ON AN ASSERTION OF ČEBYŠEV

By

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To Gabor Szegö,
on his 70th birthday on Jan. 20, 1965

1. Čebyšev announced, as well-known, (see Čebyšev [1]) in 1853 without proof the following assertion.

If p runs over the odd primes then

$$\lim_{x\to+0} \sum_p (-1)^{(p-1)/2} e^{-px} = -\infty . \tag{1.1}$$

This assertion can be interpreted in the present terminology that there are more primes of the form $4k+3$ than of $4k+1$, in Abel-summation-sense. As well known the truth of falsity of this very remarkable assertion is not yet decided; the depth of it however was exhibited by Landau (see Landau [1], [2]) and Hardy-Littlewood (see Hardy-Littlewocd [1]) who proved it is equivalent with the non-vanishing of

$$L(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} \qquad (s = \sigma + it)$$

for $\sigma > \frac{1}{2}$. As it was noted by Hardy and Littlewood l.c. the same assertion holds for $\sum_p (-1)^{(p-1)/2} \log p\, e^{-px}$.

The same result could have been proved by the same reasoning for

$$\sum_{p>3} \varepsilon_p' \cdot \log p \cdot e^{-px} \text{ and } \sum_{p>3} \varepsilon_p e^{-px}$$

with

$$\varepsilon_p' = \begin{matrix} 1 \text{ for } p \equiv 1 \bmod 3 \\ -1 \text{ for } p \equiv -1 \bmod 3 \end{matrix}$$

resp. for

$$\sum_{p>6} \varepsilon_p'' \log p \cdot e^{-px} \text{ and } \sum_{p>6} \varepsilon_p'' e^{-px}$$

with

$$\varepsilon_p'' = \begin{array}{l} 1 \text{ for } p \equiv 1 \bmod 6 \\ -1 \text{ for } p \equiv -1 \bmod 6. \end{array}$$

These are all moduli k where all primes—with exception of finitely many—are distributed in *two* residue-classes mod k, i.e. $\phi(k) = 2$.

It is quite plausible to ask what is the situation with other moduli. The next case is the case of the k's with $\phi(k) = 4$; this is realized only for the moduli

(1.2)
$$\begin{array}{llll} k = 5 & \text{with the residue-classes} & 1{,}4 & \text{and } 2{,}3 \\ k = 8 & \text{with the residue-classes} & 1 & \text{and } 3{,}5{,}7 \\ k = 10 & \text{with the residue-classes} & 1{,}9 & \text{and } 3{,}7 \\ k = 12 & \text{with the residue-classes} & 1 & \text{and } 5{,}7{,}11. \end{array}$$

The residue-classes for each modulus were written in two sets; in the first one are the quadratic residues, in the other one the non-residues of the respective modulus. We shall call them R-set resp. N-set for the sake of brevity. In the cases with $\phi(k) = 2$ only a comparison of two residue-classes from *different* sets (one from R, the other from N) was possible. The principal novelty of these cases arises from the fact that now also comparison of residue-classes of the *same* set is necessary. Since in the cases $k = 8$ and $k = 12$ this occurs only within the N-set, within the moduli in (1.2) they constitute the simpler case and in this note we shall confine ourselves to this case. Moreover it will be enough to consider the case $k = 8$ since from $k = 12$ everything goes mutatis mutandis.

The application of the method of Hardy-Landau-Littlewood shows at once that for all l-values among 3,5,7 defining

$$\varepsilon_p^{(\,)} = \begin{array}{l} 1 \text{ for } p \equiv 1 \bmod 8 \\ -1 \text{ for } p \equiv l \bmod 8 \end{array}$$

the "Abelian preponderance-relations"

$$\lim_{x \to +0} \sum_{p} \varepsilon_p^{(l)} \log p \cdot e^{-px} = -\infty \tag{1.3}$$

$$\lim_{x \to +0} \sum_{p} \varepsilon_p^{(l)} e^{-px} = -\infty \tag{1.4}$$

hold if and only if the $L(s,\chi)$-functions belonging to mod 8 do not vanish for $\sigma > \frac{1}{2}$. For the new cases, when l_1 and l_2 are any two different ones from the corresponding N-set, i.e. among 3,5,7, the classical theorem of Landau would give wi hout any conjectures that the functions

$$g_l(x) \overset{def.}{=} \sum_{p} \varepsilon_p^{(l_1,l_2)} \log p \cdot e^{-px} \tag{1.5}$$

with

$$\varepsilon_p^{(l_1,l_2)} = \begin{matrix} 1 \text{ for } p \equiv l_1 \bmod 8 \\ -1 \text{ for } p \equiv l_2 \bmod 8 \end{matrix} \tag{1.6}$$

