Further developments in the comparative prime-number theory II

(A modification of Chebyshev's assertion)

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1. Chebyshev's assertion in question (see Chebyshev [1]) states that

\[(1.1) \quad \lim_{N \to +\infty} \sum_{p \leq N} (-1)^{(p-1)/2} e^{-\pi p} = -\infty \]

if \( p \) runs through all odd primes; in other words, it says, there are more primes of the form \( 4n+3 \) than of \( 4n+1 \), at least in the above "Abelian" sense. As it was shown by Hardy-Littlewood and Landau (see Hardy-Littlewood [1], Landau [1]) this holds if and only if the function

\[(1.2) \quad L(s, \chi) = \sum_{n=1}^\infty \frac{(-1)^n}{(2n+1)^s}, \quad s = \sigma + it, \ \sigma > 0, \]

does not vanish for \( \sigma > \frac{1}{2} \). As pointed out by them, the same holds for the relation

\[(1.3) \quad \lim_{N \to +\infty} \sum_{p \leq N} (-1)^{(p-1)/2} \log p e^{-\pi p} = -\infty. \]

This aspect lends an additional interest to the comparative study of the distribution of primes in progressions (and in other forms) and suggests above all the necessity to extend (1.1) or (1.3) to general \( k' \)'s. In our paper [5] we discussed the case \( k = 8 \), the first beyond the Chebyshevian case \( k = 4 \). With the notation

\[(1.4) \quad e_k(p, t_1, t_2) = \begin{cases} 1 & \text{if } p = t_1 \pmod{8}, \\ -1 & \text{if } p = t_2 \pmod{8}, \\ 0 & \text{otherwise} \end{cases} \]
Hardy-Littlewood-Landau's argument gave (using also strongly some numerical data furnished by Dr. P. C. Haselgrove) that the relation
\[ \lim_{x \to \infty} \sum_{p < x} a_p(p, l, 1) \log p e^{-\mu x} = -\infty \quad (l = 3, 5, 7) \]
holds if and only if no \( L(s, \chi) \) function belonging to mod 8 with \( \chi \neq \chi_0 \) vanishes for \( s > \frac{1}{2} \). If \( l_1 \) and \( l_2 \) are any two of 3, 5, 7 (= quadratic non-residues mod 8), we proved i.e. without any conjectures that if \( c_0 \) (and later \( c_1, c_2, \ldots \)) denote positive numerical constants, that for 0 < \( \epsilon < c_1 \)
\[ \max_{x^{-1/2} < \epsilon < x^{-1/2}} \sum_{p < x} a_p(p, l_1, l_2) \log p e^{-\mu x} > \epsilon^{-1} e^{-\frac{1}{2} \log 1/\epsilon} \log 1/\epsilon }{ \log 1/\epsilon } } \]
(i.e. changing \( l_1 \) and \( l_2 \) also
\[ \min_{x^{-1/2} < \epsilon < x^{-1/2}} \sum_{p < x} a_p(p, l_1, l_2) \log p e^{-\mu x} < -\epsilon^{-1} e^{-\frac{1}{2} \log 1/\epsilon} \log 1/\epsilon }{ \log 1/\epsilon } } , \]
and thus there is a sign-change of the function \( \sum_{p} a_p(p, l_1, l_2) \log p e^{-\mu x} \) in every interval of the form \( [\delta^{-1/2}, \delta] \).)

