Since $0 \leqslant s < r$, from (5.8) $k \geqslant 1$ and $k = 1$ only if $s \geqslant 2$; in this case however $r \geqslant 3$ and then (5.9) is not satisfied. Hence $k \geqslant 2$ and so from (5.9) $2r^2 < 2(r+1)$, i.e. $r \leqslant 1$. Thus $r = 1$ and $s = 0$. From (5.8) $k \geqslant 3$ and from (5.9) $k < 4$ so $k = 3$. Now when $r = 1$, $s = 0$, $t = 3$ we have $(r+1)(s+t)+s+1 > 2(s+t)+2-\frac{t}{r+1}$ so the only case for which $\frac{n}{m}$ is a better bound in the range $m+2 < n < 2m$ is $r = 1$, $s = 0$, $t = 3$, $\theta = 6$, i.e. $n = 9$, $m = 6$.

References

[1] L. J. Mordell, *On the inequality* $\sum_{r=1}^{n} \frac{x_r}{x_{r+1}+x_{r+2}} \geqslant \frac{n}{2}$ *and some others,* Abh. Math. Sem. Univ. Hamburg. 22 (1958), pp. 229-241.

[2] P. H. Diananda, *On a conjecture of L. J. Mordell regarding an inequality involving quadratic forms,* Journ. London Math. Soc. 36 (1961), pp. 185-192.

Reçu par la Rédaction le 27. 4. 1964

ACTA ARITHMETICA
XI (1965)

Further developments in the comparative prime-number theory IV

(Accumulation theorems for residue-classes representing quadratic non-residues mod k)

by

S. Knapowski (Poznań) and P. Turán (Budapest)

1. In the second and third papers of this series we introduced a new approach instead of that of Chebyshev, in order to find a sense in which there are more primes $\equiv l_1 \bmod k$ than $\equiv l_2 \bmod k$ if and only if l_1 is a quadratic non-residue, l_2 quadratic residue mod k. We succeeded in obtaining results in this direction when the Haselgrove-condition is satisfied for k, i.e. when there is an $E = E(k) > 0$ such that no $L(s, \chi)$ belonging to the modulus k vanishes for(1)

(1.1) $$\sigma \geqslant \tfrac{1}{2}, \quad |t| \leqslant E(k) \quad (s = \sigma + it).$$

For the sake of brevity we shall call such k-values "good" k-values. We made a comparison in the second paper for the residue-classes

$$\equiv 1 \bmod k \quad \text{and} \quad \equiv l \bmod k$$

(l quadratic non-residue mod k) in the third one for the residue-classes

$$\equiv 1 \bmod k \quad \text{and} \quad \equiv l \bmod k$$

(l quadratic residue mod k).

In this paper we shall pass to the more general case, when we compare the residue-classes

(1.2) $$\equiv l_1 \bmod k \quad \text{and} \quad \equiv l_2 \bmod k$$

(l_1, l_2 both quadratic non-residues).

---

(1) Though no k-value is known for which this would be false, it is desirable to prove its truth at least for an infinity of k-values.

This time we succeeded only for k's satisfying a condition more stringent than (1.1). We shall suppose not only (1.1) but also with an η, satisfying with a suitably small c_1 (²) the condition

$$0 < \eta < \min\left(c_1, \left(\frac{E(k)}{6\pi}\right)^2\right) \tag{1.3}$$

the nonvanishing of all $L(s, \chi)$-functions belonging to $\operatorname{mod} k$ for

$$\sigma > \tfrac{1}{2}, \quad |t| \leqslant 2/\sqrt{\eta}. \tag{1.4}$$

On $E(k)$ we may suppose without loss of generality that

$$E(k) \leqslant \frac{1}{k^{15}}. \tag{1.5}$$

Then we shall prove the

THEOREM I. *If for $k > c_2$ with sufficiently large c_2 the condition* (1.1), (1.3), (1.4) *and* (1.5) *is satisfied, then for*

$$T > \max\left(c_3, e^{\frac{2}{\eta^4} e^{\frac{1}{4}k^{10}}}\right) \tag{1.6}$$

and for quadratic non-residue l_1 and l_2 there are x_j and ν_j $(j = 1, 2)$ with

$$T^{1-\sqrt{\eta}} \leqslant x_1, x_2 \leqslant T e^{\log^{3/4} T} \tag{1.7}$$

and

$$2\eta \log T \leqslant \nu_1, \nu_2 \leqslant 2\eta \log T + \sqrt{\log T} \tag{1.8}$$

so that

$$\sum_{p \equiv l_1 \operatorname{mod} k} \log p \, e^{-\frac{1}{\nu_1}\log^2\frac{p}{x_1}} - \sum_{p \equiv l_2 \operatorname{mod} k} \log p \, e^{-\frac{1}{\nu_1}\log^2\frac{p}{x_1}} > T^{\frac{1}{2}-4\sqrt{\eta}}.$$

2. In the first paper of this new series we proved the first "accumulation"-theorem. This states in its simplest form that for a sufficiently large c_4 for $T > c_4$ there are U_1, U_2, U_3, U_4 with

$$T e^{-\log^{11/12} T} \leqslant U_1 < U_2 \leqslant T,$$
$$T e^{-\log^{11/12} T} \leqslant U_3 < U_4 \leqslant T$$

so that

$$\sum_{\substack{n \equiv 1 (4) \\ U_1 \leqslant n \leqslant U_2}} \Lambda(n) - \sum_{\substack{n \equiv 3 (4) \\ U_1 \leqslant n \leqslant U_2}} \Lambda(n) > \sqrt{T} e^{-\log^{11/12} T}$$

