Further developments in the comparative prime-number theory III

by
S. Knapowski (Poznań) and P. Turán (Budapest)

1. As well-known, Chebyshev (see Chebyshev [1]) asserted without
a proof that (p standing for primes)

\[ \lim_{x \to \infty} \sum_{p \leq x} (-1)^{\mu(p)} e^{-\sigma p} = -\infty, \]

i.e. "there are more primes \( \equiv 3 \pmod{4} \) than \( \equiv 1 \pmod{4} \) in Abel's sense". This
is undecided until now; but as well-known (see Hardy-Littlewood [1],
Landau [1], [2]) it is equivalent to the fact that (with \( s = \sigma + it \))

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s} \neq 0, \quad \sigma > \frac{1}{2}. \]

The same could have been proved for the sum

\[ \sum_{n \equiv 0 \pmod{k}} (-1)^{\mu(n)} \log p \cdot e^{-\log n} \]

and analogously for

\[ \sum_{p \equiv 0 \pmod{k}} \log p \cdot e^{-\log p} - \sum_{p \equiv 0 \pmod{k}} \log p \cdot e^{-\log p}. \]

By these are essentially all moduli \( k \) with \( \varphi(k) = 2 \) settled. As to the next
difficult question \( \varphi(k) = 4 \), the simplest is the case \( k = 8 \). It turned out
(see Knapowski-Turán [1]) that for the functions

\[ \sum_{p \equiv 0 \pmod{k}} \log p \cdot e^{-\log p} - \sum_{p \equiv 0 \pmod{k}} \log p \cdot e^{-\log p}, \]

we have an analogous situation as before; but as a new phenomenon,
we proved that for \( 0 < \delta < e \), for each \( l_1 \neq l_2 \) among 3, 5, 7 we have

\[ \max_{s > \log l_2 / \log l_1} \left\{ \sum_{p \equiv 0 \pmod{k}} \log p \cdot e^{-\log p} - \sum_{p \equiv 0 \pmod{k}} \log p \cdot e^{-\log p} \right\} > \frac{1}{Y \delta} e^{-\log l_1 \log l_2 / \log l_1} \]
and hence also

\[
\min_{e^{-1} \leq 0 < e \leq 1} \left\{ \sum_{p \leq x} \log p \cdot e^{-\beta p} \right\} \leq -\frac{1}{\sqrt{\log x}} e^{-\alpha_1 \log(\log x) \log(3/\log x)}. \tag{1.6}\]

These results would suggest for the general \( k \) that if \( l_1 \) resp. \( l_2 \) are quadratic residue resp. non-residue \( \mod k \) (i.e. \( l_1 \) and \( l_2 \) are of opposite quadratic character) then for

\[
\sum_{p \leq x} \log p \cdot e^{-\beta p} \leq \sum_{p \leq x} \log p \cdot e^{-\beta p}
\]

we have a situation analogous to that of (1.1) or (1.3) or (1.4), whereas if \( l_1 \) and \( l_2 \) have the same quadratic character \( \mod k \), then for the function (1.7) we have a situation analogous to (1.3) and (1.6). By other words if \( l_1 \) and \( l_2 \) have opposite quadratic character \( \mod k \) then "definitive preponderance in Abel's sense" holds if and only if the generalized Riemann-conjecture holds for all \( L(s, \chi) \)-functions \( \mod k \), whereas in the case when \( l_1 \) and \( l_2 \) are of the same quadratic character, there is no definitive preponderance even in Abel's sense. A closer analysis however revealed (see Knapsowski-Turan [2]) that owing to the "small" zeros of the \( L \)-functions a proof of any of these assertions for large \( k \)'s would be difficult in particular the first. In the same paper we made the observation that if we replace the Abel-means

\[
\sum_{p \leq x} \log p \cdot e^{-\beta p} \leq \sum_{p \leq x} \log p \cdot e^{-\beta p}
\]

by

\[
\sum_{p \leq x} \log p \cdot e^{-1/2 \log p} \leq \sum_{p \leq x} \log p \cdot e^{-1/2 \log p}
\]

or with the notation

\[
e_k(a, l_1, l_2) = \begin{cases} +1 & \text{for } a \equiv l_1(k), \\ -1 & \text{for } a \equiv l_2(k), \\ 0 & \text{otherwise} \end{cases}
\]

