|}.
\]

As to the contribution of \( f_1(s) \) to \( H(r) \), we can shift the line of integration to \( \sigma > 34 \); hence this is
\[
\frac{1}{2\pi i} \int_{(34)} e^{\sigma x + \sigma y} \frac{1}{\phi(k)} \sum_{\sigma = 1}^\infty \sum_{p \leq \sqrt{T}} \log p e \frac{1}{\im \varphi |(\sigma - 1/2\pi i)/(\pi \sigma)|}.
\]

which in turn is owing to (5.8) absolutely
\[
< c_{m4} \int_{-\infty}^{\infty} e^{\delta y + \delta |b_1| |k|^{1/3}} |Y \eta |^{1/2} |d| d < c_{m4} e^{\delta y} |Y \eta |^{1/2} |d| d,
\]

6. Next we shift the line of integration in (5.6) to the line \( W \) in Lemma IV. We get the residue-sum
\[
\frac{1}{2\pi i} \int_{(34)} e^{\sigma x + \sigma y} \frac{1}{\phi(k)} \sum_{\sigma = 1}^\infty \sum_{p \leq \sqrt{T}} \log p e \frac{1}{\im \varphi |(\sigma - 1/2\pi i)/(\pi \sigma)|}.
\]

and the integral
\[
\frac{1}{2\pi i} \int_{(34)} e^{\sigma x + \sigma y} \frac{1}{\phi(k)} \sum_{\sigma = 1}^\infty \sum_{p \leq \sqrt{T}} \log p e \frac{1}{\im \varphi |(\sigma - 1/2\pi i)/(\pi \sigma)|}.
\]
The last one is, using standard estimations, absolutely less than

\[ c_{50} \phi^6(\ell \ell_1^{1/2} + \ell \ell_2^{1/2} + \ell_1^{1/2} + \ell_2^{1/2}) \log h. \]

Next we estimate the contribution of the \( c \)'s with

\[ \|c\| > 4/\sqrt{7}. \]

This is absolutely less than

\[ c_{50} \phi^6(\ell \ell_1^{1/2} + \ell \ell_2^{1/2} + \ell_1^{1/2} + \ell_2^{1/2}) \sum_{n \geq \frac{\epsilon}{\sqrt{2}} - 1} \frac{n \log h}{n} \leq c_{50} \log h \frac{\epsilon^6(\ell \ell_1^{1/2} + \ell \ell_2^{1/2} + \ell_1^{1/2} + \ell_2^{1/2})}{\sqrt{2}}. \]

Using (5.2) and (4.1), further Lemma 1, (1.3) and (1.5) this is for \( h > c_{50} \)

and suitable small \( \epsilon \) in (1.3)

\[ c_{50} \log h \frac{\epsilon^6(\ell \ell_1^{1/2} + \ell \ell_2^{1/2} + \ell_1^{1/2} + \ell_2^{1/2})}{\sqrt{2}} \leq c_{50} \log h \frac{\epsilon^6(\ell \ell_1^{1/2} + \ell \ell_2^{1/2} + \ell_1^{1/2} + \ell_2^{1/2})}{\sqrt{2}}. \]

Now let \( \varphi = \sigma^0 + i\varphi^0 \) be one of the nontrivial zeros of the \( L(s, \chi) \)-functions with \( \chi(l_1) \neq \chi(l_2) \) for which

\[ |\varphi^{1/2 + \frac{1}{2} \varphi^0 + \frac{1}{2} \varphi^0}| = \text{maximal} \]

among the \( \varphi = \sigma^0 + i\varphi^0 \)-zeros with

\[ \|\varphi\| \leq 4/\sqrt{7}. \]

Collecting (5.9), (5.10), (6.1), (6.2) and (6.3), dividing by \( \varphi^{1/2 + \frac{1}{2} \varphi^0 + \frac{1}{2} \varphi^0} \) and taking real parts, we get

\[ \left| \sum_p \varphi(p, l_1, l_2) \log p \varphi \left( \frac{1}{\sqrt{2} + \log p - \frac{n^{1/2 + \frac{1}{2} \varphi^0 + \frac{1}{2} \varphi^0}}}{\sqrt{2}} \right) \right| \]

\[ \leq V \left( \frac{1}{\sqrt{2} + \log p - \frac{n^{1/2 + \frac{1}{2} \varphi^0 + \frac{1}{2} \varphi^0}}}{\sqrt{2}} \right) \]

\[ \leq V \left( \frac{1}{\sqrt{2} + \log p - \frac{n^{1/2 + \frac{1}{2} \varphi^0 + \frac{1}{2} \varphi^0}}}{\sqrt{2}} \right) \]