(²) c_1 and later $c_2, c_3, \ldots$ denote always positive numerical constants.

and

$$\sum_{\substack{n \equiv 1 (4) \\ U_3 \leqslant n \leqslant U_4}} \Lambda(n) - \sum_{\substack{n \equiv 3 (4) \\ U_3 \leqslant n \leqslant U_4}} \Lambda(n) < -\sqrt{T} e^{-\log^{11/12} T}.$$

The corresponding problems for primes instead of prime-powers are generally more difficult. In this direction we shall prove the following

THEOREM II. *Under the conditions of Theorem I there are $\mu_1, \mu_2, \mu_3, \mu_4$ with*

$$T^{1-4\sqrt{\eta}} \leqslant \mu_1 < \mu_2 \leqslant T^{1+4\sqrt{\eta}},$$
$$T^{1-4\sqrt{\eta}} \leqslant \mu_3 < \mu_4 \leqslant T^{1+4\sqrt{\eta}}$$

so that

$$\sum_{\substack{p \equiv l_1(k) \\ \mu_1 \leqslant p \leqslant \mu_2}} 1 - \sum_{\substack{p \equiv l_2(k) \\ \mu_1 \leqslant p \leqslant \mu_2}} 1 > T^{\frac{1}{2}-5\sqrt{\eta}}$$

and

$$\sum_{\substack{p \equiv l_1(k) \\ \mu_3 \leqslant p \leqslant \mu_4}} 1 - \sum_{\substack{p \equiv l_2(k) \\ \mu_3 \leqslant p \leqslant \mu_4}} 1 < -T^{\frac{1}{2}-5\sqrt{\eta}}.$$

Since this can be derived from Theorem I following the pattern of our paper [2] of this series, we shall omit the details.

3. We shall need a number of lemmas.

LEMMA I. *If no $L(s, \chi)$-functions* $\operatorname{mod} k$ *vanish for*

$$\sigma > \tfrac{1}{2}, \quad |t| \leqslant \log^2 \varphi(k),$$

then for all $(l, k) = 1$ there exists a prime $P = P_l$ with $P \equiv l(k)$ for which (³), *with suitable c_5 and c_6,*

$$c_5 \varphi(k)^{5/2} \leqslant P \leqslant c_6 \varphi(k)^{5/2}.$$

Let with a fixed l, with $(l, k) = 1$,

$$F(s) = -\frac{1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l) \frac{L'}{L}(s+1, \chi) \tag{3.1}$$

and

$$\nu = \left[\frac{1}{10} \log \varphi(k)\right], \tag{3.2}$$

$$A = 10/\log \varphi(k). \tag{3.3}$$

(³) A weaker lemma is deduced in our paper [1] from the exact prime-number formula (p. 50). We prefer now to give an independent proof. The conditions could have been much weakened.

With these we consider the integral

$$(3.4)\qquad J = \frac{1}{2\pi i}\int\limits_{\left(\frac{1}{\log\varphi(k)}\right)}\left(e^{25s}\frac{e^{\Delta s}-e^{-\Delta s}}{2\Delta s}\right)^{\nu}F(s)\,ds.$$

Replacing $F(s)$ by its Dirichlet-series and integrating term by term, we get

$$J = \sum_{\substack{n\equiv l(k)\\ e^{(25-\Delta)\nu}\leqslant n\leqslant e^{(25+\Delta)\nu}}}\frac{\Lambda(n)}{n}\cdot\frac{1}{2\pi i}\int\limits_{(0)}\left(e^{25s}\frac{e^{\Delta s}-e^{-\Delta s}}{2\Delta s}\right)^{\nu}\frac{ds}{n^{s}}.$$

Since the integral is

$$(3.5)\qquad \frac{1}{\Delta\pi}\int\limits_{0}^{\infty}\left(\frac{\sin t}{t}\right)^{\nu}\cos\frac{25\nu-\log n}{\Delta}t\,dt \stackrel{\text{def}}{=} a_n(\nu),$$

which is positive for $e^{(25-\Delta)\nu}<n<e^{(25+\Delta)\nu}$ and 0 otherwise, further from (3.2), (3.3)

$$e^{(25+\Delta)\nu}\leqslant e\varphi(k)^{5/2},\qquad e^{(25-\Delta)\nu}\geqslant e^{-26}\varphi(k)^{5/2},$$

we have

$$(3.6)\qquad J = \sum_{\substack{n\equiv l(k)\\ e^{-26}\varphi(k)^{5/2}\leqslant n\leqslant e\varphi(k)^{5/2}}}\frac{\Lambda(n)}{n}a_n(\nu).$$

Next we replace the line

$$\sigma = 1/\log\varphi(k)$$

by the broken line ($a\leqslant\log\varphi(k)$ to be determined)