by

\[
F_k(x, l_1, l_2) = \sum_{\chi \mod k} \varphi_k(p, l_1, l_2) e^{-\frac{1}{2} \log p}
\]

with suitable \( \tau = \tau(\sigma) \), the situation changes to a certain extent. In particular it is so for "good" \( k \)-values, i.e. those for which there is an \( E = E(k) \) such that

\[
\prod_{\chi \mod k} L(s, \chi) \neq 0
\]

for

\[
\sigma > \frac{1}{2}, \quad |\tau| \leq E(k) \quad (0 < E(k) \leq \delta)
\]

(Hasegawa-condition). For such \( k \)-values (whose number is probably infinite) we showed at least that for \( l_1 = 1 \) and \( l_2 = \text{quadratic non-residue} \mod k \), the relation

\[
(1.11) \quad \lim_{x \rightarrow \infty} F_k(x, 1, 1, 1) = -\infty
\]

for all \( \varepsilon > r = r(x) \leq \log x \) holds if and only if the generalized Riemann-conjecture holds for \( k \).

2. In the present note we shall deal with the case

\[
(2.1) \quad l_1 = 1, \quad l_2 = l \text{ - quadratic residue } \mod k.
\]

For this case we shall prove in correspondence with (1.5) and (1.6) the

\[\text{THEOREM I. For "good" } k \text{'s in the case (2.1) and for}\]

\[
T > \max(e, e^\delta \log T, e^\delta e^{\log(T^{\frac{1}{2}})} \log(T))
\]

there exist \( \tau_1, \tau_2 \) in the interval

\[
(T^{-\log(T^{\frac{1}{2}})} \log(T), T^{\log(T^{\frac{1}{2}})})\]

such that for suitable

\[
(2\log T)^{1/2} \leq \tau_1 \leq (2\log T)^{1/2} + (2\log T)^{1/2}
\]

the inequalities

\[
\sum_p \varphi_k(p, l_1, 1) \log p \cdot e^{-\frac{1}{2} \log p} \geq \sqrt{T} e^{-\log(T^{\frac{1}{2}})}
\]

\[
\sum_p \varphi_k(p, l_1, 1) \log p \cdot e^{-\frac{1}{2} \log p} \leq -\sqrt{T} e^{-\log(T^{\frac{1}{2}})}
\]

hold.

This is a special case of

\[\text{THEOREM II. In the case (2.1) for "good" } k \text{'s if } \varphi_k = \beta_4 + t \gamma_k \text{ is a zero of an } L(s, \chi) \text{-function } \mod k \text{ with }\]

\[
\beta_4 \geq \frac{1}{2}, \quad \gamma_k > 0, \quad \chi(1) = 1,
\]

there exist for

\[
T > \max(e, e^\delta, e^\delta \log(T)^{1/2}, e^{\delta \log(T)})
\]

\( \tau_1, \tau_2 \)-numbers in the interval

\[
(T^{-\log(T^{\frac{1}{2}})} \log(T), T^{\log(T^{\frac{1}{2}})})\]

such that the inequalities (2.5) hold with \( \sqrt{T} \) replaced by \( T^\delta \).

However we shall confine ourselves to the proof of Theorem I; that of Theorem II follows mutatis mutandis.
As in paper II of this series we can conclude directly as to the discrepancy of primes \( \equiv 1 (k) \) and \( \equiv 1 (k) \) if \( l \) is a quadratic residue mod \( k \). So we assert the

**Theorem III.** For "good" \( k \)'s in the case (2.1) for \( T \)'s satisfying (2.2) there are \( U \)-numbers with

\[
Te^{-\log T \varepsilon} \leq U_1 < U_2 < Te^{\log T \varepsilon},
\]

resp.

\[
Te^{-\log T \varepsilon} \leq U_3 < U_4 < Te^{\log T \varepsilon},
\]

so that

\[
\sum_{U_3 \leq \chi \leq U_4} \epsilon_2(p, 0, 1, l) > \sqrt{Te^{-\log T \varepsilon}}
\]

and

\[
\sum_{U_1 \leq \chi \leq U_2} \epsilon_2(p, 0, 1, l) < - \sqrt{Te^{-\log T \varepsilon}}.
\]

Since the proof runs exactly like that in our paper II, we omit the details.

