ACTA ARITHMETICA
X (1964)

Further developments in the comparative prime-number theory II

(A modification of Chebyshev's assertion)

by

S. KNAPOWSKI (Poznań) and P. TURÁN (Budapest)

1. Chebyshev's assertion in question (see Chebyshev [1]) states that

$$\lim_{x\to+\infty}\sum_{p>2}(-1)^{(p-1)/2}e^{-p/x} = -\infty \tag{1.1}$$

if p runs through all odd primes; in other words, it says, there are more primes of the form $4n+3$ than of $4n+1$, at least in the above "Abelian" sense. As it was shown by Hardy-Littlewood and Landau (see Hardy-Littlewood [1], Landau [1]) this holds if and only if the function

$$L(s,\chi_1) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}, \qquad s = \sigma+it,\ \sigma>0, \tag{1.2}$$

does not vanish for $\sigma > \frac{1}{2}$. As pointed out by them, the same holds for the relation

$$\lim_{x\to+\infty}\sum_{p>2}(-1)^{(p-1)/2}\log p\, e^{-p/x} = -\infty. \tag{1.3}$$

This aspect lends an additional interest to the comparative study of the distribution of primes in progressions (and in other forms) and suggests above all the necessity to extend (1.1) or (1.3) to general k's. In our paper [3] we discussed the case $k = 8$, the first beyond the Chebyshevian case $k = 4$. With the notation

$$\varepsilon_8(p,l_1,l_2) = \begin{cases} 1 & \text{if} \quad p\equiv l_1 \pmod 8, \\ -1 & \text{if} \quad p\equiv l_2 \pmod 8, \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

Hardy-Littlewood-Landau's argument gave (using also strongly some numerical data furnished by Dr. P. C. Haselgrove) that the relation

$$(1.5)\qquad \lim_{x\to+\infty}\sum_p \varepsilon_8(p,1,l)\log p\, e^{-p/x} = -\infty \qquad (l=3,5,7)$$

holds if and only if no $L(s,\chi)$ function belonging to mod 8 with $\chi\neq\chi_0$ vanishes for $\sigma>\frac{1}{2}$. If l_1 and l_2 are any two of 3, 5, 7 (= quadratic non-residues mod 8), we proved l.c. without any conjectures that if c_1 (and later $c_2, c_3, \ldots$) denote positive numerical constants, that for $0<\delta<c_1$

$$(1.6)\qquad \max_{\delta^{-1/3}\leqslant x\leqslant\delta^{-1}}\sum_p \varepsilon_8(p,l_1,l_2)\log p\, e^{-p/x} > \delta^{-1/2}e^{-22\frac{\log 1/\delta\log_3 1/\delta}{\log_2 1/\delta}}$$

(i.e. changing l_1 and l_2 also

$$\min_{\delta^{-1/3}\leqslant x\leqslant\delta^{-1}}\sum_p \varepsilon_8(p,l_1,l_2)\log p\, e^{-p/x} < -\delta^{-1/2}e^{-22\frac{\log 1/\delta\log_3 1/\delta}{\log_2 1/\delta}},$$

and thus there is a sign-change of the function $\sum_p \varepsilon_8(p,l_1,l_2)\log p\, e^{-p/x}$ in every interval of the form $[\delta^{-1/3},\delta^{-1}]$).

2. Let us analyse the situation for general k. Putting with $l_1\equiv l_2 \bmod k$

$$(2.1)\qquad f_{l_1,l_2}(w) \overset{\text{def}}{=} \frac{1}{\varphi(k)}\sum_\chi \big(\bar\chi(l_1)-\bar\chi(l_2)\big)\left\{\frac{L'}{L}(w,\chi)-\frac{L'}{L}(2w,\chi^2)\right\},$$

we start from the integral

$$(2.2)\qquad J=\frac{1}{2\pi i}\int_{(2)}\Gamma(w)x^w f_{l_1,l_2}(w)\,dw$$

(due essentially to Hardy-Littlewood). Since for $\mathrm{Re}\,w>1$ we have from (2.1)

$$(2.3)\qquad f_{l_1,l_2}(w) = \sum_{p\equiv l_2(k)}\frac{\log p}{p^w}-\sum_{p\equiv l_1(k)}\frac{\log p}{p^w}+$$

$$+\Bigg\{\sum_{\substack{p,\alpha\\ p^\alpha\equiv l_2(k)\\ \alpha\geqslant 3}}\frac{\log p}{p^{\alpha w}}-\sum_{\substack{p,\alpha\\ p^\alpha\equiv l_1(k)\\ \alpha\geqslant 3}}\frac{\log p}{p^w}+\sum_{\substack{p,\alpha\\ p^{2\alpha}\equiv l_1(k)\\ \alpha\geqslant 2}}\frac{\log p}{p^{2\alpha w}}-\sum_{\substack{p,\alpha\\ p^{2\alpha}\equiv l_2(k)\\ \alpha\geqslant 2}}\frac{\log p}{p^{2\alpha w}}\Bigg\}$$

$$\overset{\text{def}}{=}\sum_{p\equiv l_2(k)}\frac{\log p}{p^w}-\sum_{p\equiv l_1(k)}\frac{\log p}{p^w}+f^*_{l_1,l_2}(w),$$

where $f^*_{l_1,l_2}(w)$ is regular e.g. for $\mathrm{Re}\,w\geqslant\frac{2}{5}$ and satisfies here the inequality

$$(2.4)\qquad |f^*_{l_1,l_2}(w)|\leqslant c_2,$$

we get from (2.2) (adapting the notation (1.4) for general k-moduli)

$$J=\sum_p \varepsilon_k(p,l_2,l_1)\log p\, e^{-p/x}+\frac{1}{2\pi i}\int_{(2/5)}\Gamma(w)x^w f^*_{l_1,l_2}(w)\,dw$$
$$=\sum_p \varepsilon_k(p,l_2,l_1)\log p\, e^{-p/x}+O(x^{2/5})$$

(O meant uniformly in x and k). On the other hand, shifting the line of integration to the left, the "essential" part of J is

$$(2.5)\qquad \frac{1}{\varphi(k)}\sum_\chi\big(\bar\chi(l_2)-\bar\chi(l_1)\big)\sum_{\varrho(\chi)}\Gamma(\varrho)x^\varrho+\frac{\Gamma(\frac{1}{2})}{2\varphi(k)}\sqrt{x}\sum_\chi{}'\big(\bar\chi(l_1)-\bar\chi(l_2)\big)-$$
$$-\frac{1}{2\varphi(k)}\sum_\chi\big(\bar\chi(l_1)-\bar\chi(l_2)\big)\sum_{\varrho(\chi^2)}\Gamma(\varrho/2)x^{\varrho/2},$$

where dash means that the summation is to be extended to all real characters (with $\chi(l_1)\neq\chi(l_2)$) and summation over $\varrho(\chi)$ means that it is to be extended to all non-trivial zeros of $L(s,\chi)$ with a fixed χ. Let us consider the "most suspicious" case for preponderance, when l_1 is a quadratic residue, l_2 a non-residue mod k, and suppose the truth of the Riemann-Piltz conjecture. In this case the third term is unessential, the second term as we shall see in section 8 is

$$(2.6)\qquad \geqslant\frac{2^r}{2\varphi(k)}\sqrt{\pi x},$$

where r stands for the number of different odd prime-factors of k. As to the critical first sum in (2.5), trivial treatment gives only the upper bound

$$\frac{2\sqrt{x}}{\varphi(k)}\sum_\chi\sum_{\varrho(\chi)}|\Gamma(\varrho)|$$

which certainly supersedes the value in (2.6) if k is sufficiently large and nothing better can be accomplished at present (owing to the factor x^ϱ).