K_1: $\sigma = 1/\log\varphi(k),\quad t\leqslant -a\log\varphi(k),$

K_2: $-9/20\leqslant\sigma\leqslant 1/\log\varphi(k),\quad t=-a\log\varphi(k),$

K_3: $\sigma=-9/20,\quad -a\log\varphi(k)\leqslant t\leqslant a\log\varphi(k),$

K_4: $-9/20\leqslant\sigma\leqslant 1/\log\varphi(k),\quad t=a\log\varphi(k),$

K_5: $\sigma=1/\log\varphi(k),\quad t\geqslant a\log\varphi(k).$

Denoting the respective integrals by $I_1, I_2, \ldots, I_5$, we have by standard estimations concerning L-functions for $|I_1|$ and $|I_5|$ the upper bound

$$c_8 e^{\frac{(25+\Delta)\nu}{\log\varphi(k)}}\int\limits_{a\log\varphi(k)}^{\infty}\frac{\log\varphi(k)}{(\Delta t)^{\nu}}dt < c_9 a^2\frac{\log\varphi(k)}{(10a)^{\frac{1}{10}\log\varphi(k)}},$$

for $|I_2|$ and $|I_4|$

$$c_{10}e^{\frac{(25+\Delta)\nu}{\log\varphi(k)}}\frac{a\log\varphi(k)}{(10a)^{\frac{1}{10}\log\varphi(k)}} < c_{11}\frac{a\log\varphi(k)}{(10a)^{\frac{1}{10}\log\varphi(k)}},$$

and finally for $|I_3|$

$$c_{12}\log\varphi(k)\,e^{-\frac{9}{20}(25-\Delta)\nu}\left(\tfrac{5}{2}\right)^{\nu} < c_{13}\varphi(k)^{\frac{1}{10}\log\frac{5}{2}-\frac{9}{8}}\log\varphi(k).$$

Now we choose a so large that

$$\tfrac{1}{10}\log(10a) = 1{,}1,\qquad a=\tfrac{1}{10}e^{11};$$

this can be done if $a\log\varphi(k)\leqslant\log^2\varphi(k)$, i.e. $k>c_{14}$. Since

$$\tfrac{1}{10}\log\tfrac{5}{2}<\tfrac{1}{10}$$

and the residuum at $s=0$ is $1/\varphi(k)$, we get from (3.6)

$$(3.7)\qquad \sum_{\substack{n\equiv l(k)\\ e^{-26}\varphi(k)^{5/2}\leqslant n\leqslant e\varphi(k)^{5/2}}}\frac{\Lambda(n)}{n}a_n(\nu) > \frac{1}{\varphi(k)}-\frac{c_{15}}{\varphi(k)^{41/40}} > \frac{1}{2}\cdot\frac{1}{\varphi(k)}$$

if $k>c_{16}$. Since the contribution of the prime-powers p^{β} ($\beta\geqslant 2$) to (3.7) is owing to (3.5) at most

$$c_{17}\log^2\varphi(k)\sum_{p>e^{-13}\varphi(k)^{5/4}}\frac{1}{p^2} < c_{18}\frac{\log^2\varphi(k)}{\varphi(k)^{5/4}} < \frac{1}{4\varphi(k)}$$

if $k>c_{19}$. Hence for $k>c_{20}$ the assertion is proved with $c_5=e^{-25}$, $c_6=e$. From this Lemma I follows easily generally.

Further we need the

LEMMA II. *If α_ν and β_ν are real, further*

$$|\alpha_\nu|\geqslant U\quad(\leqslant\tfrac{1}{2})$$

further with a $\gamma>1$

$$\sum_{\nu}\frac{1}{1+|\alpha_\nu|^{\gamma}}\leqslant V\quad(<\infty)$$

and $\Delta>1/U$, then each real interval of length Δ contains a ξ with the property that for each ν-index the inequality

$$\{\alpha_\nu\xi+\beta_\nu\} > \frac{1}{24V}\cdot\frac{1}{1+|\alpha_\nu|^{\gamma}}$$

holds (4).

For the proof see our paper [2].

(4) $\{x\}$ stands as usual for the distance of x from the nearest integer.

Also we need the

LEMMA III. *Let m be positive and z_j's with*

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_h| \geqslant \ldots \geqslant |z_n|$$

such that with a $0 < \varkappa \leqslant \pi/2$

$$\varkappa \leqslant |\operatorname{arc} z_j| \leqslant \pi; \quad j = 1, 2, \ldots, n,$$

the index h we define by

$$|z_h| > \frac{4N}{m+N(3+\pi/\varkappa)} \tag{3.8}$$

where N is an upper bound for n. Then there are integers ν_1 and ν_2 with

$$m \leqslant \nu_1, \nu_2 \leqslant m+N(3+\pi/\varkappa)$$

so that

$$\operatorname{Re} \sum_{j=1}^{n} d_j z_j^{\nu_1} \geqslant \frac{B}{3N} \left(\frac{N}{24e\big(m+N(3+\pi/\varkappa)\big)}\right)^{2N} \left(\frac{|z_h|}{2}\right)^{m+N(3+\pi/\varkappa)}$$

and

$$\operatorname{Re} \sum_{j=1}^{n} d_j z_j^{\nu_2} \leqslant -\frac{B}{3N} \left(\frac{N}{24e\big(m+N(3+\pi/\varkappa)\big)}\right)^{2N} \left(\frac{|z_h|}{2}\right)^{m+N(3+\pi/\varkappa)};$$

here B stands for

$$B = \min_{v \geqslant h} \left|\operatorname{Re} \sum_{j=1}^{v} d_j\right|. \tag{3.9}$$

For the proof see our paper [3].