5. For the proof we shall need some lemmata.

**Lemma I.** Let for a positive \( m \) and \( n \leq N \) the \( \xi \)'s with

\[
1 = |\xi_1| > |\xi_2| > \cdots > |\xi_k| > \cdots > |\xi_n|
\]

are such that with a \( 0 < x \leq \pi/2 \)

\[
x \leq |\text{arc} \xi_j| \leq \pi \quad (j = 1, \ldots, n);
\]

further, let \( k \) resp. \( h \) be defined by

\[
|\xi_1| > \frac{4N}{m + N(3 + \pi/x)}
\]

resp. by

\[
\begin{cases}
|\xi_2| < |\xi_1| - \frac{2N}{m + N(3 + \pi/x)}, & \text{if there is such an } h_1; \\
h_1 = n & \text{otherwise}
\end{cases}
\]

and finally

\[
B \overset{\text{def}}{=} \min_{\xi_0 \in (x, \chi)} \text{Re} \left( \sum_{\xi \in \chi} b_{\xi} \right).
\]

Then there are integer \( v_1 \) and \( v_2 \) with

\[
m \leq v_1 < v_2 < m + N(3 + \pi/x)
\]

such that

\[
\text{Re} \sum_{\xi \in \chi} b_{\xi} \xi^t \geq \frac{B}{2N + 1} \left( \frac{N}{24(m + N(3 + \pi/x))} \right)^{1/2} \left( \frac{|\xi_1|}{2} \right)^{1/2} \left( \frac{N}{24(m + N(3 + \pi/x))} \right)^{1/2}.
\]

and

\[
\text{Re} \sum_{\xi \in \chi} b_{\xi} \xi^t \geq - \frac{B}{2N + 1} \left( \frac{N}{24(m + N(3 + \pi/x))} \right)^{1/2} \left( \frac{|\xi_2|}{2} \right)^{1/2} \left( \frac{N}{24(m + N(3 + \pi/x))} \right)^{1/2}.
\]

The proof of this lemma one can find in Knapowski-Turán [3] as Theorem 4.1.

**Lemma II.** If \( a_1, a_2, \ldots, \beta_1, \beta_2, \ldots \) are real with

\[
|a_i| > U \quad (> 0),
\]

further

\[
\Delta > 1 / U
\]

and with a \( \gamma > 1 \)

\[
\sum_{i} \frac{1}{1 + |a_i|^\gamma} < \Gamma \quad (< \infty),
\]

then every real interval of length \( \Delta \) contains a \( \xi \)-value such that for all \( \nu \)-indices the inequality

\[
\{a_\nu + \beta_\nu \} \geq \frac{1}{24 \Gamma^{1/2} (1 + |a_\nu|^\gamma)}
\]

holds ((\( a \) denoting the distance of \( x \) from the next integer),

For the proof of this lemma, see Knapowski-Turán [4].

**Lemma III.** For any given \( k \) modulus there exists a broken line \( W \) in the vertical strip \( 0 \leq \nu \leq 1 \) symmetrical to the real axis, consisting alternately of vertical resp. horizontal segments, each horizontal strip of width 1 containing at most one of the horizontal segments and on which for all \( \chi \)'s mod \( k \) the inequalities

\[
\left| \frac{L'}{L} (s, \chi) \right| \leq c_1 \log^2 k (2 + |l|),
\]

\[
\left| \frac{L'}{L} (2s, \chi) \right| \leq c_1 \log^2 k (2 + |l|)
\]

hold.

The proof of this lemma is contained mutatis mutandis in the book of the second of us (see Turán [1]).