change sign infinitely often when $x \to +0$. Using other ideas we proved (see Knapowski-Turán [1]) that if c_1 (and later $c_2, \cdots$) is a suitable positive numerical constan., than (*) for $0 < \delta < c_1$ without any conjectures the inequality

$$\max_{\delta \leqq x \leqq \delta^{1/3}} \sum_{p} \varepsilon_p^{(l_1,l_2)} \log p \cdot e^{-px}$$

$$> \frac{1}{\sqrt{\delta}} \exp\left(-22 \frac{\log(1/\delta) \log_3(1/\delta)}{\log_2(1/\delta)} \right) \tag{1.7}$$

(and, what follows automatically by changing l_1 and l_2

(*) We remark that $\log_\nu (1/\delta)$ stands always for ν-times iterated logarithm.

$$\min_{\delta < x < \delta^{1/3}} \sum_p \varepsilon^{(l_1,l_2)} \log p \cdot e^{-px}$$

(1.8)
$$< -\frac{1}{\sqrt{\delta}} \exp\left(-22 \frac{\log(1/\delta)\log_3(1/\delta)}{\log_2(1/\delta)}\right)$$

holds. These imply of course that all functions

$$\sum_p \varepsilon_p^{(l_1,l_2)} \log p \cdot e^{-px}$$

change sign in every interval of the form $(\delta, \delta^{1/3})$ if only $0 < \delta < c_1$. However both methods fail to give the corresponding theorem for the "properly Čebyšev-functions"

(1.9)
$$F_{l_1,l_2}(x) \overset{def}{=} \sum_p \varepsilon_p^{(l_1,l_2)} e^{-px}.$$

In what follows we are going to give a first contribution to this difficult question.

2. We observed that the difficulties can be essentially reduced to theorems of Markoff's type. More exactly we mean the following type of problems. For a fixed positive integer n we are given a sequence of complex numbers

(2.1)
$$0 = \lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_n$$

and for

(2.2)
$$f(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$$

we define the "λ-derivative of $f(x)$" by

(2.3)
$$f_\lambda(x) = \sum_{\nu=0}^{n} \lambda_\nu a_\nu x^\nu .$$

Then the problems in question ask with prescribed $\gamma \geqq -1$ and $a < b$ what is

$$\max_{a \leqq x \leqq b} \operatorname{Re} x^{\gamma} f_{\lambda}(x) \tag{2.4}$$

if $f(x)$ is of form (2.2) and restricted by various conditions. We get back A. Markoff's classical theorem by choosing

$$\gamma = -1, \quad \lambda_{\nu} = \nu \quad (\nu = 1, 2, \cdots, n)$$

and restricting the polynomials $f(x)$ by

$$\max_{a \leqq x \leqq b} |f(x)| = 1 \text{ or } \operatorname*{osc}_{a \leqq x \leqq b} f(x) = 2. \tag{2.5}$$

In our intended application we need

$$(a,b) = (0,b), \gamma = 0 \text{ and } \lambda_{\nu} = \log(\nu + 1) \qquad (\nu = 1, 2, \cdots, n).$$

We realized that a good upper bound for $\max_{(0,b)} \operatorname{Re} x^{\gamma} f_{\lambda}(x)$ would do the purpose (though the *exact* solution would be in itself much more satisfactory) and conjectured that in the case (2.5) a power of $\log n$ will give an upper bound. Now it was G. Szegö who by the aid of the integral formula

$$\log \nu = \int_0^{\infty} \frac{e^{-t} - e^{-\nu t}}{t} \, dt \qquad \nu = 1, 2, \cdots \tag{2.6}$$

and the subsequent representation

$$f^*(x) \overset{def}{=} \sum_{\nu=0}^{n} \log(\nu + 1) a_{\nu} x^{\nu} = \int_0^{\infty} \frac{e^{-t}}{t} \{f(x) - f(xe^{-t})\} \, dt \tag{2.7}$$

proved the following theorem.

If for $0 \leqq x \leqq b$ the inequality

$$|f(x)| \leqq \mu_0$$

holds, so we have here the inequality (independently of b)

$$|f^*(x)| \leqq 4\mu_0 \log(en).$$

He will give his proofs for this and related theorems in a separate paper.