2. Let us analyze the situation for general \( k \). Putting with \( l_i \equiv l_i \mod k \)
\[ \omega_{i,k}(w) = \frac{1}{\varphi(k)} \sum_x \frac{1}{x} \left( \chi(l_i - x(l_i)) \frac{L'}{L}(we, \chi) - \frac{L'}{L}(2we, \chi^2) \right) \]
we start the integral
\[ J = \frac{1}{2\pi i} \int \Gamma(\epsilon) \omega_{i,k}(w) dw \]
(due essentially to Hardy-Littlewood). Since for \( \Re w > 1 \) we have from
\[ f_{i,k}(w) = \sum_{\nu \equiv l_i \mod k} \log p e^{-\mu x} + \sum_{\nu \equiv l_i \mod k} \log p e^{-\mu x} - \sum_{\nu \equiv l_i \mod k} \log p e^{-\mu x} \]
\[ = \sum_{p \equiv l_i \mod k} \log p e^{-\mu x} + \sum_{p \equiv l_i \mod k} \log p e^{-\mu x} - \sum_{p \equiv l_i \mod k} \log p e^{-\mu x} + f_{i,k}(w), \]
where \( f_{i,k}(w) \) is regular e.g. for \( \Re w > 1 \) and satisfies here the inequality
\[ f_{i,k}(w) \leq c_2, \]
we get from (2.2) (adapting the notation (1.4) for general \( k \)-moduli)
\[ J = \sum_p a_p(p, l_1, l_2) \log p e^{-\mu x} + \frac{1}{2\pi i} \int \Gamma(\epsilon) \omega_{i,k}(w) dw \]
\[ = \sum_p a_p(p, l_1, l_2) \log p e^{-\mu x} + O(a^2) \]
(\( O \) means uniformly in \( x \) and \( k \)). On the other hand, shifting the line of integration to the left, the "essential" part of \( J \) is
\[ \frac{1}{\varphi(k)} \sum_x \left( \chi(l_i - x(l_i)) \sum_{s \in \mathbb{Q}} \gamma(s) e^{s w} + \frac{\Gamma(\epsilon)}{\varphi(k)} \int \sum_x (\chi(l_i - x(l_i)) - \frac{1}{2\pi i} \sum_{s \in \mathbb{Q}} \chi(\xi), x) \gamma(s) e^{s w} \right) \]
where dash means that the summation is to be extended to all real characters (with \( \chi(l_i) \neq \chi(l_i) \)) and summation over \( \varphi(k) \) means that it is to be extended to all non-trivial zeros of \( L(s, \chi) \) with a fixed \( \chi \). Let us consider the "most suspicious" case for preponderance, when \( l_i \) is a quadratic residue, \( l_i \) a non-residue \( k \) and suppose the truth of the Riemann-Furtwängler conjecture. In this case the third term is unessential, the second term as we shall see in section 3 is
\[ \geq \frac{1}{2\pi i} \Gamma(w), \]
where \( r \) stands for the number of different odd prime-factors of \( k \). As to the critical first sum in (2.5), trivial treatment gives only the upper bound
\[ \frac{1}{\varphi(k)} \sum_x \sum_{s \in \mathbb{Q}} \left| \gamma(s) \right| \]
which certainly supersedes the value in (2.8) if \( k \) is sufficiently large and nothing better can be accomplished at present (owing to the factor \( \varphi(k) \)).

3. Beside this difficulty also another fact makes it desirable to find an appropriate modification of Chebyshev's problem. The relation (1.5) could be replaced by
\[ \sum_{p} a_p(p, l, 1) \log p e^{-\mu x} < -\epsilon_0 \varphi(x) \]
If \( x > q_k \) and the Riemann–Piltz conjecture is true for the \( L \)-functions mod \( 8 \) with \( \chi \neq \chi_k \) and by an inequality of type (1.6) if it is false. Hence everything remains true if in the sum
\[
\sum_{\mathcal{P}} \varepsilon_k(p, l_1, l_2) \lambda(p) e^{-\sigma \log^2 p}
\]
we drop the terms with
\[
p < x^{12 - \varepsilon}
\]
and trivially dropping those with
\[
p > 10 x \log x.
\]
In other words, putting
\[
y = 10 x \log x
\]
such a preponderance-behaviour was exhibited for all sufficiently large \( y \)'s for primes of the form \( 8n + l_1 \) and \( 8n + l_2 \) in the interval
\[
(\sqrt{y}, y)
\]
To push the lower bound in (3.4) above \( \sqrt{y} \), i.e. to strengthen the “accumulation”, however desirable, seems hopeless at present, even in the Chebyshevian case \( k = 4 \).