4. Now we can turn to the proof of our theorem. If \( l \) (with \( (l, k) = 1 \)) is a quadratic residue mod \( k \), then the solutions of \( x^2 = l (k) \) form obviously a coset according to the subgroup formed by the solutions of \( x^2 = 1 (k) \).
in the multiplicative group of reduced residue-classes mod \( k \). Let the solutions of \( x^2 \equiv l \pmod{k} \) resp. \( x^2 \equiv 1 \pmod{k} \) be

\[
e_1, e_2, \ldots, e_s \quad \text{resp.} \quad \beta_1, \beta_2, \ldots, \beta_t,
\]

Then

\[
-\frac{1}{\varphi(k)} \sum_x \frac{L'}{L}(s, x) = \sum_{p \equiv l(\mod{k})} \log p \frac{p^s}{p^2} + \sum_{p \equiv l(\mod{k})} \log p \frac{p^{-2s}}{p^s} + f_1(s) = \sum_{p \equiv l(\mod{k})} \log p \frac{p^s}{p^2} + \sum_{p \equiv l(\mod{k})} \log p \frac{p^{-2s}}{p^s} + f_1(s) = \sum_{p \equiv l(\mod{k})} \log p \frac{p^s}{p^2} = \sum_{p \equiv l(\mod{k})} \frac{1}{\varphi(k)} \sum_x \chi(a) \frac{L'}{L}(2s, x) + f_1(s),
\]

where generally \( f_1(s) \) stand for functions regular for \( \sigma > 0.34 \) and satisfying here the inequality

\[
f_1(s) \leq c_1.
\]

c, c', c_1, c_2, c_3, c_4, c_5 denote positive numerical constants. (4.2) gives the identity

\[
\frac{1}{\varphi(k)} \sum_x \frac{L'}{L}(s, x) = \frac{1}{\varphi(k)} \sum_x \frac{L'(a)}{L'(a)} (2s, x) \quad \text{def} \quad \Phi_x(s, l) = \sum_p \log p \frac{p^s}{p^2} + f_2(s)
\]

for \( \sigma > 0.34 \). Now with a \( T \) in (2.2) let

\[
D \overset{\text{def}}{=} (3 \log T)^{1/3}
\]

5. Next we consider all \( \phi = a_\phi + i b_\phi \) zeros of all \( L(s, \chi) \) functions mod \( k \) satisfying

\[
|\chi| \leq 2y\overline{D}
\]

and apply Lemma II with \( \gamma = \frac{11}{10} U = \frac{1}{6\pi} \sum L(k) \) to the numbers

\[
a_\phi = \frac{\psi}{4\pi} \quad \text{and} \quad b_\phi = \frac{t}{8\pi},
\]

\[
\beta = \frac{1}{8\pi} \text{Im}(\phi) \quad \text{and} \quad \frac{1}{32\pi} \text{Im}(\phi),
\]

Then one can choose evidently \(^{(1)}\)

\[
V = c_4 \log k
\]

and thus Lemma II assures the existence of a \( b_\phi \) in

\[
\left( 3 \leq \frac{D}{2} \leq \right) D - \frac{10\pi}{2\pi} b_\phi \leq b_\phi \leq D
\]

so that for all \( \phi \)’s in \( (5.1) \)

\[
\frac{1}{2\pi} b_\phi + \frac{1}{2\pi} \text{Im}(\phi) \geq \frac{c_1}{2\pi} b_\phi + \frac{1}{2\pi} \text{Im}(\phi) \geq \frac{c_1}{2\pi} b_\phi + \frac{1}{2\pi} \text{Im}(\phi)
\]

i.e.

\[
\frac{c_1}{2\pi} b_\phi + \frac{1}{2\pi} \text{Im}(\phi) \leq \frac{1}{2\pi} \text{Im}(\phi) \leq \frac{1}{2\pi} \text{Im}(\phi)
\]

Since from (2.3) and (4.3) we have

\[
\text{Im}(\phi) \leq (\log(T))^{1/3} D
\]

we get the inequalities

\[
\frac{c_1}{2\pi} b_\phi + \frac{1}{2\pi} \text{Im}(\phi) \leq \frac{1}{2\pi} \text{Im}(\phi) \leq \frac{1}{2\pi} \text{Im}(\phi)
\]

for all \( \phi \)’s in \( (5.1) \).)