3. Beside this difficulty also another fact makes it desirable to find an appropriate modification of Chebyshev's problem. The relation (1.5) could be replaced by

$$(3.1)\qquad \sum_p \varepsilon_8(p,1,l)\log p\, e^{-p/x} < -c_3\sqrt{x}$$

if $x > c_4$ and the Riemann-Piltz conjecture is true for the L-functions mod 8 with $\chi \neq \chi_0$ and by an inequality of type (1.6) if it is false. Hence everything remains true if in the sum

$$\sum_p \varepsilon_8(p, l_1, l_2) \log p \, e^{-p/x}$$

we drop the terms with

(3.2) $$p < x^{1/2-\varepsilon}$$

(and trivially dropping those with

(3.3) $$p > 10x \log x).$$

In other words, putting

$$y = 10x \log x$$

such a preponderance-behaviour was exhibited for all sufficiently large y's for primes of the form $8n+l_1$ and $8n+l_2$ in the interval

(3.4) $$(y^{1/2-\varepsilon}, y)$$

To push the lower bound in (3.4) above $\sqrt{y}$, i.e. to strengthen the "accumulation", however desirable, seems hopeless at present, even in the Chebyshevian case $k = 4$.

4. The main result of this paper can be expressed shortly that replacing the factors $e^{-p/x}$ by

(4.1) $$e^{-\frac{1}{r}\log^2\frac{p}{x}}$$

with a suitable ("small") $r = r(x)$, one can come much closer to both desideratums. This holds in particular for the "good" k's, i.e. for those for which with an $E(k)$ no $L(s, \chi)$-function mod k vanishes for

(4.2) $$0 < \sigma < 1, \quad |t| \leqslant E(k)$$

(Haselgrove-property). This property has been verified in a number of cases, notably for all $k \leqslant 10$. The extension of them seems very desirable to us; particularly, a proof that Haselgrove-property holds for an infinity of k's would be of great significance. Out of our results the most complete are those comparing primes $\equiv 1$, resp. $\equiv l \bmod k$ when l is a quadratic non-residue mod k (which obviously comprises the Chebyshevian case). We formulate first

THEOREM I. *For any fixed "good" k and for all quadratic non-residues $l \bmod k$, $(l, k) = 1$, the relation*

$$\lim_{x\to+\infty} \sum_p \varepsilon_k(p, l, 1) \log p \, e^{-\frac{1}{r}\log^2\frac{p}{x}} = +\infty$$

for every $r = r(x)$ satisfying $a_1(k) \leqslant r \leqslant \log x$ is true if and only if none of the $L(s, \chi)$-functions mod k *with $\chi \neq \chi_0$ vanishes for $\sigma > \frac{1}{2}$.*

Slightly more generally we formulate

THEOREM II. *For any fixed "good" k and fixed quadratic non-residue $l \bmod k$, $(l, k) = 1$,*

$$\lim_{x\to+\infty} \sum_p \varepsilon_k(p, l, 1) \log p \, e^{-\frac{1}{r}\log^2\frac{p}{x}} = +\infty$$

for every $r = r(x)$ satisfying $a_1(k) \leqslant r \leqslant \log x$ is true if and only if none of the $L(s, \chi)$-functions mod k *with $\chi(l) \neq 1$ vanishes for $\sigma > \frac{1}{2}$.*

To deduce Theorem I from Theorem II we have only to remark that if for a character χ^*, for all non-residues l, $\chi^*(l) = 1$, then $\chi^* = \chi_0$. Namely if a is an arbitrary quadratic residue mod k, $(a, k) = 1$ and l is an arbitrary non-residue mod k with $(l, k) = 1$, then $al = l' =$ non-residue, i.e. $\chi^*(a) = \chi^*(l')\overline{\chi}(l) = 1$.

5. In turn, Theorem II will be a consequence of Theorem III and IV. Here we shall assume (which goes without loss of generality) that

(5.1) $$E(k) \leqslant \sqrt{\log k}/k.$$

THEOREM III. *If for a "good" k and a prescribed quadratic non-residue l no $L(s, \chi)$ with $\chi(l) \neq 1$ vanishes for $\sigma > \frac{1}{2}$, then for suitable c_4, c_5, c_6 and*

$$r_0 = c_4 \frac{\log k}{E(k)^2},$$

the inequality

$$\sum_p \varepsilon_k(p, l, 1) \log p \, e^{-\frac{1}{r}\log^2\frac{p}{x}} > c_5\sqrt{x}$$

holds whenever

$$r_0 \leqslant r \leqslant \log x$$

and

$$x > c_6 k^{50}.$$

Since the contribution of primes p with

$$p > x e^{10\sqrt{r\log x}} \quad \text{and} \quad p < x e^{-10\sqrt{r\log x}}$$

is $o(\sqrt{x})$, Theorem III asserts under the given circumstances the preponderance of primes $\equiv l(k)$ over those $\equiv 1(k)$ in the given sense in the interval

(5.2) $$(x e^{-10\sqrt{r\log x}}, \; x e^{10\sqrt{r\log x}}).$$

Putting

$$y = x e^{10\sqrt{r \log x}}$$

this means a preponderance of primes $\equiv l(k)$ over those $\equiv 1(k)$ in the intervals

$$(5.3) \qquad (y e^{-20\sqrt{r \log y}},\ y)$$

for all sufficiently large y's.