4. The main result of this paper can be expressed shortly by replacing the factors \( e^{-\frac{1}{2} \log^2 p} \) by
\[
e^{-\frac{1}{2} \log^2 p + \frac{1}{2} \log E(k) - \frac{1}{2} \log k}
\]
with a suitable (“small”) \( r = r(a) \), one can come much closer to both desiderataums. This holds in particular for the “good” \( k \)'s, i.e. for those for which with an \( E(k) \) no \( L(s, \chi) \) function mod \( k \) vanishes for
\[
0 < \sigma < 1, \quad \lambda(l) = E(k)
\]
(Hasselgrove-property). This property has been verified in a number of cases, notably for all \( k \leq 10 \). The extension of them seems very desirable to us; particularly, a proof that Hasselgrove-property holds for an infinity of \( k \)'s would be of great significance. Out of our results the most compete are those comparing primes \( \equiv 1 \mod k \) when \( l \) is a quadratic non-residue mod \( k \) (which obviously comprises the Chebyshevian case). We formulate first

**Theorem 1.** For any fixed “good” \( k \) and for all quadratic non-residues \( l \mod k \), \( (l, k) = 1 \), the relation
\[
\lim_{x \to +\infty} \sum_{\mathcal{P}} \varepsilon_k(p, l, 1) \lambda(p) e^{-\frac{1}{2} \log^2 p} = +\infty
\]
for every \( r = r(a) \) satisfying \( a_k(k) \leq r \leq \log x \) is true if and only if none of the \( L(s, \chi) \)-functions mod \( k \) with \( \chi \neq \chi_k \) vanishes for \( \sigma > \frac{1}{2} \).

Slightly more generally we formulate

**Theorem 2.** For any fixed “good” \( k \) and fixed quadratic non-residues \( l \mod k \), \( (l, k) = 1 \),
\[
\lim_{x \to +\infty} \sum_{\mathcal{P}} \varepsilon_k(p, l, 1) \lambda(p) e^{-\frac{1}{2} \log^2 p} = +\infty
\]
for every \( r = r(a) \) satisfying \( a_k(k) \leq r \leq \log x \) is true if and only if none of the \( L(s, \chi) \)-functions mod \( k \) with \( \chi(l) \neq 1 \) vanishes for \( \sigma > \frac{1}{2} \).

To deduce Theorem I from Theorem II we have only to remark that if for a character \( \chi \), for all non-residues \( l, \chi(l) = 1 \), then \( \chi = \chi_1 \).

Namely if \( \alpha \) is an arbitrary quadratic residue mod \( k \), \( (a, k) = 1 \) and \( l \) is an arbitrary non-residue mod \( k \) with \( (l, k) = 1 \), then \( \alpha l = l' = \text{non-residue}, \) i.e. \( \chi(l') = \chi(l') \chi(l) = 1 \).

5. In turn, Theorem II will be a consequence of Theorem III and IV. Here we shall assume (which goes without loss of generality) that
\[
E(k) < r \log k / k.
\]

**Theorem III.** If for a “good” \( k \) and a prescribed quadratic non-residue \( l \) no \( L(s, \chi) \) with \( \chi(l) = 1 \) vanishes for \( \sigma > \frac{1}{2} \), then for suitable \( c_1, c_2, c_3 \) and
\[
r_k = c_1 \log k
\]
the inequality
\[
\sum_{\mathcal{P}} \varepsilon_k(p, l, 1) \lambda(p) e^{-\frac{1}{2} \log^2 p} > c_2 \sqrt{\varepsilon}
\]
holds whenever
\[
r_k \leq r \leq \log x
\]
and
\[
x > c_3 k^2
\]
Since the contribution of primes \( p \) with
\[
p > x e^{10 \log x} \quad \text{and} \quad p < x e^{-10 \log x}
\]
is \( o(1) \), Theorem III asserts under the given circumstances the preponderance of primes \( \equiv 1 \mod k \) over those \( \equiv 1 \mod k \) in the given sense in the interval
\[
(x e^{-10 \log x}, x e^{10 \log x})
\]
Putting

\[ y = 2e^{\ln(\ln^2 y)} \]

this means a preponderance of primes \( \equiv l(k) \) over those \( \equiv 1(k) \) in the intervals

(5.3)

\[ (y e^{-\sqrt{\ln(\ln y)}}, y) \]

for all sufficiently large \( y \)'s.