6. Fixing \( h_\phi \) that way, let \( r \) be an integer to be determined later so that

\[
D^2 \leq r \leq D^2 + D^{10}
\]

and consider the integral

\[
L(r) = \int_0^{r} s z(s, l) ds.
\]

\(^{(1)}\) Here we used the fact that the number of zeros of any \( L(s, \chi) \) in the half-strip \( \lambda < t < 1-\lambda, \sigma > 0 \) is at most \( c_4 \log(2 + |\lambda|) \).
Using the integral-formula (see e.g. Knappowsk-Turán [2])

$$\frac{1}{2\pi i} \int_{L} e^{(\sigma+ib)t-\frac{1}{2}t^{2}} dt = \frac{1}{V^{\nu}} e^{1/2} \left(\sigma+ib+\frac{1}{2}\right)^{\nu}$$

we get from (4.4) and (6.2)

$$I_{\nu}(r) = \frac{r^{\nu}}{V^{\nu}} \sum_{p} \epsilon_{p}(p, l, 1) e^{-\frac{1}{2} y \log_{p}(m_{l})} + \frac{1}{2\pi i} \int_{c_{0}+i\infty}^{c_{0}+i\infty} f_{\lambda}(s) ds.$$

Using (4.3) we get for the absolute-value of this last integral—shifting it to the vertical line $\sigma = 0.54$—the upper bound

$$c_{2} \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\sigma y \log_{p}(m_{l})} d\sigma < c_{2} e^{\frac{1}{2} \log_{p}(m_{l})},$$

i.e. owing to (4.5), (5.2) and (6.1)

$$< e_{1} e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})} e^{\nu} < \frac{1}{2\pi} e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})},$$

i.e. if only $c$ in (2.2) is sufficiently large. On the other hand, inserting in (6.2) the left-side for $\Phi_{\nu}(\sigma, l)$ and shifting the line of integration to $W$ we get

$$I_{\nu}(r) = \frac{1}{q(k)} \sum_{l} (1 - \frac{1}{2} \log_{q}(l)) \sum_{\nu} e^{\nu \log_{p}(m_{l})} + \frac{1}{2\pi i} \int_{c_{0}+i\infty}^{c_{0}+i\infty} f_{\lambda}(s) ds,$$

where $\Sigma'$ resp. $\Sigma''$ means that the respective summation must be extended only to those $\phi$'s for which $\phi$ resp. $g$ is right from $W$. For the absolute value of the last integral in (6.5) Lemma III gives the upper bound

$$c_{2} k^{2} \int_{-\infty}^{\infty} e^{\sigma y \log_{p}(m_{l})} d\sigma < c_{2} k^{2} \log^{2} k e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})},$$

which in turn is owing to (2.2), (5.2) and (6.1)

$$< e_{1} e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})} e^{\nu} < \frac{1}{2\pi} e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})},$$

(6.5), (6.6), (6.4) and (6.3) give, taking real parts the inequality

$$\left| \sum_{l} e_{l}(p, l, 1) \log_{p}(m_{l}) e^{\frac{1}{2} \log_{p}(m_{l})^{2}} \right| < 1 \frac{r^{2}}{2\pi} \Re \left\{ \sum_{x} \frac{1}{q(k)} \sum_{\nu} e^{\nu \log_{p}(m_{l})} \sum_{x} \frac{1}{q(k)} \sum_{\nu} e^{\nu \log_{p}(m_{l})} \right\} < c_{2} k^{2}.$$

7. Now we estimate (trivially) the contribution of $\phi$'s satisfying

$$\left| \epsilon_{l} \right| > 21 \tilde{D}.$$

Using the footnote on p. 121 this contribution is absolutely

$$\left| \epsilon_{l} \right| e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})} \left| \epsilon_{l} \right| e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})} \left| \epsilon_{l} \right| e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})}$$

$$< c_{1} e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})} \left| \epsilon_{l} \right| e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})} \left| \epsilon_{l} \right| e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})}$$

$$< c_{12} e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})} \left| \epsilon_{l} \right| e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})} \left| \epsilon_{l} \right| e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})}$$

Let $G$ be the domain right from $W$ satisfying $|\epsilon_{l}| < 21 \tilde{D}$ and

$$\max_{\epsilon_{l}(p, l, 1)} \left| \epsilon_{l} \right| e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})} \left| \epsilon_{l} \right| e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})} \left| \epsilon_{l} \right| e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})}$$

$$\left| \epsilon_{l} \right| e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})} \left| \epsilon_{l} \right| e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})} \left| \epsilon_{l} \right| e^{\frac{r^{2}}{2} + \epsilon_{B} \log_{p}(m_{l})}$$