THEOREM IV. *If for a "good" k and a quadratic non-residue l there is an $L(s, \chi_1)$ with $\chi_1(l) \neq 1$ such that*

$$(5.4) \qquad L(\varrho_0, \chi_1) = 0. \quad \varrho_0 = \beta_0 + i\gamma_0, \quad \beta_0 > \tfrac{1}{2}, \quad \gamma_0 > 0,$$

then for all T with

$$(5.5) \qquad T > \max\left(c_7,\ e^{\pi^7 E(k)^{-7}},\ e^{e^k},\ e^{\left(\frac{4+\gamma_0^2}{\beta_0 - 1/2}\right)^{21}}\right)$$

there exist integers r_1 and r_2 with

$$(5.6) \qquad 2\log^{5/7} T - 4\log^{4/7} T \leqslant r_1, r_2 \leqslant 2\log^{5/7} T + 4\log^{2/3} T$$

and x_1, x_2 with

$$(5.7) \qquad T \leqslant x_1, x_2 \leqslant T e^{4\log^{20/21} T}$$

such that

$$\sum_p \varepsilon_k(p, l, 1) \log p\, e^{-\frac{1}{r_1}\log^2\frac{p}{x_1}} \geqslant T^{\beta_0} e^{-(1+\gamma_0^2)\log^{5/7} T}$$

and

$$\sum_p \varepsilon_k(p, l, 1) \log p\, e^{-\frac{1}{r_2}\log^2\frac{p}{x_2}} \leqslant -T^{\beta_0} e^{-(1+\gamma_0^2)\log^{5/7} T}.$$

Again the contribution of primes p with

$$p > T e^{\log^{41/42} T} \quad \text{and} \quad p < T e^{-\log^{41/42} T}$$

is $o(\sqrt{T})$; hence the theorem asserts roughly that under the given circumstances there are "densely" (x_1, r_1), resp. (x_2, r_2)-pairs such that the intervals $(x_1 e^{-r_1}, x_1 e^{r_1})$ contain "much more" primes $\equiv l(k)$ than $\equiv 1(k)$ and also "densely" intervals of type $(x_2 e^{-r_2}, x_2 e^{r_2})$ with "much more" primes $\equiv 1(k)$ than $\equiv l(k)$.

But we can express this state of affairs in a much more pregnant form. This is given in

THEOREM V. *Under the restrictions* (5.4) *and* (5.5) *there exist U_1, U_2, U_3, U_4 with*

$$(5.8) \qquad T e^{-5\log^{20/21} T} \leqslant U_1 < U_2 \leqslant T e^{5\log^{20/21} T},$$

$$(5.9) \qquad T e^{-5\log^{20/21} T} \leqslant U_3 < U_4 \leqslant T e^{5\log^{20/21} T}$$

such that

$$(5.10) \qquad \sum_{\substack{U_1 \leqslant p \leqslant U_2 \\ p \equiv l(k)}} 1 - \sum_{\substack{U_1 \leqslant p \leqslant U_2 \\ p \equiv 1(k)}} 1 > T^{\beta_0} e^{-(2+\gamma_0^2)\log^{5/7} T}$$

and

$$(5.11) \qquad \sum_{\substack{U_3 \leqslant p \leqslant U_4 \\ p \equiv l(k)}} 1 - \sum_{\substack{U_3 \leqslant p \leqslant U_4 \\ p \equiv 1(k)}} 1 < -T^{\beta_0} e^{-(2+\gamma_0^2)\log^{5/7} T}.$$

6. We shall deduce this theorem from Theorem IV right now. Putting

$$(6.1) \qquad \sum_{\substack{p \leqslant x \\ p \equiv l(k)}} 1 - \sum_{\substack{p \leqslant x \\ p \equiv 1(k)}} 1 \stackrel{\text{def}}{=} g(x),$$

the first assertion of Theorem IV can be written in the form

$$\int_0^\infty e^{-\frac{1}{r_1}\log^2\frac{r}{x_1}} \log r\, dg(r) \geqslant T^{\beta_0} e^{-(1+\gamma_0^2)\log^{5/7} T}$$

or

$$(6.2) \qquad \int_0^\infty g(r) e^{-\frac{1}{r_1}\log^2\frac{r}{x_1}} \cdot \frac{1}{r}\left\{\frac{2}{r_1}\log\frac{r}{x_1}\log r - 1\right\} dr \geqslant T^{\beta_0} e^{-(1+\gamma_0^2)\log^{5/7} T}.$$

As to the integral on the left, putting, with a suitable $0 < \vartheta < 1$ to be determined later,

$$\xi_1 = x_1 e^{-\log^\vartheta x_1}, \qquad \xi_2 = x_1 e^{\log^\vartheta x_1},$$

we split it into

$$(6.3) \qquad \int_0^{\xi_1} + \int_{\xi_1}^{\xi_2} + \int_{\xi_2}^\infty \stackrel{\text{def}}{=} J_1 + J_2 + J_3.$$

First we remark that owing to (5.6) and (5.7), choosing c_7 in (5.5) sufficiently large, it follows easily that the only zero $r^* > 1$ of the equation

$$\frac{2}{r_1}\log r \log\frac{r}{x_1} - 1 = 0$$

satisfies the inequality

$$(6.4) \qquad x_1 < r^* < 2x_1.$$

Using also the trivial inequality

$$|g(r)| \leqslant r,$$

we get at once

$$|J_3| < \int_{\xi_2}^{\infty} r\left(-e^{-\frac{1}{r_1}\log^2\frac{r}{x_1}}\log r\right)' dr \tag{6.5}$$

$$= \left(x_1 e^{\log^{\vartheta} x_1 - \frac{1}{r_1}\log^{2\vartheta} x_1}\right)\log \xi_2 + \int_{\xi_2}^{\infty} e^{-\frac{1}{r_1}\log^2\frac{r}{x_1}}\log r\, dr.$$

Choosing ϑ so that

$$2\vartheta - \tfrac{5}{7} > 1,$$

i.e.

$$\vartheta > \tfrac{6}{7}, \tag{6.6}$$

the first term in (6.5) is bounded. As to the second, it is

$$= x_1 \int_{\log^{\vartheta} x_1}^{\infty} e^{-\frac{y^2}{r_1}+y}(y+\log x_1)\, dy,$$

and owing to the inequalities

$$y+\log x_1 < e^{y}, \qquad y < y^2/4r_1,$$

valid in our range, a fortiori

$$< x_1 \int_{\log^{\vartheta} x_1}^{\infty} e^{-y^2/2r_1} dy < x_1 \int_{\log^{\vartheta} x_1}^{\infty} \frac{y}{r_1} e^{-y^2/2r_1} dy = x_1 e^{-\log^{2\vartheta} x_1/2r_1},$$

which is bounded again owing to (6.6.). Hence $|J_3|$ is bounded and the same holds for $|J_1|$ (even simpler). As to J_2 in (6.3), we write it as

$$\int_{\xi_1}^{r^*} + \int_{r^*}^{\xi_2} \tag{6.7}$$

and hence

$$J_2 \leqslant -\min_{\xi_1 \leqslant r \leqslant r^*} g(r) \cdot \int_{\xi_1}^{r^*} \left(e^{-\frac{1}{r_1}\log^2\frac{r}{x_1}}\log r\right)' dr + \tag{6.8}$$

$$+ \max_{r^* \leqslant r \leqslant \xi_2} g(r) \cdot \int_{r^*}^{\xi_2} \left(-e^{-\frac{1}{r_1}\log^2\frac{r}{x_1}}\log r\right)' dr$$

$$\leqslant \{\max_{r^* \leqslant r \leqslant \xi_2} g(r) - \min_{\xi_1 \leqslant r \leqslant r^*} g(r)\} e^{-\frac{1}{r_1}\log^2\frac{r^*}{x_1}}\log r^* + e^{-\frac{1}{r_1}\log^2\frac{\xi_1}{x_1}} \cdot 2\xi_2 \log \xi_2.$$

Choosing U_2, resp. U_1, as values for which the last max and min are attained respectively and also fact that the last term in (6.7) is bounded, the first assertion of Theorem V follows at once from (6.2), (6.5), (6.8), (6.4) and (5.7), choosing e.g. $\vartheta = \frac{13}{14}$. The second half goes analogously.