**Theorem IV.** If for a "good" \( k \) and a quadratic non-residue \( l \) there is an \( L(e^{\gamma}, \gamma) \) with \( \chi_l(0) \neq 1 \) such that

(5.4)

\[ L(e^{\gamma}, \gamma) = 0, \quad \phi = \beta_0 + i\gamma_0, \quad \beta_0 > \frac{1}{2}, \quad \gamma_0 > 0, \]

then for all \( T \) with

(5.5)

\[ T > \max \left \{ \epsilon_1, e^{\gamma_0(1+\gamma_0)}, \epsilon^+, e^{\gamma_0(1+\gamma_0)} \right \} \]

there exist integers \( r_1 \) and \( r_2 \) with

(5.6)

\[ 2\log^3 T - 4\log^{4/3} T < r_1, r_2 \leq 2\log^2 T + 4\log^{4/3} T \]

and \( z_1, z_2 \) with

(5.7)

\[ T < z_1, z_2 < T e^{\ln(\ln^3 y)} \]

such that

\[ \sum_{p \leq T} e(p, l, 1) \log p e^{-\frac{1}{r_1} \log z_1^{1/4} z_1^{1/4}} < T^{1 - \frac{1}{2} \log z_2^{1/4} z_2^{1/4}} \]

and

\[ \sum_{p \leq T} e(p, l, 1) \log p e^{-\frac{1}{r_2} \log z_2^{1/4} z_2^{1/4}} < T^{1 - \frac{1}{2} \log z_1^{1/4} z_1^{1/4}}. \]

Again the contribution of primes \( p \) with

\[ p > T e^{\ln(\ln^3 y)} \quad \text{and} \quad p < T e^{-\ln(\ln^3 y)} \]

is \( o(T^2) \); hence the theorem asserts roughly that under the given circumstances there are "densely" \((z_1, r_1)\), resp. \((z_2, r_2)\)-pairs such that the intervals \((z_1 e^{-r_1}, z_1 e^{r_1})\) contain "much more" primes \( \equiv l(k) \) than \( \equiv 1(k) \) and also "densely" intervals of type \((z_2 e^{-r_2}, z_2 e^{r_2})\) with "much more" primes \( \equiv 1(k) \) than \( \equiv l(k) \).

But we can express this state of affairs in a much more pregnant form. This is given in

**Theorem V.** Under the restrictions (5.4) and (5.5) there exist \( U_1, U_2, U_3, U_4 \) with

(5.8)

\[ T e^{-\ln(\ln^3 y)} \leq U_1, U_2 \leq T e^{\ln(\ln^3 y)} \]

(5.9)

\[ T e^{-\ln(\ln^3 y)} \leq U_3, U_4 \leq T e^{\ln(\ln^3 y)} \]

such that

\[ \sum_{r \leq r_1} 1 - \sum_{r \leq r_2} 1 > T^{1 - \frac{1}{2} \log z_2^{1/4} z_2^{1/4}} \]

and

\[ \sum_{r \geq r_3} 1 - \sum_{r \geq r_4} 1 < - T^{1 - \frac{1}{2} \log z_1^{1/4} z_1^{1/4}}. \]

6. We shall deduce this theorem from Theorem IV right now. Putting

(6.1)

\[ \sum_{r \leq r_1} 1 - \sum_{r \leq r_2} 1 \overset{\text{def}}{=} g(x), \]

the first assertion of Theorem IV can be written in the form

\[ \int g(x) e^{rac{-1}{r_1} \log r} \log r \, dr \geq T^{1 - \frac{1}{2} \log z_2^{1/4} z_2^{1/4}} \]

or

\[ \int g(x) e^{rac{-1}{r_1} \log r} \frac{1}{r_1} \log r \log r \, dr \geq T^{1 - \frac{1}{2} \log z_2^{1/4} z_2^{1/4}}. \]

As to the integral on the left, putting, with a suitable \( 0 < \theta < 1 \) to be determined later,

\[ \xi_1 = x e^{-\theta \log x}, \quad \xi_2 = x e^{\theta \log x}, \]

we split it into

(6.3)

\[ \int \frac{1}{r_1} + \int \frac{1}{r_1} + \int \frac{1}{r_1} = J_1 + J_2 + J_3. \]