Hence from (6.7), (7.2) and (7.3) we get the inequality

$$\left| \sum_{l} e_{l}(p, l, 1) \log_{p}(m_{l}) e^{\frac{1}{2} \log_{p}(m_{l})^{2}} \right| < 1 \frac{r^{2}}{2\pi} \Re \left\{ \sum_{x} \frac{1}{q(k)} \sum_{\nu} e^{\nu \log_{p}(m_{l})} \sum_{x} \frac{1}{q(k)} \sum_{\nu} e^{\nu \log_{p}(m_{l})} \right\} + \sum_{x} \sum_{\nu} e^{\nu \log_{p}(m_{l})} \sum_{x} \frac{1}{q(k)} \sum_{\nu} e^{\nu \log_{p}(m_{l})} \right\} < c_{3} k^{2}.$$

8. Until now the integer $r$ was subjected only to the restriction (6.1); now we shall determine it using our lemmata. Let us denote the expression

$$\Re \left\{ \sum_{x} e^{\nu \log_{p}(m_{l})} \right\} < c_{3} k^{2}.$$
in (7.4) by \( Z(r) \); we shall try to use Lemma I with

\[ e^{i\frac{3}{2} + 3\phi_0 - \frac{1}{2} \log (3 + \phi_0)} \]

resp.

\[ e^{i\frac{3}{2} + 3\phi_0 - \frac{1}{2} \log (3 + \phi_0)} \]

as \( \alpha \)-vectors, calling them \( \alpha_i \)'s of first (resp. of second) category. Correspondingly we shall choose as \( b_i \)-coefficients the numbers

\[ \frac{1 - \chi(l)}{\varphi(k)} \]

resp.

\[ \frac{\chi(a_i) - \chi(b_i)}{2\varphi(k)} \]

and call them \( b_i \)'s of first resp. of second category. First we have to verify

\[ \max_i |x_i| = 1. \]  

For the \( z_i \)'s of first category this is evident from the definition of \( g_1 \). To verify it also for the \( z_i \)'s of second category we remark first that owing to a theorem of Siegel (see Siegel [1]) there is a \( \xi \) such that each \( L(s, \chi) \) has a zero in the domain

\[ \sigma \geq \frac{1}{2}, \quad |t| \leq \xi; \]

this holds especially for the \( L \)-functions belonging to \( \chi \)'s with \( \chi(l) \neq 1 \). Denoting by \( \xi_i = \xi_1 + \xi_2 \) any of such zeros we have (\( M \) in (7.3))

\[ M \geq \frac{1}{2} \left| \frac{\xi_1 + \xi_2}{\xi_1 + \xi_2} \right| = \frac{1}{2} \left| \frac{\xi_1^2 - \xi_1^2}{\xi_1^2 - \xi_2^2} \right| \]

In order to show that for the \( z_i \)'s of second category even the sharper inequality

\[ |z_i| \leq e^{-i} \]

holds, it suffices owing to (3.2) to show

\[ \frac{\xi_1^2 - \xi_1^2}{\xi_1^2 + \xi_2^2} \leq \frac{1}{2} \xi_1^2 \xi_2^2 \]

or a fortiori

\[ 1 + b_i \sigma_i < b_i - \xi_i - 8. \]

But owing to the classical theorem we have

\[ c_8 < \max \left\{ 1 - \frac{\xi_1}{\log (2 + |1|)}, 1 - \frac{\xi_1}{k} \right\}, \]

i.e. in \( G \), using also (2.2),

\[ c_8 < \max \left\{ 1 - \frac{\xi_1}{\log (2 + |1|)}, 1 - \frac{\xi_1}{k} \right\} < 1 - \frac{c_1}{\log D}; \]

hence if \( c \) in (2.2) is sufficiently large, using (5.2) we get

\[ c_4 \frac{b_0}{\log D} > \frac{c_4}{2}, \quad \frac{D}{\log D} > 0 + c_4, \]

i.e.

\[ \frac{1}{4} + b_0 \sigma_i \leq \frac{1}{4} + b_0 - c_4 \frac{b_0}{\log D} < b_0 - c_4 - 8 \]

and (8.9) — whence (8.8) and (8.5) — holds indeed.