7. Theorem III will be again a special case of

THEOREM VI. *If for a "good" k, prescribed quadratic residue l_1 and quadratic non-residue l_2 mod k no $L(s, \chi)$ with $\chi(l_1) \neq \chi(l_2)$ vanishes for $\sigma > \frac{1}{2}$, then for suitable c_4, c_5, c_6 and*

$$r_0 = c_4 \frac{\log k}{E(k)^2} \tag{7.1}$$

the inequality

$$\sum_{p} \varepsilon_k(p, l_2, l_1) \log p\, e^{-\frac{1}{r}\log^2\frac{p}{x}} > c_5 \sqrt{x}$$

holds whenever

$$r_0 \leqslant r \leqslant \log x$$

and

$$x > c_6 k^{50}. \tag{7.2}$$

We cannot prove at present a similar generalization of Theorem IV. Hence we have to prove only Theorems IV and VI; the former will be the more difficult one.

In the Appendix we shall make some simple remarks on the comparison of primes of two progressions

$$\equiv l_1(k_1), \quad \text{resp.} \quad \equiv l_2(k_2),$$
$$(l_1, k_1) = (l_2, k_2) = 1, \quad k_1 \neq k_2.$$

8. We shall need the one-sided theorem (see Turán [1]) which we state as

LEMMA 1. *If*

$$|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n| \tag{8.1}$$

and with a $0 < \varkappa \leqslant \pi/2$ we have

$$\varkappa \leqslant |\arg z_j| \leqslant \pi, \tag{8.2}$$

further for the complex d_j-numbers we have

$$\min_{\lambda} \operatorname{Re} \sum_{j=1}^{\lambda} d_j \geqslant D > 0, \tag{8.3}$$

then for each $m > 0$ we have integer ν_1 and ν_2 with

$$m \leqslant \nu_1, \quad \nu_2 \leqslant m + n(3+\pi/\varkappa) \tag{8.4}$$

such that [1]

$$\operatorname{Re} \sum_{j=1}^{n} d_j z_j^{\nu_1} > \left(\frac{n}{8e(m+n(3+\pi/\varkappa))}\right)^{2n} \frac{D}{3n} |z_1|^{\nu_1}$$

(1) As in all applications, we know only an upper bound N for n. Completing, if necessary, the z_j's by zeros, we obtain at once that n can everywhere be replaced by N.

and

$$\operatorname{Re}\sum_{j=1}^{n} d_j z_j^{r_2} < -\left(\frac{n}{8e\big(m+n(3+\pi/\varkappa)\big)}\right)^{2n}\frac{D}{3n}|z_1|^{r_2}.$$

Further we shall need a lemma due in a somewhat weaker form to the first of us (see Knapowski [1]).

Lemma 2. *Let $\beta_1, \beta_2, \ldots$ be a real sequence and $\alpha_1, \alpha_2, \ldots$ another one such that with a positive U and $\gamma > 1$ we have*

$$|\alpha_\nu| \geqslant U, \tag{8.5}$$

$$\sum_{\nu}\frac{1}{1+|\alpha_\nu|^\gamma} \leqslant V \quad (<\infty). \tag{8.6}$$

Then, if only

$$\Delta > 1/U, \tag{8.7}$$

there exists a ξ with

$$\tau \leqslant \xi \leqslant \tau+\Delta \tag{8.8}$$

such that for all ν's the inequality

$$\frac{1}{24V}\cdot\frac{1}{1+|\alpha_\nu|^\gamma} \leqslant \alpha_\nu\xi+\beta_\nu-[\alpha_\nu\xi+\beta_\nu] \leqslant 1-\frac{1}{24V}\cdot\frac{1}{1+|\alpha_\nu|^\gamma} \tag{8.9}$$

holds.

For a short proof of this lemma we first fix ν and consider $\alpha_\nu x+\beta_\nu$. If x runs over the interval (8.8) then $(\alpha_\nu x+\beta_\nu)$ runs over an interval of length $\alpha_\nu\Delta$ which contains at most

$$1+|\alpha_\nu|\Delta$$

points with integer abscissae. Fixing an arbitrary one, λ say, the x-values satisfying the inequality

$$|\alpha_\nu x+\beta_\nu-\lambda| \leqslant \frac{1}{24V}\cdot\frac{1}{1+|\alpha_\nu|^\gamma}$$

(and (8.8)) form an interval of length

$$\leqslant \frac{1}{12V}\cdot\frac{1}{1+|\alpha_\nu|^\gamma}\cdot\frac{1}{|\alpha_\nu|}$$

and hence for a fixed ν the total measure of "bad" x-values is (²)

$$\leqslant \frac{1}{12V}\cdot\frac{1}{1+|\alpha_\nu|^\gamma}\cdot\frac{1}{|\alpha_\nu|}\{2+|\alpha_\nu|\Delta\}.$$

(²) Taking also in account that two more "bad" half-intervals may belong to our interval (8.8).

Summing for ν the measure of the set of "bad" x-values is at most

$$\frac{1}{12V}\sum_{\nu}\frac{2+|\alpha_\nu|\Delta}{|\alpha_\nu|(1+|\alpha_\nu|^\gamma)} = \frac{1}{12V}\Big\{\sum_{|\alpha_\nu|\leqslant 1}+\sum_{|\alpha_\nu|>1}\Big\} \overset{\text{def}}{=} \frac{1}{12V}\{S_1+S_2\}. \tag{8.10}$$

For S_1 we have from (8.5) and (8.7)

$$S_1 = \sum_{|\alpha_\nu|\leqslant 1}\left(\frac{2}{|\alpha_\nu|}+\Delta\right) \leqslant \sum_{|\alpha_\nu|\leqslant 1}\left(\frac{2}{U}+\Delta\right) \tag{8.11}$$

$$\leqslant 3\Delta\sum_{|\alpha_\nu|\leqslant 1}1 \leqslant 3\Delta\sum_{|\alpha_\nu|\leqslant 1}\frac{2}{1+|\alpha_\nu|^\gamma} \leqslant 6\Delta V.$$

For S_2 we have, owing to

$$1 \leqslant |\alpha_\nu|\Delta,$$

the inequality

$$S_2 < 3\Delta\sum_{|\alpha_\nu|>1}\frac{1}{1+|\alpha_\nu|^\gamma} \leqslant 3\Delta V.$$

From this, (8.10) and (8.11) we get for the measure of the set of "bad" x-values in (8.8) the upper bound

$$\tfrac{3}{4}\Delta < \Delta,$$

which proves Lemma 2.