First we remark that owing to (5.6) and (5.7), choosing \( \theta \) in (5.5) sufficiently large, it follows easily that the only zero \( r^* > 1 \) of the equation

\[ \frac{2}{r_1} - \log r \log r - 1 = 0 \]

satisfies the inequality

(6.4)

\[ \xi_2 < r^* < 2 \xi_1. \]

Using also the trivial inequality

\[ |g(r)| \leq r, \]

Therefore the only zero \( r^* > 1 \) of the equation

\[ \frac{2}{r_1} - \log r \log r - 1 = 0 \]

satisfies the inequality

(6.4)

\[ \xi_2 < r^* < 2 \xi_1. \]
we get at once
\[ |J_\theta| < \int \left( -e^{-\frac{1}{2} \log \theta} \log r \right) dr \]
\[ = 2 \log \frac{\log 2}{\log \theta} + \int \left( -e^{-\frac{1}{2} \log \theta} \log r \right) dr. \]
Choosing \( \theta \) so that
\[ 2 \theta - \frac{1}{2} > 1, \]
i.e.
\[ \theta > \frac{3}{4}, \]
the first term in (6.5) is bounded. As to the second, it is
\[ = 2 \log \frac{\log 2}{\log \theta} + \int \left( y + \log y \right) dy, \]
and owing to the inequalities
\[ y + \log y < e^y, \quad y < \theta \log \theta, \]
valid in our range, a fortiori
\[ < 2 \log \frac{\log 2}{\log \theta} + \int \frac{y}{\log y} dy, \]
which is bounded again owing to (6.6). Hence \( |J_\theta| \) is bounded and the same holds for \( |J_\varphi| \) (even simpler). As to \( J_2 \) in (6.3), we write it as
\[ J_2 = -\min_{t < c} g(r) \int \left( e^{\frac{1}{2} \log r - \frac{1}{2} \log \theta} \log r \right) dr + \]
\[ + \max_{t < c} g(r) \int \left( -e^{\frac{1}{2} \log r - \frac{1}{2} \log \theta} \log r \right) dr \]
\[ \leq \left\{ \max_{t < c} g(r) - \min_{t < c} g(r) \right\} e^{\frac{1}{2} \log r - \frac{1}{2} \log \theta} \log r + e^{-\frac{1}{2} \log r - \frac{1}{2} \log \theta} \log r, \]
Choosing \( U_2 \), resp. \( U_1 \), as values for which the last max and min are attained respectively and also fact that the last term in (6.7) is bounded, the first assertion of Theorem V follows at once from (6.2), (6.5), (6.8), (6.4), and (5.7), choosing e.g. \( \theta = \frac{3}{4} \). The second half goes analogously.

7. Theorem III will be again a special case of THEOREM VI. If for a “good” \( k \), prescribed quadratic residues \( l_1, \) and quadratic non-residues \( l_2 \) mod \( k \) so \( L(s, 2) \) with \( \chi(l_1) \neq \chi(l_2) \) vanishes for \( \sigma > \frac{1}{2}, \) then for suitable \( c_1, c_2, c_3 \), and
\[ r = c_1 \log k \]
the inequality
\[ \sum \left( \log e \right) e^{\frac{1}{2} \log r - \frac{1}{2} \log \theta} > c_3 r \]
holds whenever
\[ r \leq r \leq \log x \]
and
\[ x > c_3 k^{3/2}. \]

We cannot prove at present a similar generalization of Theorem IV. Hence we have to prove only Theorems IV and VI; the former will be the more difficult one.

In the Appendix we shall make some simple remarks on the comparison of primes of two progressions
\[ l_1 (l_1) \text{ resp. } l_2 (l_2), \]
\[ (l_1, l_2) = (l_1, k) = 1, \quad k_1 \neq k_2. \]

8. We shall need the one-sided theorem (see Turán [1]) which we state as

**Lemma 1.** If
\[ |2| > |2| > \ldots > |2| \]
and with a \( 0 < x < \pi/2 \) we have
\[ x \leq \arg z \leq \pi, \]
for the complex \( d_i \)-numbers we have
\[ \min \sum_{i=1}^{n} d_i > r > 0, \]
then for each \( m > 0 \) we have integer \( v_1 \) and \( v_2 \) with
\[ m < v_1 < m + \arg z \cdot m, \]
such that
\[ \sum_{i=1}^{n} d_i v_i > \left( \frac{m}{8 \pi (m + \arg z \cdot m)} \right)^{1/3} r. \]