\[ \times \left[ \frac{\varphi(l_1) - \varphi(l_2)}{\varphi(l_1)} \varphi(l_2) \right] \leq c_{50} e^{\frac{1}{11} \varphi^0} \varphi^0 \varphi^0 \leq c_{50} e^{\frac{1}{11} \varphi^0} \varphi^0 \varphi^0. \]}
is a zero with the maximal imaginary part \( \leq k^b \). We have to verify (3.8) for our choice of \( \sigma_0 \). First we assert that for all \( \sigma_0 \)'s belonging to \( \varphi \)'s with

\begin{equation}
|\epsilon_0| > \epsilon^{(1)}
\end{equation}

we have

\begin{equation}
|\epsilon_0| < |\epsilon_0|.
\end{equation}

In order to prove it, we remark first that owing to the form of \( \sigma_0 \)'s it suffices to show

\[|\epsilon^{(2)} + \epsilon^{(3)}| < |\epsilon^{(2)} + \epsilon^{(3)}| |\epsilon^{(1)}|,
\]

i.e.

\begin{equation}
\sigma_0^2 - \sigma_0 + 2b_0 \sigma_0 < \sigma_0^2 - \sigma_0^2 + 2b_0 \sigma_0^2.
\end{equation}

Owing to (8.2) and the definition of \( \sigma_0^{(1)} \) we have certainly

\begin{equation}
\sigma_0^{(1)} = \frac{1}{2}.
\end{equation}

If

\[|\epsilon_0| \leq \sqrt{\frac{1}{4} + 1/\eta + k^{10}}
\]

then (8.1) implies

\[\sigma_0 = \frac{1}{2}.
\]

Hence (8.5) takes in this case the form

\[1 - \sigma_0 + b_0 < 1 - \sigma_0^2 + b_0,
\]

which is true according to (8.3). If

\[|\epsilon_0| > \sqrt{\frac{1}{4} + 1/\eta + k^{10}},
\]

then we have to use (5.2) and (8.6)

\[\sigma_0^2 - \sigma_0 + 2b_0 \sigma_0 < 1 - \sigma_0^2 + 2b_0 \sigma_0 < 1 - \sigma_0^2 + 2b_0 \sigma_0^2,
\]

again. Thus (8.3) and (8.4) is proved. Since \( |\epsilon_0| \geq |\epsilon_0| \), (8.3) and (8.4) implies

\[|\epsilon_0| = \epsilon^{(1)} \leq k^b,
\]

i.e. from (8.3)

\begin{equation}
\epsilon^{(1)} = \frac{1}{2}.
\end{equation}

For the applicability of Lemma III we have to verify (3.8). Since

\begin{equation}
\epsilon^{(1)} = \frac{1}{2} \leq \frac{1}{2} |\epsilon_0|,
\end{equation}

the definition of \( \sigma_0 \) and (8.6) give at once

\[|\epsilon_0| \geq \epsilon^{(2)} + \epsilon^{(3)} |\epsilon^{(1)} + \epsilon^{(2)}| = \frac{1}{2} |\epsilon_0^2| > \frac{1}{4} |\epsilon_0|,
\]

Since from (7.2), (7.1), (5.4) and (1.6)

\[
\frac{4N}{m + N (3 + \pi /a)} \leq \frac{2 \pi \log T - \eta^{-1} \log (3 + \pi /a)}{2 \pi \log T - \eta^{-1} \log (3 + \pi /a)} < \frac{1}{\eta \log T} < e^{\frac{1}{2} \log}.
\]

(3.8) is satisfied at our choices.

9. Further we assert that for all \( \sigma_0 \)'s belonging to \( \varphi \)'s with

\[|\epsilon_0| \leq \epsilon^{(1)}
\]

the inequality

\begin{equation}
|\epsilon_0| \geq |\epsilon_0|
\end{equation}

holds. Namely, owing to the definition of \( \sigma_0 \)'s and (8.6), the inequality

(9.2) is certainly true if

\[\frac{1}{2} - \sigma_0 + b_0 \geq \frac{1}{2} - \sigma_0 + b_0,
\]

which holds for \( \varphi \)'s with (9.1) indeed. Hence the inequality

\[|\epsilon_0| > |\epsilon_0|
\]

holds exactly for \( \sigma_0 \)'s belonging to \( \varphi \)'s with

\begin{equation}
|\epsilon_0| > \epsilon^{(1)}.
\end{equation}

This gives the possibility to find a positive lower bound for \( B \) in (3.9). Namely, owing to the definitions of \( \sigma_0 \) and Lemma V, all \( \sigma_0 \)'s with

\[|\epsilon_0| \leq k^b - \epsilon^{(1)}
\]

occur in \( B \); applying Lemma VI we get

\[B > c_D P_1 \log^2 P_1 - c_{\text{as}} \sum_{n=\text{as}} \log k \pi \epsilon^{(2)} + \text{as} \epsilon^{(1)} ,
\]

i.e. for \( k > c_{\text{as}} \) — using (4.1) and Lemma I —

\begin{equation}
B > k^b - c_{\text{as}} k^b \int_{k^{10}}^{e^{-\pi \text{as}}} e^{-\epsilon^{(2)} \log \text{as}} \text{d} \epsilon > 1.
\end{equation}