We shall further need the

Lemma 3. *In the vertical strip*

$$\frac{1}{100} \leqslant \sigma \leqslant \frac{1}{50}$$

for an arbitrary modulus k there exists a broken line H consisting alternately of horizontal and vertical segments, each horizontal strip of width 1 containing at most one of the horizontal segments and on which for each $L(s,\chi)$ belonging to mod k *the inequalities*

$$\left|\frac{L'}{L}(s,\chi)\right| \leqslant c_7\varphi(k)\log^2 k(1+|t|),$$

$$\left|\frac{L'}{L}(2s,\chi)\right| \leqslant c_7\varphi(k)\log^2 k(1+|t|)$$

hold.

Since the proof of this lemma follows *mutatis mutandis* the (simple) one given in Appendix III of the book of one of us (see Turán [2]), we shall omit it here.

9. Now we turn to the proof of Theorem VI. We have fixed l_1-quadratic residue and l_2-quadratic non-residue mod k and suppose that none of the $L(s, \chi)$-functions belonging to k and for which

$$\chi(l_1) \neq \chi(l_2) \tag{9.1}$$

vanishes in the half-plane

$$\sigma > \tfrac{1}{2}. \tag{9.2}$$

Suppose $r \geqslant r_0$ and $b \geqslant 2$ (b to be determined later); we start from the integral (r_0 from (7.1), $f_{l_1,l_2}(w)$ defined in (2.1))

$$J_4 = \frac{1}{2\pi i} \int_{(2)} e^{(w+b)^2 r/4} f_{l_1,l_2}(w)\,dw. \tag{9.3}$$

Using (2.3), we get

$$\begin{aligned} J_4 = &\sum_{p \equiv l_2(k)} \log p \frac{1}{2\pi i} \int_{(2)} e^{(w+b)^2 r/4 - w\log p}\,dw - \\ &- \sum_{p \equiv l_1(k)} \log p \frac{1}{2\pi i} \int_{(2)} e^{(w+b)^2 r/4 - w\log p}\,dw + \frac{1}{2\pi i} \int_{(2/5)} e^{(w+b)^2 r/4} f^*_{l_1,l_2}(w)\,dw. \end{aligned} \tag{9.4}$$

(2.4) gives for the absolute value of the last integral the upper bound

$$c_2 \frac{1}{\pi} \int_0^\infty e^{\{(2/5+b)^2 - v^2\} r/4}\,dv = \frac{c_2}{\sqrt{\pi r}} e^{(2/5+b)^2 r/4}. \tag{9.5}$$

Since

$$\begin{aligned} \frac{1}{2\pi i} \int_{(2)} e^{(w+b)^2 r/4 - w\log p}\,dw &= e^{b^2 r/4 - \frac{1}{r}(\log p - rb/2)^2} \cdot \frac{1}{2\pi i} \int_{(0)} e^{w^2 r/4}\,dw \\ &= \frac{1}{\sqrt{\pi r}} e^{rb^2/4 - \frac{1}{r}(\log p - rb/2)^2}, \end{aligned}$$

we obtain from this, (9.3), (9.4) and (9.5)

$$\left| J_4 - \frac{e^{rb^2/4}}{\sqrt{\pi r}} \sum_p \varepsilon_k(p, l_2, l_1) \log p\, e^{-\frac{1}{r}(\log p - rb/2)^2} \right| \leqslant \frac{c_8}{\sqrt{r}} e^{(2/5+b)^2 r/4}. \tag{9.6}$$

10. Shifting the line of integration to the broken line H, given by Lemma 3, we get

$$\begin{aligned} J_4 = &\frac{1}{\varphi(k)} \sum_{\chi} \big(\bar\chi(l_1) - \bar\chi(l_2)\big) \sum_{\varrho(\chi)} e^{(\varrho+b)^2 r/4} + \frac{e^{(1/2+b)^2 r/4}}{2\varphi(k)} \sum_{\substack{\chi \\ \chi \text{ real}}} \big(\bar\chi(l_1) - \bar\chi(l_2)\big) - \\ &- \frac{1}{2\varphi(k)} \sum_{\chi} \big(\bar\chi(l_1) - \bar\chi(l_2)\big) \sum_{\varrho(\chi^2)} e^{(\varrho/2+b)^2 r/4} + \frac{1}{2\pi i} \int_{(H)} e^{(w+b)^2 r/4} f_{l_1,l_2}(w)\,dw. \end{aligned} \tag{10.1}$$

We shall repeatedly use the fact that the number of non-trivial zeros of any of the $L(s, \chi)$'s in the horizontal strip

$$X \leqslant t < X+1$$

cannot exceed

$$c_9 \log k (1 + |X|). \tag{10.2}$$

Hence the first sum in (10.1) is absolutely

$$\begin{aligned} &\leqslant 4 \Big(c_9 \log k \cdot e^{\{(1/2+b)^2 - E(k)^2\} r/4} + \sum_{\mu=1}^{\infty} c_9 \log k (1+\mu) \cdot e^{\{(1/2+b)^2 - \mu^2\} r/4} \Big) \\ &< c_{10} \log k \cdot e^{\{(1/2+b)^2 - E(k)^2\} r/4}. \end{aligned} \tag{10.3}$$

Similarly the third sum in (10.1) is absolutely

$$< c_{11} \log k \cdot e^{(1/4+b)^2 r/4}. \tag{10.4}$$

Using Lemma 3 and (10.2), one gets easily for the absolute value of the integral on the right of (10.1) the upper bound

$$c_{12} k \log k \cdot e^{(1/50+b)^2 r/4}. \tag{10.5}$$

In order to evaluate the remaining sum on the right of (10.1), we remark that if

$$k = 2^{\omega_0} p_1^{\omega_1} p_2^{\omega_2} \ldots p_j^{\omega_j}, \quad 2 < p_1 < p_2 < \ldots < p_j,$$

g_ν are primitive roots mod $p_\nu^{\omega_\nu}$ ($\nu = 1, 2, \ldots, j$) and for a given n the δ_ν's are determined by

$$n \equiv g_\nu^{\delta_\nu} (\operatorname{mod} p_\nu^{\omega_\nu}), \quad \nu = 1, 2, \ldots, j,$$