We shall use further the

LEMMA IV. *There exists a broken line W in the vertical strip $\frac{1}{5} \leqslant \sigma \leqslant \frac{1}{4}$ consisting alternately of horizontal and vertical segments so that each horizontal strip of width* 1 *contains at most one horizontal segment and on W for all L-functions* mod k *the inequality*

$$\left|\frac{L'}{L}(s, \chi)\right| \leqslant c_{21} \varphi(k) \log^2 k (2+|t|)$$

holds.

A proof of this lemma follows *mutatis mutandis* that of the appendix III of paper of the second of us [1]. We shall also need the following simple consequence of a theorem of Siegel ([1]) which we state as

LEMMA V. *For a suitable c_{22} all L-functions (for all $k \geqslant 1$) have a zero in all parallelograms (τ real)*

$$\tfrac{1}{2} \leqslant \sigma \leqslant 1, \quad \tau \leqslant t \leqslant \tau + c_{22}.$$

4. Finally we shall need the

LEMMA VI. *In the notation of Lemma* I, *with $(l, k) = (l_1, k) = 1$, $l \not\equiv l_1 \bmod k$ and*

$$b_0 = 2P_l^2 \log^3 P_l, \qquad r_0 = \frac{1}{P_l^2 \log^2 P_l}, \tag{4.1}$$

we have for $k > c_{23}$ the inequality

$$\frac{1}{\varphi(k)} \operatorname{Re} \sum_{\chi} \big(\bar\chi(l_1) - \bar\chi(l)\big) \sum_{\varrho(\chi)}{}' e^{r_0(\varrho^2+2b_0\varrho)/4} > c_{24} P_l \log^2 P_l,$$

where Σ' means that the summation is to be extended only to the nontrivial zeros $\varrho = \varrho(\chi)$ of $L(s, \chi)$ right to W.

For the proof we shall define

$$f(s) = \frac{1}{\varphi(k)} \sum_{\chi} \big(\bar\chi(l_1) - \bar\chi(l)\big) \frac{L'}{L}(s, \chi) \tag{4.2}$$

and we start with the integral

$$J_1 = \frac{1}{2\pi i} \int_{(2)} e^{r_0(s+b_0)^2/4} f(s)\, ds. \tag{4.3}$$

Since for positive λ's we have (see our paper [2])

$$\frac{1}{2\pi i} \int_{(2)} e^{r_0(s+b_0)^2/4-\lambda s} ds = \frac{1}{\sqrt{\pi r_0}} e^{\frac{r_0}{4} b_0^2 - \frac{1}{r_0}\left(\lambda - \frac{r_0 b_0}{2}\right)^2}, \tag{4.4}$$

inserting the Dirichlet-series of $f(s)$ we get, using the notation

$$\varepsilon_k(n, l, l_1) = \begin{cases} 1 & \text{if} \quad n \equiv l(k), \\ -1 & \text{if} \quad n \equiv l_1(k), \\ 0 & \text{otherwise}, \end{cases} \tag{4.5}$$

the relation

$$J_1 = \frac{e^{r_0 b_0^2/4}}{\sqrt{\pi r_0}} \sum_{n} \Lambda(n) \varepsilon_k(n, l, l_1) e^{-\frac{1}{r_0}\left(\log n - \frac{r_0 b_0}{2}\right)^2}. \tag{4.6}$$

Hence J_1 is real. The contribution of $n = P_l$ to the sum is obviously $\log P_l$. The contribution of the terms $n < P_l$ to the sum is

$$\begin{aligned} &< \sum_{n \leqslant P_l - 1} \log n\, e^{-\frac{1}{r_0}\left\{\log(P_l - 1) - \frac{r_0 b_0}{2}\right\}^2} \\ &< P_l \log P_l\, e^{-P_l^2 \log^2 P_l \log^2(1 - 1/P_l)} \\ &< P_l \log P_l\, e^{-\log^2 P_l} < c_{25}. \end{aligned}$$

The contribution of the n's with $n > P_l$ is

$$< \sum_{n>P_l} \log n \cdot e^{-\frac{1}{r_0}\log^2\frac{n}{P_l}} = \sum_{P_l+1\leqslant n\leqslant P_l^2} + \sum_{n>P_l^2} < 2\log P_l \int_{P_l}^{\infty} e^{-\frac{1}{r_0}\log^2\frac{x}{P_l}} dx + c_{26} < c_{27}$$

and hence

$$J_1 > \frac{e^{r_0 b_0^2/4}}{\sqrt{\pi}} P_l \log P_l (\log P_l - c_{28}). \tag{4.7}$$

Shifting the line of integration to the line W we get

(4.8)

$$J_1 = \operatorname{Re} J_1 = \frac{1}{\varphi(k)} \operatorname{Re} \sum_{\chi} \big(\bar{\chi}(l_1) - \bar{\chi}(l)\big) \sum_{\varrho(\chi)}{}' e^{r_0(\varrho+b_0)^2/4} + \operatorname{Re} \frac{1}{2\pi i} \int_{(W)} e^{r_0(s+b_0)^2/4} f(s)\, ds.$$

The last integral is, using Lemma IV, absolutely

$$< c_{28} \log^2 k\, e^{r_0(b_0+1/4)^2/4} < c_{29} \log^2 P_l e^{r_0 b_0^2/4} e^{r_0 b_0/8} < c_{30} e^{r_0 b_0^2/4} P_l^{3/4} \log^2 P_l.$$

Collecting all these and dividing by $e^{r_0 b_0^2/4}$, the lemma follows if $k > c_{23}$.