(1) As in all applications, we know only an upper bound \( N \) for \( n \). Completing, if necessary, the \( n \)-th by zeroes, we obtain at once that \( n \) can everywhere be replaced by \( N \).
and
\[ \text{Re} \sum_{j=1}^{n} d_j z^j = -\left( \frac{n}{\text{Re}(m+n(3+\pi/4))} \right)^{2n} \frac{D}{3n} |x|^2. \]

Further we shall need a lemma due in a somewhat weaker form to the first of us [see Knapowski (1)].

**LEMMA 2.** Let \( \beta_1, \beta_2, \ldots \) be a real sequence and \( a_1, a_2, \ldots \) another one such that with a positive \( U \) and \( r \geq 1 \) we have
\[ |a_j| \geq U, \]
\[ \sum_j \frac{1}{1+|a_j|^r} \leq V \quad (< \infty). \]

Then, if only
\[ A > 1/U, \]
there exists a \( \xi \) with
\[ r \leq \xi \leq r + A \]
\[ \text{such that for all } \nu \text{ the inequality} \]
\[ \frac{24V}{1+|a_j|^r} \leq a_j \xi + \beta_j - [a_j \xi + \beta_j] \leq 1 - \frac{1}{24V} \cdot \frac{1}{1+|a_j|^r} \]
holds.

For a short proof of this lemma we first fix \( \nu \) and consider \( a_j \xi + \beta_j \). If \( a_j \) runs over the interval (8.8) then \( (a_j \xi + \beta_j) \) runs over an interval of length \( A, A \) which contains at most
\[ 1 + |a_j|. \]
points with integer abscissae. Fixing an arbitrary one, \( \lambda \), say, the \( a_j \)-values satisfying the inequality
\[ |a_j \xi + \beta_j - \lambda| \leq \frac{1}{24V} \cdot \frac{1}{1+|a_j|^r} \]
(and (8.8)) form an interval of length
\[ \frac{1}{12V} \cdot \frac{1}{1+|a_j|^r} \cdot \frac{1}{|a_j|} \]
and hence for a fixed \( \nu \) the total measure of "bad" \( a_j \)-values is (1)
\[ \frac{1}{12V} \cdot \frac{1}{1+|a_j|^r} \cdot \frac{1}{|a_j|} (2 + |a_j| A). \]

Summing for \( \nu \) the measure of the set of "bad" \( a_j \)-values is at most
\[ A \sum |a_j| \leq A \sum (1 + |a_j|^r) \leq \frac{1}{12V} \cdot \frac{1}{1+|a_j|^r} \cdot \frac{1}{|a_j|} (2 + |a_j| A). \]

For \( S_1 \), we have from (8.10) and (8.9)
\[ S_1 = A \sum |a_j| \leq A \sum \left( \frac{1}{U} \right)^r \]
\[ \leq 3d \sum_{|a_j| > 1} \frac{1}{1+|a_j|^r} \leq 3d V. \]

For \( S_2 \), we have, owing to
\[ 1 \leq |a_j|, \]
the inequality
\[ S_2 < 3d \sum_{|a_j| > 1} \frac{1}{1+|a_j|^r} \leq 3d V. \]

This from (8.10) and (8.11) we get for the measure of the set of "bad" \( a_j \)-values in (8.8) the upper bound
\[ \frac{1}{4A} < A, \]
which proves Lemma 2.

We shall further need the

**LEMMA 3.** In the vertical strip
\[ \frac{1}{100} \leq \sigma \leq \frac{1}{50} \]
for an arbitrary modulus \( k \) there exists a broken line \( \Pi \) consisting alternately of horizontal and vertical segments, each horizontal strip of width 1 containing at most one of the horizontal segments and on which for each \( L(x, z) \) belonging to mod \( k \) the inequalities
\[ \left| \frac{L'}{L}(x, z) \right| \leq e_\psi(k) \log^2 k (1 + |t|), \]
\[ \left| \frac{L'}{L}(2x, z) \right| \leq e_\psi(k) \log^2 k (1 + |t|) \]
hold.