further δ_0', δ_0'' are for $\omega_0 \geqslant 2$ determined by

$$n \equiv (-1)^{\delta_0'} 5^{\delta_0''} (\operatorname{mod} 2^{\omega_0}),$$

$$0 \leqslant \delta_0' \leqslant 1, \quad 0 \leqslant \delta_0'' \leqslant \tfrac{1}{2}\varphi(2^{\omega_0}) - 1,$$

then the real characters have the form

$$\chi(n) = (-1)^{a_1\delta_1 + a_2\delta_2 + \ldots + a_j\delta_j + a_{j+1}\delta_0' + a_{j+2}\delta_0''};$$

here

$$0 \leqslant a_\nu \leqslant 1, \quad \nu = 1, 2, \ldots, j+2,$$

if $\omega_0 \geqslant 3$,

$$0 \leqslant a_\nu \leqslant 1, \ \nu = 1, 2, \ldots, j+1, \quad a_{j+2} = 0,$$

if $\omega_0 = 2$ and

$$0 \leqslant a_\nu \leqslant 1, \quad \nu = 1, 2, \ldots, j, \quad a_{j+1} = a_{j+2} = 0,$$

if $\omega_0 \leqslant 1$. Hence for a fixed n in the case $\omega_0 \geqslant 3$

$$\sum_{\chi \text{ real}} \chi(n) = 0$$

if a single of $\delta_1, \delta_2, \ldots, \delta_j, \delta_0', \delta_0''$ is odd, i.e. if n is a quadratic non-residue mod k; this holds evidently also for $\omega_0 \leqslant 2$. If $\omega_0 \geqslant 3$ and n is a quadratic residue mod k, then all δ_j's nad δ_0', δ_0'' are even and hence

$$\sum_{\chi \text{ real}} \chi(n) = 2^{j+2};$$

correspondingly

$$\sum_{\chi \text{ real}} \chi(n) = 2^{j+1}, \quad \text{resp. } 2^j,$$

if $\omega_0 = 2$, resp. $\omega_0 \leqslant 1$. Hence the second sum in (10.1) is

$$\geqslant \frac{1}{2\varphi(k)} e^{(1/2+b)^2 r/4} \cdot 2^j \geqslant \frac{1}{2\varphi(k)} e^{(1/2+b)^2 r/4}.$$

Collecting all these, (9.6) gives

$$(10.6) \quad \frac{e^{rb^2/4}}{\sqrt{\pi r}} \sum_p \varepsilon_k(p, l_2, l_1) \log p \, e^{-\frac{1}{r}(\log p - br/2)^2} \geqslant \frac{1}{2\varphi(k)} e^{(1/2+b)^2 r/4} - $$
$$- c_{13} \log k \left\{ e^{(1/2+b)^2 r/4 - E(k)^2 r/4} + e^{(1/4+b)^2 r/4} + k e^{(1/50+b)^2 r/4} + \frac{1}{\sqrt{r}} e^{(2/5+b)^2 r/4} \right\}.$$

If c_4 in (7.1) is sufficiently large, then (10.6) assumes the form

(10.7)

$$\sum_p \varepsilon_k(p, l_2, l_1) \log p \, e^{-\frac{1}{r}(\log p - br/2)^2} \geqslant \sqrt{\pi r} \left\{ \frac{e^{rb/4}}{4k} - 2c_{13} \log k \left(e^{rb/5} + k e^{rb/100}\right) \right\} e^{r/16}.$$

Choosing (³)

$$(10.8) \qquad b = 2 \frac{\log x}{r}$$

and making c_6 in (7.2) sufficiently large we have

$$rb/4 > 25 \log k + \tfrac{1}{2} \log c_6 > 25 \log k + \log(32 c_{13})^8$$

from which

$$k < e^{rb/100}$$

(³) Here, since $b \geqslant 2$, we come to the restriction $r \leqslant \log x$.

and

$$4c_{13} \log k < \frac{1}{8k} e^{rb/20}$$

easily follow. This, (10.7), (10.8) and (7.1) result, using also (5.1),

$$\sum_p \varepsilon_k(p, l_2, l_1) \log p \, e^{-\frac{1}{r} \log^2 \frac{p}{x}} > \frac{1}{8} \cdot \frac{\sqrt{\pi r}}{k} \sqrt{x} > c_5 \sqrt{x}, \qquad \text{q.e.d.}$$

11. The proof of Theorem IV is more difficult. We start again from integral (9.3) with $b > 100$, with integer $r \geqslant 2$ instead of $r/4$, $l_1 = 1$ and $l_2 = l$, where $(l, k) = 1$; then (9.6) gives

$$(11.1) \quad \left| J_4 - \frac{e^{rb^2}}{2\sqrt{\pi r}} \sum_p \varepsilon_k(p, l, 1) \log p \cdot e^{-\frac{1}{4r}(\log p - 2br)^2} \right| \leqslant \frac{c_8}{2\sqrt{r}} e^{r(2/5+b)^2}.$$

For J_4 we have again the representation (10.1); nevertheless, since now the truth of the Riemann-Piltz-conjecture is not supposed, the sums regarding the ϱ's must be replaced by $\sum'$, where the prime indicates that the summation extends to the ϱ's resp. $\varrho/2$'s right from H. For the integral on the right the estimation (10.5) holds again. The second sum on the right of (10.1) is trivially absolutely $\leqslant e^{r(1/2+b)^2}$ and the same holds for the third sum. All in all we have (a bit roughly)

(11.2)

$$\left| e^{rb^2} \sum_p \varepsilon_k(p, l, 1) \log p \cdot e^{-\frac{1}{4r}(\log p - 2br)^2} - \frac{2\sqrt{\pi r}}{\varphi(k)} \operatorname{Re} \sum_{\substack{\chi \\ \chi(l) \neq 1}} (1 - \bar{\chi}(l)) \sum_{\varrho(\chi)}{}' e^{r(\varrho+b)^2} \right|$$
$$\leqslant c_{14} \sqrt{r} \, e^{r(1/2+b)^2} k \log k.$$

We shall estimate roughly the contribution of the ϱ's with

$$\left(\frac{\pi}{2} >\right) \qquad |\arg(\varrho + b)| \geqslant \frac{\pi}{3}$$

using (10.2). This is absolutely

$$\leqslant c_{15} \sqrt{r} \sum_{\mu \geqslant b\sqrt{3}} \log(k\mu) \cdot e^{-r((1+b)^2 - \mu^2)}$$
$$< c_{16} \sqrt{r} \sum_{\mu \geqslant b\sqrt{3}} \log(k\mu) \cdot e^{-r\mu^2/2} < c_{17} e^{-rb^2} \log k.$$