Finally we have to verify (not to violate (5.5)) that

\[N (3 + \pi /a) < c_{\text{as}} \eta^{-1/5},
\]

but this is true owing to (7.1) and (7.2). Hence choosing \( r \) as \( r_1 \), resp. \( r_2 \) of Lemma III we obtain

\begin{equation}
m \leq r_1, r_2 \leq m + c_{\text{as}} \eta^{-1/5}.
\end{equation}
and using also (1.6) —

\[
Z(r_3) > \frac{1}{3} \left( \frac{1}{24e(2\pi \log T + c_{31})} \right) ^{\frac{1}{m + 3b_3a - \frac{3}{5}}}
\]

\[
> \frac{1}{\nu \log T} \left( \frac{1}{\log T} \right) ^{\frac{m - 3b_3a}{m + 3b_3a - \frac{3}{5}}}
\]

\[
> e^{-\frac{2\pi}{2\nu}} e^{-\frac{3b_3a}{m + 3b_3a - \frac{3}{5}}}
\]

and analogous negative upper bound for \(Z(r_2)\). Hence from (6.6) we get

\[
(6.6) \quad S_1 \equiv \sum_p \sum_{l_1, l_2} \log p \frac{e^{-\frac{R}{\nu} \{l_1(l_2 + l_1), k_1, k_2, k_3\}}}{n_1 m_1 + n_2 m_2 - \frac{3}{5}}.
\]

From the definition of \(S_1\), (9.5), (5.2) and (8.7), we get

\[
|m_1 + 3b_3a| \geq \frac{1}{2} e^{-\frac{R}{\nu} \{l_1(l_2 + l_1), k_1, k_2, k_3\}}
\]

\[
> e^{-\frac{2\pi}{2\nu}} e^{-\frac{3b_3a}{m + 3b_3a - \frac{3}{5}}}
\]

and using (1.3), (1.5), (5.4) and (1.6) —

\[
> \frac{1}{2} - \frac{1}{2\nu} e^{-\frac{R}{\nu}} > \frac{1}{2} - \frac{1}{2\nu} e^{-\frac{R}{\nu}}.
\]

Further we get from (4.1), Lemma I, (9.5), (5.2), (1.3), (1.5) and (1.6) for \(b > c_{31}\)

\[
\phi(b_3a + b_3b + b_3c) < e^{1b_3a \log b_3a + \frac{3}{5} b_3b + \frac{3}{5} b_3c} < e^{2b_3a - \frac{3}{5} b_3b + \frac{3}{5} b_3c}
\]

and hence, if \(c_3\) in (1.6) is sufficiently large, from (9.7) and (9.8)

\[
S_1 > \frac{1}{2} - \frac{1}{2\nu} e^{-\frac{R}{\nu}}
\]

and analogously

\[
S_2 \equiv \sum_p \sum_{l_1, l_2} \log p \frac{e^{-\frac{R}{\nu} \{l_1(l_2 + l_1), k_1, k_2, k_3\}}}{n_1 m_1 + n_2 m_2}
\]

Further putting

\[
(r_1 b_1 + r_2 b_2) / 2 = \log s_1
\]

we have from (9.5) and (5.3)

\[
\log s_1 > \frac{r_1 b_1}{2} > \frac{\eta \log T}{2} \left( \frac{1}{\nu} \right) (1 - \frac{1}{\nu}) = (1 - \frac{1}{\nu}) \log T
\]

and from (4.1) and (1.6)

\[
\log s_1 < \left( 2 \log P_f + \frac{1}{\eta} (2\eta \log T + c_{31} \nu^{-\frac{3}{5}}) \right) \frac{1}{2}
\]

\[
< \log T + c_{31} (\log h + \nu^{-\frac{3}{2}}) < \log T + \log \frac{1}{\nu}
\]

and analogously for \(s_2\). Finally for \(r_1\) and \(r_2\), (9.5) and (1.6) give

\[
2\nu \log T \leq r_1, r_2 \leq 2\nu \log T + \nu \log T
\]

which completes the proof of Theorem I.

References

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