Since the proof of this lemma follows mustas mustaide the (simple) one given in Appendix III of the book of one of us (see Turán [2]), we shall omit it here.
9. Now we turn to the proof of Theorem VI. We have fixed $l$, quadratic residue and $l$, quadratic non-residue mod $k$ and suppose that none of the $L(s, \chi)$-functions belong to $k$ and for which

$$\chi(l_1) \neq \chi(l_2)$$

vanishes in the half-plane

$$\sigma > \frac{1}{2}.$$

Suppose $r \geq 2$ and $b > 2$ ($b$ to be determined later); we start from the integral (7.1), $f_{j_1}(\omega)$ defined in (2.3)

$$J_1 = \frac{1}{2\pi i} \int \frac{e^{2\pi i l_1 \omega / k} f_{j_1}(\omega) d\omega}{\prod_{p \neq l_1, l_2} (e^{2\pi i \omega / p} - 1)}.$$  

Using (2.3), we get

$$J_1 = \sum_{n \leq \alpha} \log^2 p \frac{1}{2\pi i} \int \frac{e^{2\pi i l_1 \omega / k} f_{j_1}(\omega) d\omega}{\prod_{p \neq l_1, l_2} (e^{2\pi i \omega / p} - 1)} - \sum_{n \leq \alpha} \log^2 p \frac{1}{2\pi i} \int \frac{e^{2\pi i l_1 \omega / k} f_{j_1}(\omega) d\omega}{\prod_{p \neq l_1, l_2} (e^{2\pi i \omega / p} - 1)} + \frac{1}{2\pi i} \int \frac{e^{2\pi i l_1 \omega / k} f_{j_1}(\omega) d\omega}{\prod_{p \neq l_1, l_2} (e^{2\pi i \omega / p} - 1)}.$$

(2.4) gives the absolute value of the last integral the upper bound

$$\alpha \int \frac{1}{2\pi i} \int \frac{e^{2\pi i l_1 \omega / k} f_{j_1}(\omega) d\omega}{\prod_{p \neq l_1, l_2} (e^{2\pi i \omega / p} - 1)} = \alpha \frac{1}{2\pi i} \int \frac{e^{2\pi i l_1 \omega / k} f_{j_1}(\omega) d\omega}{\prod_{p \neq l_1, l_2} (e^{2\pi i \omega / p} - 1)}.$$

Since

$$\int \frac{e^{2\pi i l_1 \omega / k} f_{j_1}(\omega) d\omega}{\prod_{p \neq l_1, l_2} (e^{2\pi i \omega / p} - 1)} = e^{2\pi i l_1 \omega / k} f_{j_1}(\omega) d\omega,$$

we obtain from this, (9.3), (9.4) and (9.5)

$$\left| J_1 \right| \leq \frac{1}{2\pi i} \sum_{p \neq l_1, l_2} \alpha \log^2 p \leq \frac{1}{2\pi i} \int \frac{e^{2\pi i l_1 \omega / k} f_{j_1}(\omega) d\omega}{\prod_{p \neq l_1, l_2} (e^{2\pi i \omega / p} - 1)}.$$

10. Shifting the line of integration to the broken line $H$, given by Lemma 3, we get

$$J_1 = \frac{1}{2\pi i} \sum \frac{1}{p(k)} \int \frac{e^{2\pi i l_1 \omega / k} f_{j_1}(\omega) d\omega}{\prod_{p \neq l_1, l_2} (e^{2\pi i \omega / p} - 1)} + \frac{1}{2\pi i} \int \frac{e^{2\pi i l_1 \omega / k} f_{j_1}(\omega) d\omega}{\prod_{p \neq l_1, l_2} (e^{2\pi i \omega / p} - 1)}.$$

We shall repeatedly use the fact that the number of non-trivial zeros of any of the $L(s, \chi)$'s in the interval $X < \frac{1}{2} < X + 1$ cannot exceed

$$c_0 \log k(1 + |X|).$$

Hence the first sum in (10.1) is absolutely

$$< 4 \sum_{n=1}^{\infty} \alpha \log k(1 + \mu) e^{2\pi i \omega / n} e^{2\pi i \omega / n},$$