5. Now we can turn to the proof of Theorem I. First we apply Lemma II for

$$\alpha_\nu = \frac{1}{4\pi} t_\varrho, \qquad \beta_\nu = \frac{1}{8\pi} \operatorname{Im} \varrho^2$$

with

$$\gamma = \tfrac{11}{10}, \quad U = E(k)/5\pi, \quad \Delta = 6\pi/E(k); \tag{5.1}$$

one can evidently choose

$$V = c_{31} k \log k.$$

Owing to (1.3) we have

$$\Delta < 1/\sqrt{\eta};$$

hence Lemma II gives the existence of a b_1 with

$$\frac{1}{\eta} - \frac{1}{\sqrt{\eta}} \leqslant b_1 \leqslant \frac{1}{\eta} \tag{5.2}$$

such that for all ϱ's

$$\left\{\frac{1}{2\pi} \cdot \frac{1}{4} \big(2b_1 \operatorname{Im} \varrho + \operatorname{Im}(\varrho^2)\big)\right\} > \frac{c_{31}}{k \log k} \cdot \frac{1}{1+|t_\varrho|^{11/10}},$$

i.e.

$$|\operatorname{arc} e^{(\varrho^2+2b_1\varrho)/4}| > \frac{c_{32}}{(1+|t_\varrho|^{11/10}) k \log k}. \tag{5.3}$$

Let further be

$$m = 2\eta \log T \tag{5.4}$$

and the integer r be restricted momentaneously only by

$$(10 \leqslant) \quad m \leqslant r \leqslant m + c_{33}/\eta^{6/5}. \tag{5.5}$$

Then we start from the integral

$$H(\nu) \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{(2)} e^{r_0(s+b_0)^2/4 + r(s+b_1)^2/4} f(s)\, ds, \tag{5.6}$$

r_0, s_0 from (4.1), $f(s)$ being as in (4.2), with $l = l_2$ however. Obviously $f(s)$ can be written in the form

$$f(s) = \sum_p \varepsilon_k(p, l_2, l_1) \frac{\log p}{p^s} + f_1(s), \tag{5.7}$$

where — using the quadratic non-residuacity of l_1 and l_2 — $f_1(s)$ is regular for $\sigma \geqslant 0{,}34$ and here

$$|f_1(s)| \leqslant c_{34}. \tag{5.8}$$

The contribution of the sum in (5.7) to $H(\nu)$ is, owing to the integral-formula quoted in (4.4)

$$\frac{e^{(r_0 b_0^2 + r b_1^2)/4}}{\sqrt{\pi(r+r_0)}} \sum_p \varepsilon_k(p, l_2, l_1) \log p\, e^{-\frac{(\log p - (r_0 b_0 + r b_1)/2)^2}{r_0 + r}}. \tag{5.9}$$

As to the contribution of $f_1(s)$ to $H(\nu)$, we can shift the line of integration to $\sigma = 0{,}34$; hence this is

$$\frac{1}{2\pi i} \int_{(0,34)} e^{r_0(s+b_0)^2/4 + r(s+b_1)^2/4} f_1(s)\, ds,$$

which in turn is owing to (5.8) absolutely

$$\leqslant c_{35} \int_{-\infty}^{\infty} e^{\frac{r_0}{4}(b_0+0{,}34)^2 + \frac{r}{4}(b_1+0{,}34)^2 - \frac{r_0+r}{4} t^2} dt < c_{36} e^{r_0(b_0+0{,}34)^2/4 + r(b_1+0{,}34)^2/4}. \tag{5.10}$$

6. Next we shift the line of integration in (5.6) to the line W in Lemma IV. We get the residue-sum

$$\frac{1}{\varphi(k)} \sum_{\chi} \big(\bar{\chi}(l_1) - \bar{\chi}(l_2)\big) \sum_{\substack{\varrho(\chi)\text{ right} \\ \text{from } W}}{}' e^{r_0(\varrho+b_0)^2/4 + r(\varrho+b_1)^2/4} \tag{6.1}$$

and the integral

$$\frac{1}{2\pi i} \int_{(W)} e^{r_0(s+b_0)^2/4 + r(s+b_1)^2/4} f(s)\, ds.$$

The last one is, using standard estimations, absolutely less than

$$c_{37}e^{r_0(b_0+1/4)^2/4+r(b_1+1/4)^2/4}\log k. \tag{6.2}$$

Next we estimate the contribution of the ϱ's with

$$|t_\varrho| > 4/\sqrt{\eta}.$$

This is absolutely less than

$$c_{38}e^{r_0(b_0+1)^2/4+r(b_1+1)^2/4}\sum_{n>\frac{4}{\sqrt{\eta}}-1} e^{-\frac{r_0+r}{4}n^2}\log kn$$

$$< c_{39}\log\frac{k}{\eta}e^{\frac{r_0(b_0+1)^2}{4}+\frac{r}{4}(b_1+1)^2-\frac{r_0+r}{4}\cdot\frac{16}{\eta}}.$$