Hence from (11.2), we get

$$(11.3)\qquad \Big|\sum_{p}\varepsilon_k(p,l,1)\log p\cdot e^{-\frac{1}{4r}(\log p-2br)^2}-$$
$$-\frac{2\sqrt{\pi r}}{\varphi(k)}\operatorname{Re}\sum_{\substack{\chi\\ \chi(l)\neq 1}}\big(1-\overline{\chi}(l)\big)\sum_{\substack{\varrho(\chi)\\ |\operatorname{Im}\varrho|\leqslant(b+1)\sqrt{3}}}{}'\,(e^{\varrho^2+2b\varrho})^r\Big|$$
$$\leqslant c_{18}\sqrt{r}\,e^{r/4+rb}k\log k.$$

12. For the sum

$$(12.1)\qquad \operatorname{Re}\sum_{\substack{\chi\\ \chi(l)\neq 1}}\big(1-\overline{\chi}(l)\big)\sum_{\substack{\varrho(\chi)\\ |\operatorname{Im}\varrho|\leqslant(b+1)\sqrt{3}}}{}'\,(e^{\varrho^2+2b\varrho})^r\overset{\text{def}}{=}Z(r)$$

we shall give "large positive" lower bound, resp. "large negative" upper bound choosing appropriately r in Lemma 1; the fulfilledness of the critical argument-condition (4.2) will be secured by a proper choice of b. The rôle of the z_j's will obviously be played by the numbers $e^{\varrho^2+2b\varrho}$; hence putting

$$\varrho=\sigma_\varrho+it_\varrho$$

we have

$$\arg z_j=2t_\varrho b+\operatorname{Im}(\varrho^2)=2\pi\Big(\frac{t_\varrho}{\pi}b+\frac{1}{2\pi}\operatorname{Im}(\varrho^2)\Big).$$

We apply Lemma 2 with

$$\beta_j=\frac{1}{2\pi}\operatorname{Im}(\varrho^2),\qquad \alpha_j=\frac{t_\varrho}{\pi}.$$

Then we choose

$$\gamma=\frac{11}{10},\qquad U=\frac{1}{\pi}E(k)$$

so that

$$V=c_{19}k\log k.$$

For the T's in (5.5) we define τ of Lemma 2 by

$$(12.2)\qquad T=e^{\tau^{7/2}},\qquad \tau=\log^{2/7}T$$

and choose $\Delta=\sqrt{\tau}$. The restriction (8.7) of this lemma is owing to (5.5) satisfied. Hence we may choose $b=b_0$ as ξ of this lemma; thus for all j's

$$\frac{2\pi}{24c_{19}k\log k}\cdot\frac{1}{1+\Big|\frac{t_\varrho}{\pi}\Big|^{11/10}}\leqslant\arg z_j\bmod 2\pi\leqslant 2\pi-\frac{2\pi}{24c_{19}k\log k}\cdot\frac{1}{1+\Big|\frac{t_\varrho}{\pi}\Big|^{11/10}}.$$

Since from (11.3)

$$|t_\varrho|\leqslant(b_0+1)\sqrt{3}$$

and from (12.2) and (5.5)

$$(12.3)\qquad \tau>e^{2k/7},$$

we get, using also

$$(14.2)\qquad \tau\leqslant b_0\leqslant\tau+\sqrt{\tau},$$

for all z_j's in $Z(r)$ the estimation

$$(\pi\geqslant)\qquad |\arg z_j|\geqslant\frac{2\pi}{24c_{19}\log^2\tau}\cdot\frac{1}{1+\Big(\frac{\sqrt{3}}{\pi}(\tau+\sqrt{\tau}+1)\Big)^{11/10}}>\tau^{-10/9}$$

(if c_7 in (5.5) is sufficiently large). Hence we can choose as $\varkappa$ of Lemma 1

$$(12.5)\qquad \varkappa=\tau^{-10/9}.$$

13. Now we apply Lemma 1. The rôle of d_j's will obviously be played by the numbers $1-\overline{\chi}(l)$ and hence as D we can choose

$$(13.1)\qquad D=8k^{-2}.$$

Owing to (10.2), (12.3) and (12.4), we have

$$n<c_{20}kb_0\log(kb_0)<c_{21}\tau\log^2\tau,$$

and hence N can be chosen as

$$(13.2)\qquad N=c_{21}\tau\log^2\tau.$$

As m of Lemma 1 we choose

$$(13.3)\qquad m=\frac{\tau^{7/2}}{2b_0}.$$

Then Lemma 1 gives the existence of integers r_1 and r_2 with

$$(13.4)\qquad \frac{\tau^{7/2}}{2b_0}\leqslant r_1,\ r_2\leqslant\Big(\frac{\tau^{7/2}}{2b_0}+c_{22}\tau^{19/9}\log^2\tau\Big)\leqslant\frac{\tau^{7/2}}{2b_0}+\tau^{7/3},$$

so that

$$(13.5)\qquad Z(r_1)>\Big(\frac{c_{23}}{\tau^{3/2}}\Big)^{2c_{21}\tau\log^2\tau}\frac{c_{24}}{\tau\log^4\tau}|z_1|^{r_1}>e^{-\tau\log^4\tau}\cdot|z_1|^{r_1}$$

(choosing c_7 in (5.5) sufficiently large) and

$$(13.6)\qquad Z(r_2)<-e^{-\tau\log^4\tau}\cdot|z_1|^{r_2}.$$

14. We have to give lower bounds for $|z_1|^{r_1}$ and $|z_1|^{r_2}$. The first is owing to (13.4) and (12.2)

$$(14.1)\qquad \geqslant |e^{\varrho_0^2+2b_0\varrho_0}|^{r_1} > (e^{2b_0r_1})^{\beta_0}\cdot e^{-\gamma_0^2 r_1} > T^{\beta_0}e^{-\gamma_0^2\tau^{5/2}} > T^{\beta_0}e^{-\gamma_0^2\log^{5/7}T}$$

and the same for $|z_1|^{r_2}$. This, (13.5), (12.2) and (11.3) give, putting

$$(14.2)\qquad e^{2b_0r_\nu} = x_\nu, \quad \nu = 1, 2,$$

the inequality

$$\sum_p \varepsilon_k(p, l, 1)\log p\cdot e^{-\frac{1}{4r_1}\log^2\frac{p}{x_1}}$$
$$> \frac{\sqrt{r_1}}{k}\big(T^{\beta_0}e^{-(1+\gamma_0^2)\log^{5/7}T}\cdot 2\sqrt{\pi} - c_{18}e^{r_1/4}\sqrt{x_1}\,k^2\log k\big).$$

(13.4), (12.3), (12.4) and (12.2) give

$$c_{18}e^{r_1/4}\sqrt{x_1}\,k^2\log k < c_{18}\log^3\tau\cdot e^{\tau^{5/2}}\cdot e^{\frac{1}{2}\tau^{7/2}+(\tau+\sqrt{\tau})\tau^{7/3}} < \sqrt{T}\,e^{2\log^{20/21}T}$$

if c_7 is sufficiently large, i.e.