9. The number of terms in \( Z(r) \) is owing to the footnote on p. 121

\[ \leq c_4 \varphi(k)! \left| \frac{\log k}{1} \right| \leq 1 \left| \frac{\log D}{2} \right| ! \leq \frac{N}{X}. \]

What will play the role of \( \alpha \)? From (5.4) and \( |t| \leq (21 \bar{D}) \) we could choose as \( x \)

\[ \frac{c_8}{\frac{1}{1} + \left| \frac{21 \bar{D}}{1} \right| \log D} > c_8 \bar{D}^{-1 + 2 \log D}. \]

Hence

(9.2)

\[ x = \bar{D}^{-2}. \]

For \( m \) we choose

(9.3)

\[ m = \bar{D}^2. \]

As to \( \lambda \), we choose

(9.4)

\[ \lambda = 1; \]

then (3.3) is obviously satisfied if \( c \) in (2.2) is sufficiently large. As to \( z_1 \), we shall choose it so that no \( b_i \) of second category should contribute to \( B \). This choice is fulfilled if \( z_1 \) is the absolutely greatest among the \( z_i \)'s of second category. Then we have owing to (8.8)

\[ |z_1| \leq e^{-i} < \frac{6 - \frac{1}{i}}{i} < \frac{1 - 21 \bar{D}^{1 + 2 \log D}}{D} < |z_1| - \frac{2N}{m + N (3 + \pi |\alpha|)}, \]
i.e. (3.4) is fulfilled too. Now in $B$ we have only $b_j$'s with nonnegative real part, i.e.

\begin{equation}
B \gg \min_{p \leq 1} \text{Re} \left( \frac{1 - \gamma(p)}{\varphi(p)} \right) \geq \frac{8}{k_b} > \frac{8}{\log^2 D}.
\end{equation}

With the above choices the interval $[m, m+N(3+\pi/\kappa)]$ is certainly contained in the interval (6.1), i.e. $r$ can be chosen according to Lemma I. Hence $r = r_1$ and $r_2$ can be determined so that

\begin{equation}
(2 \log T)^{\beta_2} \leq r_1, \quad r_2 \leq (2 \log T)^{\beta_3} + (2 \log T)^{\beta_3}.
\end{equation}

and

\begin{equation}
Z(r_1) > \left( \frac{\sqrt{D} \log D}{24(D^2 + 4D^2 \log^2 D)} \right)^{2 \beta_3} \frac{8}{\log^2 D},
\end{equation}

\begin{equation}
\frac{1}{3 \sqrt{D} \log^2 D} \left( \frac{1}{2} \right)^{2 \beta_3} \sim e^{-2 \beta_3} = e^{-2 \log T^{\beta_3}}
\end{equation}

and analogously

\begin{equation}
Z(r_2) < -e^{-2 \log T^{\beta_3}}.
\end{equation}

10. To complete the proof we have to give a lower bound to

\begin{equation}
M_f = \gamma \pi \left( e^{\gamma} \delta_{\gamma} \right)^{1/2}, \quad j = 1, 2.
\end{equation}

Owing to the maximal-definition of $q_2$ and that of $q_3$ we get for $j = 1, 2$

\begin{equation}
M_f \geq \frac{1}{2} \pi \log T \left( e^{\gamma} \delta_{\gamma} \right)^{1/2}
\end{equation}

and the second factor, using (8.6), (8.6), (5.2), (4.5) and (2.3),

\begin{equation}
= \frac{\gamma}{2} \left( e^{\gamma} \delta_{\gamma} \right)^{1/2} \geq e^{-2 \log T \left( e^{\gamma} \delta_{\gamma} \right)^{1/2}} \geq e^{-2 \log T \left( e^{\gamma} \delta_{\gamma} \right)^{1/2}} \geq e^{-2 \pi \log T \left( e^{\gamma} \delta_{\gamma} \right)^{1/2}} \geq \sqrt{T} e^{-2 \pi \log T \left( e^{\gamma} \delta_{\gamma} \right)^{1/2}};
\end{equation}

hence

\begin{equation}
M_f \geq \sqrt{T} e^{-2 \pi \log T \left( e^{\gamma} \delta_{\gamma} \right)^{1/2}}.
\end{equation}

Putting in (7.4) for $j = 1, 2$

\begin{equation}
b_1 r = \frac{b_2 r_2}{2} = \log \sigma_1,
\end{equation}

the proof is finished. (3.3) presents no difficulties.