Using (5.2), (4.1), further Lemma I, (1.3) and (1,5) this is for $k > c_{40}$ and suitably small c_1 in (1.3)

$$\begin{aligned}(6.3)\qquad &< c_{41}\log\frac{k}{\eta}\, e^{(r_0b_0^2+rb_1^2)/4}\, e^{r_0(2b_0+1-16/\eta)/4}\, e^{r(2b_1+1-15/\eta)/4}\\ &< c_{42}\log\frac{1}{\eta}\, e^{(r_0b_0^2+rb_1^2)/4}\, e^{r_0(c_{43}k^6-15/\eta)/4}\, e^{r(3/\eta-15/\eta)/4}\\ &< c_{44}\log\frac{1}{\eta}\, e^{(r_0b_0^2+rb_1^2)/4}\, e^{-(r_0+r)/4\eta}.\end{aligned}$$

Now let $\varrho_0 = \sigma^{(0)}+it^{(0)}$ be one of the nontrivial zeros of the $L(s,\chi)$-functions with $\chi(l_1)\neq\chi(l_2)$ for which

$$|e^{(\varrho^2+2b_1\varrho)/4}| = \text{maximal} \tag{6.4}$$

among the $\varrho = \sigma_\varrho+it_\varrho$-zeros with

$$|t_\varrho| \leqslant 4/\sqrt{\eta}. \tag{6.5}$$

Collecting (5.9), (5.10), (6.1), (6.2) and (6.3), dividing by $e^{(r_0b_0^2+rb_1^2)/4}$ and taking real parts, we get

$$\begin{aligned}(6.6)\qquad &\Big|\sum_p \varepsilon_k(p,l_2,l_1)\log p\, e^{-\frac{1}{r_0+r}\left(\log p-\frac{r_0b_0+rb_1}{2}\right)^2}-\\ &-\sqrt{\pi(r+r_0)}\,|e^{(\varrho_0^2+2b_1\varrho_0)/4}|^r\ \mathrm{Re}\sum_\chi \sum_{\substack{\varrho(\chi)\\ \varrho\text{ right from }W\\ |t_\varrho|\leqslant 4/\sqrt{\eta}}} \frac{\overline{\chi}(l_1)-\overline{\chi}(l_2)}{\varphi(k)}\, e^{r_0(\varrho^2+2b_0\varrho)/4}\times\\ &\times\left(e^{(\varrho^2+2b_1\varrho)/4-\mathrm{Re}(\varrho_0^2+2b_1\varrho_0)/4}\right)^r\Big| \leqslant c_{45}e^{0,17(r_0b_0+rb_1)+(r_0+r)/4}.\end{aligned}$$

7. We denote the expression

$$\mathrm{Re}\sum_\chi \sum_{\substack{\varrho(\chi)\\ \varrho\text{ right from }W\\ |t_\varrho|\leqslant 4/\sqrt{\eta}}} \frac{\overline{\chi}(l_1)-\overline{\chi}(l_2)}{\varphi(k)}\, e^{r_0(\varrho^2+2b_0\varrho)/4}\left(e^{(\varrho^2+2b_1\varrho)/4-\mathrm{Re}(\varrho_0^2+2b_1\varrho_0)/4}\right)^r$$

in (6.6) by $Z(r)$ and shall determine r by using properly Lemma III. The role of the d_j's will be played by the numbers

$$\frac{\overline{\chi}(l_1)-\overline{\chi}(l_2)}{\varphi(k)}\, e^{r_0(\varrho^2+2b_0\varrho)/4}$$

that of the z_j's by the numbers

$$e^{(\varrho^2+2b_1\varrho)/4-\mathrm{Re}(\varrho_0^2+2b_1\varrho_0)/4};$$

the condition $\max_j |z_j| = 1$ is obviously fulfilled. From (5.3) and (6.5) we have

$$|\mathrm{arc}\, z_j| \geqslant \frac{c_{32}}{\{1+(4/\sqrt{\eta})^{11/10}\}k\log k}$$

and owing to (1.5) and (1.3) *a fortiori*

$$> c_{45}\frac{\eta^{11/20+1/30}}{\log(1/\eta)} > \eta^{3/5} \overset{\text{def}}{=} \varkappa. \tag{7.1}$$

Further we have from (1.5) and (1.3)

$$n < c_{46}\frac{1}{\sqrt{\eta}}\log\frac{k}{\eta} < c_{47}\frac{1}{\sqrt{\eta}}\log\frac{1}{\eta} < \eta^{-11/20} \overset{\text{def}}{=} N. \tag{7.2}$$