$$\sum_p \varepsilon_k(p, l, 1)\log p\cdot e^{-\frac{1}{4r_1}\log^2\frac{p}{x_1}}$$
$$> \frac{\sqrt{r_1}}{k}T^{\beta_0}e^{-(1+\gamma_0^2)\log^{5/7}T}\{2 - e^{-(\beta_0-1/2)\log T+(3+\gamma_0^2)\log^{20/21}T}\}.$$

But owing to (5.5)

$$(\beta_0-\tfrac{1}{2})\log T > (4+\gamma_0^2)\log^{20/21}T;$$

taking also in account that

$$\sqrt{r_1} > \left(\frac{\tau^{7/2}}{\tau+\sqrt{\tau}}\cdot\frac{1}{2}\right)^{1/2} > \tau > k, \quad \text{i.e.}\quad \frac{\sqrt{r_1}}{k} > 1,$$

we have

$$\sum_p \varepsilon_k(p, l, 1)\log p\cdot e^{-\frac{1}{4r_1}\log^2\frac{p}{x_1}} > T^{\beta_0}e^{-(1+\gamma_0^2)\log^{5/7}T}$$

and analogously

$$\sum_p \varepsilon_k(p, l, 1)\log p\cdot e^{-\frac{1}{4r_2}\log^2\frac{p}{x_2}} < -T^{\beta_0}e^{-(1+\gamma_0^2)\log^{5/7}T}.$$

Further, from (14.2), (13.4) and (12.2) we have

$$x_\nu \geqslant e^{\tau^{7/2}} = T$$

and

$$x_\nu \leqslant e^{\tau^{7/2}+\tau^{7/3}2(\tau+\sqrt{\tau})} \leqslant T\,e^{4\log^{20/21}T}$$

indeed. Since finally

$$r_\nu \geqslant \frac{\tau^{7/2}}{2b_0} \geqslant \frac{\tau^{7/2}}{2(\tau+\sqrt{\tau})} \geqslant \frac{\tau^{5/2}}{2}\left(1-\frac{1}{\sqrt{\tau}}\right) > \frac{1}{2}\tau^{5/2}-\tau^2 = \frac{1}{2}\log^{5/7}T-\log^{4/7}T$$

and

$$r_\nu \leqslant \frac{\tau^{7/2}}{2b_0}+\tau^{7/3} \leqslant \frac{1}{2}\tau^{5/2}+\tau^{7/2} = \frac{1}{2}\log^{5/7}T+\log^{2/3}T,$$

(5.6) is obviously shown too and the proof is complete.

Appendix

As remarked by G. G. Lorentz, the comparison of primes in the progressions $\equiv l_1 \bmod k$ and $\equiv l_2 \bmod k$ with $\varphi(k_1) = \varphi(k_2)$ leads to still more difficult problems. Here we shall make only such remarks which are immediate corollaries of our previous work. The simplest case is obviously

$$k_1 = 3, \quad k_2 = 4.$$

We want to compare first the progressions $(3v+1)$ and $(4v+1)$. As easy to see we have

$$\pi(x, 3, 1)-\pi(x, 4, 1) = \pi(x, 12, 7)-\pi(x, 12, 5)$$

and both 7 and 5 are quadratic non-residues mod 12. Since for the modulus 12 the Haselgrove-condition is satisfied, Theorem 1.1 of ours (see Knapowski-Turán [1]) gives *mutatis mutandis*

COROLLARY 1. *For $T > c_{19}$ we have*

$$\max_{T^{1/3}\leqslant x\leqslant T}\{\pi(x, 3, 1)-\pi(x, 4, 1\} > \sqrt{T}\,e^{-23\frac{\log T\log_3 T}{\log_2 T}}$$

and

$$\min_{T^{1/3}\leqslant x\leqslant T}\{\pi(x, 3, 1)-\pi(x, 4, 1)\} < -\sqrt{T}\,e^{-23\frac{\log T\log_3 T}{\log_2 T}}.$$

Comparing the progressions $(3v+2)$ and $(4v+1)$, we have evidently

$$\pi(x, 3, 2)-\pi(x, 4, 1) = \pi(x, 12, 11)-\pi(x, 12, 1)+1;$$

since 1 resp. 11 are quadratic residues, resp. non-residues mod 12, this case is different from the previous one. Nevertheless Theorem 5 of our paper [2] leads to the following

COROLLARY 2. *For* $T > c_{20}$ *we have*

$$\max_{\log_3 T \leqslant x \leqslant T} \frac{\log x}{\sqrt{x}} \{\pi(x,3,2)-\pi(x,4,1)\} > 1$$

and

$$\min_{\log_3 T \leqslant x \leqslant T} \frac{\log x}{\sqrt{x}} \{\pi(x,3,2)-\pi(x,4,1)\} < -1.$$

For

$$\pi(x,4,3)-\pi(x,3,2),$$

resp.

$$\pi(x,4,3)-\pi(x,3,1)$$

we have evidently the same behaviour.

These remarks settle the case with

$$\varphi(k_1) = \varphi(k_2) = 2.$$

The next case, when

$$\varphi(k_1) = \varphi(k_2) = 4$$

(which is essentially only the case of $k_1 = 5$, $k_2 = 8$) seems to be more difficult; we hope to return to it later.

References

P. L. Chebyshev

[1] *Lettre de M. le professeur Tchébychev à M. Fuss sur un nouveaux théorème rélatif aux nombres premiers contenus dans les formes* $4n+1$ *et* $4n+3$, Bull. de la Classe phys. de l'Adac. Imp. des Sciences St. Petersburg 11 (1853), p. 208.

G. H. Hardy and J. E. Littlewood

[1] *Contributions to the theory of the distribution of primes*, Acta Math. 41 (1918), pp. 119-196.

E. Landau

[1] *Über einige ältere Vermutungen und Behauptungen in der Primzahltheorie*, Math. Zeitschr. 1 (1919), pp. 1-24.

S. Knapowski and P. Turán

[1] *Comparative prime-number theory IV*, Acta Math. Hung. 14 (1963), pp. 30-42.
[2] *Comparative prime-number theory II*, Acta Math. Hung. 13 (1962), pp. 315-342.
[3] *Comparative prime-number theory VIII*, Acta Math. Hung. 14(1963), pp. 251-268.

S. Knapowski

[1] *On sign-changes in the remainder-term in the prime-number formula*, Journ. London Math. Soc. (1961), pp. 451-460.

P. Turán

[1] *On some further one-sided theorems of new type in the theory of diophantine approximation*, Acta Math. Hung. 12 (1961), pp. 455-468.
[2] *Eine neue Methode in der Analysis und deren Anwendungen*, Akad. Kiadó, Budapest, 1953. A completely rewritten new edition will appear in the Interscience Tracts series.

Reçu par la Rédaction le 26. 10. 1963