Actually we shall need his theorem in the following form: If with a $d \geqq 0$ we have for $y \geqq d\ (>0)$

$$\left|\sum_{\nu=0}^{n} b_{\nu+1} e^{-\nu y}\right| \leqq \mu_1 \tag{2.8}$$

then we have for the same y-values the inequality

$$\left|\sum_{\nu=1}^{n+1} b_\nu \log \nu \cdot e^{-\nu y}\right| \leqq 4\mu_1 \log (en). \tag{2.9}$$

3. Having this theorem one can proceed as follows. Let with our δ's

$$M \leqq \frac{1}{\sqrt{\delta}} \exp\left(-23 \frac{\log (1/\delta) \log_3 (1/\delta)}{\log_2 (1/\delta)}\right) \tag{3.1}$$

and suppose for some of our l_1, l_2's and $F_{l_1,l_2}(x)$-function from (1.9) the inequality

$$\max_{\delta \leqq x \leqq \delta^{1/3}} |F_{l_1,l_2}(x)| \leqq M \tag{3.2}$$

holds. Since for $x \geqq \delta^{1/3}$ we have evidently

$$\sum_p \varepsilon_p^{(l_1,l_2)} e^{-px} \leqq \sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}} < \frac{2}{\delta^{1/3}}$$

this gives

$$\max_{x \geqq \delta} |F_{l_1,l_2}(x)| \leqq M + 2\delta^{-1/3}. \tag{3.3}$$

Next we remark that for $x \geqq \delta$, $0 < \delta \leqq c_2$ we have

$$\sum_{p > (1/\delta)\log^2(1/\delta)} \varepsilon_p^{(l_1,l_2)} e^{|-px|} < \sum_{n > (1/\delta)\log^2(1/\delta)} e^{-nx} < e^{-(1/2)\log^2/(1/\delta)}$$

and thus from (3.3) for $0 < \delta < c_3$

$$\max_{x \geqq \delta} \left| \sum_{p \leqq (1/\delta)\log^2(1/\delta)} \varepsilon_p^{(l_1,l_2)} e^{-px} \right| \leqq M + 3\delta^{-1/3}.$$

Now we apply Szegö's Theorem (2.8)-(2.9) with

$$\sum_{\nu=0}^{n} b_{\nu+1} e^{-\nu y} = \sum_{p \leqq (1/\delta \log^2(1/\delta)} \varepsilon_p^{(l_1,l_2)} e^{-(p-1)y}$$

$$\mu_1 = 2M + 4\delta^{-1/3}$$

$$d = \delta \quad .$$

This gives for $x \geqq \delta$ the inequality

$$\left| \sum_{p \leqq (1/\delta)\log^2(1/\delta)} \varepsilon_p^{(l_1,l_2)} \log p \cdot e^{-py} \right| < 8(M + 2\delta^{-1/3}) \log \left\{ \frac{e}{\delta} \log^2 \frac{1}{\delta} \right\}$$

and hence for $\delta < c_4$ for $x \geqq \delta$

$$\left| \sum_{p} \varepsilon_p^{(l_1,l_2)} \log p\, e^{-py} \right| \leqq 8(M + 2\delta^{-1/3}) \log \left\{ \frac{e}{\delta} \log^2 \frac{1}{\delta} \right\}$$

$$+ \sum_{n > (1/\delta)\log^2(1/\delta)} \log n \cdot e^{-n\delta} < 17(M + \delta^{-1/3}) \log \frac{1}{\delta}.$$

But for $\delta < c_5$ this contradicts to either (1.7) or (1.8). Hence (3.1)–(3.2) lead to a contradiction and hence we got the following

Theorem I. *For any $l_1 \neq l_2$ among 3,5,7 for $0 < \delta < c_5$ we have the inequality*

$$\max_{\delta \leqq x \leqq \delta^{1/3}} \left| \sum_{p \equiv l_1 \bmod 8} e^{-px} - \sum_{p \equiv l_2 \bmod 8} e^{-px} \right| \geqq \frac{1}{\sqrt{\delta}} \exp\left(-23 \frac{\log(1/\delta) \log_3(1/\delta)}{\log_2(1/\delta)} \right).$$

To the more difficult problem of one-sided theorems we hope to return.

In the case when one of the l's is 1, a combination of the above reasoning with the theorem of Hardy-Littlewood-Landau leads to the following

Theorem II. *For $l \neq 1, k = 4$ or 8 and $0 < \delta < c_6$ we have the inequality*

$$\max_{\delta \leqq x \leqq \delta^{1/3}} \left| \sum_{p \equiv 1 \bmod k} e^{-px} - \sum_{p \equiv l \bmod k} e^{-px} \right| \geqq \frac{1}{\sqrt{\delta}} \exp\left(-23 \frac{\log(1/\delta) \log_3(1/\delta)}{\log_2(1/\delta)} \right).$$

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(Received August 13, 1963)