8. Now we shall use the fact that no $L(s,\chi)\bmod k$ vanishes in the domain (1.4) which implies on the one hand — using (1.3) and (1.5) — the non-vanishing in

$$\sigma > \tfrac{1}{2},\qquad |t| \leqslant \sqrt{\tfrac{3}{4}+1/\eta+k^{10}}, \tag{8.1}$$

and, on the other hand, the non-vanishing in

$$\sigma > \tfrac{1}{2},\qquad |t| \leqslant k^5. \tag{8.2}$$

We choose as z_h the quantity

$$z_h = e^{(\varrho^{(1)2}+2b_1\varrho^{(1)})/4-\mathrm{Re}(\varrho_0^2+2b_1\varrho_0)/4}$$

where

$$\varrho^{(1)} = \sigma^{(1)}+it^{(1)}$$

is a zero with the maximal imaginary part $\leqslant k^5$. We have to verify (3.8) for our choice of z_h. First we assert that for all z_j's belonging to ϱ's with

$$|t_\varrho| > t^{(1)} \tag{8.3}$$

we have

$$|z_j| < |z_h|. \tag{8.4}$$

In order to prove it, we remark first that owing to the form of z_j's it suffices to show

$$|e^{(\varrho^2+2b_1\varrho)/4}| < |e^{(\varrho^{(1)^2}+2b_1\varrho^{(1)})/4}|,$$

i.e.

$$\sigma_\varrho^2 - t_\varrho^2 + 2b_1\sigma_\varrho < \sigma^{(1)^2} - t^{(1)^2} + 2b_1\sigma^{(1)}. \tag{8.5}$$

Owing to (8.2) and the definition of $\varrho^{(1)}$ we have certainly

$$\sigma^{(1)} = \tfrac{1}{2}. \tag{8.6}$$

If

$$|t_\varrho| \leqslant \sqrt{\tfrac{3}{4} + 1/\eta + k^{10}}$$

then (8.1) implies

$$\sigma_\varrho = \tfrac{1}{2}.$$

Hence (8.5) takes in this case the form

$$\tfrac{1}{4} - t_\varrho^2 + b_1 < \tfrac{1}{4} - t^{(1)^2} + b_1$$

which is true according to (8.3). If

$$|t_\varrho| > \sqrt{\tfrac{3}{4} + 1/\eta + k^{10}},$$

then we have owing to (5.2) and (8.6)

$$\sigma_\varrho^2 - t_\varrho^2 + 2b_1\sigma_\varrho \leqslant 1 - t_\varrho^2 + 2b_1 < \tfrac{1}{4} - k^{10} + b_1 \leqslant \sigma^{(1)^2} - t^{(1)^2} + 2b_1\sigma^{(1)}$$

again. Thus (8.3) and (8.4) is proved. Since $|z_1| \geqslant |z_h|$, (8.3) and (8.4) implies

$$|t^{(0)}| \leqslant t^{(1)} \leqslant k^5,$$

i.e. from (8.2)

$$\sigma^{(0)} = \tfrac{1}{2}. \tag{8.7}$$

For the applicability of Lemma III we have to verify (3.8). Since (8.7) implies

$$e^{\mathrm{Re}(\varrho_0^2+2b_1\varrho_0)/4} < e^{\frac{1}{4}\left(\frac{1}{4}+b_1\right)};$$

the definition of z_h and (8.6) give at once

$$|z_h| \geqslant e^{\frac{1}{4}\left(\frac{1}{4}-t^{(1)^2}+b_1\right)-\frac{1}{4}\left(\frac{1}{4}+b_1\right)} = e^{-\frac{1}{4}t^{(1)^2}} > e^{-\frac{1}{4}k^{10}}.$$

Since from (7.2), (7.1), (5.4) and (1.6)

$$\frac{4N}{m+N(3+\pi/\varkappa)} = \frac{4\eta^{-11/20}}{2\eta\log T + \eta^{-11/20}(3+\pi\eta^{-3/5})} < \frac{2}{\eta^2\log T} < e^{-\frac{1}{4}k^{10}},$$

(3.8) is satisfied at our choices.

9. Further we assert that for all z_j's belonging to ϱ's with

$$|t_\varrho| \leqslant t^{(1)} \tag{9.1}$$

the inequality

$$|z_j| \geqslant |z_h| \tag{9.2}$$

holds. Namely, owing to the definition of z_j's and (8.6), the inequality (9.2) is certainly true if

$$\tfrac{1}{4} - t_{\varrho_j}^2 + b_1 \geqslant \tfrac{1}{4} - t^{(1)^2} + b_1$$

which holds for ϱ's with (9.1) indeed. Hence the inequality

$$|z_h| > |z_j|$$

holds *exactly* for z_j's belonging to ϱ's with

$$|t_\varrho| > t^{(1)}. \tag{9.3}$$

This gives the possibility to find a positive lower bound for B in (3.9). Namely, owing to the definitions of z_h and Lemma V, all d_j's with

$$|t_\varrho| \leqslant k^5 - c_{22}$$

occur in B; applying Lemma VI we get

$$B > c_{24}P_l\log^2 P_l - c_{48}\sum_{n\geqslant k^5-c_{22}} \log kn\, e^{r_0(2b_0+1-n^2)/4},$$

i.e. for $k > c_{49}$ — using (4.1) and Lemma I —

$$> k^{5/2} - c_{50}k^3 \int_{\frac{1}{2}k^5}^{\infty} e^{-x^2/k^6}\log kx\, dx > 1. \tag{9.4}$$

Finally we have to verify (not to violate (5.5)) that

$$N(3+\pi/\varkappa) < c_{33}\eta^{-6/5};$$

but this is true owing to (7.1) and (7.2). Hence choosing r as ν_1, resp. ν_2 of Lemma III we obtain

$$m \leqslant \nu_1,\ \nu_2 \leqslant m + c_{33}\eta^{-6/5} \tag{9.5}$$

and — using also (1.6) —

$$Z(\nu_1) > \frac{\eta^{11/20}}{3}\left(\frac{1}{24e(2\eta\log T + c_{33}\eta^{-6/5})}\right)^{2\eta-11/20}\left(\frac{|z_h|}{2}\right)^{m+c_{33}\eta^{-6/5}}$$

$$> \frac{1}{\sqrt{\log T}}\left(\frac{1}{\log T}\right)^{2\eta-11/20}\left(\frac{|z_h|}{2}\right)^{m+c_{33}\eta^{-6/5}}$$

$$> e^{-\sqrt{\log T}}2^{-2m}|z_h|^{m+c_{33}\eta^{-6/5}} > T^{-4\eta}|z_h|^{m+c_{33}\eta^{-6/5}}$$

and analogous negative upper bound for $Z(\nu_2)$. Hence from (6.6) we get

$$S_1 \stackrel{\text{def}}{=} \sum_p \varepsilon_k(p, l_2, l_1)\log p\, e^{-\frac{(\log p-(r_0b_0+\nu_1b_1)/2)^2}{r_0+\nu_1}} \tag{9.6}$$

$$> T^{-4\eta}\left|e^{(\varrho_0^2+2b_1\varrho_0)/4}\right|^{\nu_1}|z_h|^{m+c_{33}\eta^{-6/5}} - c_{45}\, e^{0,17(r_0b_0+\nu_1b_1)+(r_0+\nu_1)/4}.$$

From the definition of z_h, (9.5), (5.2) and (8.7), we get

$$\left|e^{(\varrho_0^2+2b_1\varrho_0)/4}\right|^{\nu_1}|z_h|^{m+c_{33}\eta^{-6/5}}$$

$$= e^{\frac{1}{4}\left(\frac{1}{4}-t(1)^2+b_1\right)(m+c_{33}\eta^{-6/5})}\, e^{\frac{1}{4}\left(\frac{1}{4}-t(0)^2+b_1\right)(\nu_1-m-c_{33}\eta^{-6/5})}$$

$$> e^{\frac{1}{4}\left(-k^{10}+\frac{1}{\eta}-\frac{1}{\sqrt{\eta}}\right)m-\frac{1}{4}\left(\frac{1}{4}+\frac{1}{\eta}\right)c_{33}\eta^{-6/5}}$$

and — using (1.3), (1.5), (5.4) and (1.6) —

$$> T^{\frac{1}{2}-\sqrt{\eta}}e^{-1/\eta^3} > T^{\frac{1}{2}-2\sqrt{\eta}}. \tag{9.7}$$

Further we get from (4.1), Lemma I, (9.5), (5.2), (1.3), (1.5) and (1.6) for $k > c_{51}$

$$e^{0,17(r_0b_0+\nu_1b_1)+(r_0+\nu_1)/4} < e^{0,18(2\eta\log T+c_{33}\eta^{-6/5})\frac{1}{\eta}} < T^{0,36+\eta} \tag{9.8}$$

and hence, if c_3 in (1.6) is sufficiently large, from (9.7) and (9.8)

$$S_1 > T^{\frac{1}{2}-4\sqrt{\eta}} \tag{9.9}$$

and analogously

$$S_2 \stackrel{\text{def}}{=} \sum_p \varepsilon_k(p, l_2, l_1)\log p\, e^{-\frac{(\log p-(r_0b_0+\nu_2b_1)/2)^2}{r_0+\nu_2}} < -T^{\frac{1}{2}-4\sqrt{\eta}}.$$

Further putting

$$(r_0b_0+\nu_1b_1)/2 = \log x_1$$

we have from (9.5) and (5.2)

$$\log x_1 > \frac{\nu_1 b_1}{2} > \eta\log T\left(\frac{1}{\eta}-\frac{1}{\sqrt{\eta}}\right) = (1-\sqrt{\eta})\log T$$

and from (4.1) and (1.6)

$$\log x_1 < \left\{2\log P_l + \frac{1}{\eta}(2\eta\log T + c_{33}\eta^{-6/5})\right\}\frac{1}{2}$$

$$< \log T + c_{52}(\log k + \eta^{-5/2}) < \log T + \log^{3/4} T$$

and analogously for x_2. Finally for ν_1 and ν_2 (9.5) and (1.6) give

$$2\eta\log T \leqslant \nu_1, \nu_2 \leqslant 2\eta\log T + \sqrt{\log T}$$

which completes the proof of Theorem I.

References

S. Knapowski and P. Turán

[1] *Comparative prime-number theory V*, Acta Math. Acad. Sci. Hung. 14 (1963), pp. 43 - 64.

[2] *Further developments in the comparative prime-number theory II*, Acta Arith. 10 (1964), pp. 293 - 313.

[3] *Comparative prime-number theory III*, Acta Math. Acad. Sci. Hung. 13 (1962), pp. 343 - 364.

C. L. Siegel

[1] *On the zeros of Dirichlet L-functions*, Ann. of Math. 46 (1945), pp. 409 - 422.

P. Turán

[1] *Eine neue Methode in der Analysis und deren Anwendungen*, Akad. Kiadó, Budapest, 1953. A completely rewritten English edition in the Interscience Tracts Series is under preparation.

Reçu par la Rédaction le 20. 7. 1964