Similarly the third sum in (10.1) is absolutely

$$< c_1 \log k (1 + \mu) e^{2\pi i \omega / n}. $$

Using Lemma 3 and (10.2), one gets easily for the absolute value of the integral on the right of (10.1) the upper bound

$$c_0 \log k (1 + \mu) e^{2\pi i \omega / n},$$

In order to evaluate the remaining sum on the right of (10.1), we remark that if

$$k = 2^{a_2} p_1^{a_1} p_2^{a_2} \ldots p_l^{a_l} = 2 < p_1 < p_2 < \ldots < p_l,$$

$g_n$ are primitive roots mod $p_n^\nu$ ($\nu = 1, 2, \ldots, j$) and for a given $n$ the $\delta_n$'s are determined by

$$n = g_1^{\delta_1}(\text{mod } p_1^\nu), \quad n = 1, 2, \ldots, j,$$

further $\delta_n$ for $a_n \geq 2$ determined by

$$n = (-1)^{a_n} \delta_n (\text{mod } 2^n),$$

and then the real characters have the form

$$\chi(n) = (-1)^{a_n} \delta_n (\nu = 1, 2, \ldots, j),$$

here

$$0 \leq a_n \leq 1, \quad \nu = 1, 2, \ldots, j + 2,$$

if $a_n \geq 3,

$$0 \leq a_n \leq 1, \quad \nu = 1, 2, \ldots, j + 1, \quad a_{n+1} = 0,$$

if $a_n = 2$ and

$$0 \leq a_n \leq 1, \quad \nu = 1, 2, \ldots, j, \quad a_{n+2} = 0.$$
If $a_0 \leq 1$. Hence for a fixed $n$ in the case $a_0 \geq 3$

$$\sum_{x \text{real}} \chi(n) = 0$$

If a single of $\delta_1, \delta_2, \ldots, \delta_l, \delta_k^0$ is odd, i.e. if $n$ is a quadratic non-residue mod $k$; this holds evidently also for $a_0 \leq 2$. If $a_0 \geq 3$ and $n$ is a quadratic residue mod $k$, then all $\delta_i$'s and $\delta_j, \delta_k^0$ are even and hence

$$\sum_{x \text{real}} \chi(n) = 2^{l+2};$$

correspondingly

$$\sum_{x \text{real}} \chi(n) = 2^{l+3}, \quad \text{resp. } 2^l,$$

if $a_0 = 2$, resp. $a_0 = 1$. Hence the second sum in (10.1) is

$$\frac{1}{2p(k)} e^{\left[\frac{1}{2} \log x - \log p \right] i} \geq \frac{1}{2p(k)} e^{\left(\frac{1}{2} - \frac{1}{2} \log x \right) i}.$$

Collecting all these, (9.6) gives

$$\frac{\varepsilon e^{1/4}}{\sqrt{x}} \sum_p a_1(p, l_1, l_1) \log p e^{-\frac{1}{2} \left(\log x - \log p \right)^2} \geq \frac{1}{2p(k)} e^{\left(\frac{1}{2} - \frac{1}{2} \log x \right) i} - \varepsilon \log k \left(\frac{\varepsilon e^{1/4}}{\sqrt{x}} \right) e^{\left(\frac{1}{2} - \frac{1}{2} \log x \right) i}.$$

If $c_k$ in (7.1) is sufficiently large, then (10.6) assumes the form

$$\sum_p a(p, l_1, l_1) \log p e^{-\frac{1}{2} \left(\log x - \log p \right)^2} \geq \frac{\varepsilon e^{1/4}}{\sqrt{x}} \left(\varepsilon e^{1/4} + 2c_k \log k \left(\varepsilon e^{1/4} + k \varepsilon \log x \right) \right) e^{1/4}.$$

Choosing (1)

$$b = 2 \frac{\log x}{r}$$

and making $c_k$ in (7.2) sufficiently large we have

$$r k > 25 \log k + \frac{1}{r} \log c_k > 25 \log k + \log \left(32 c_k^3 \right)$$

from which

$$k < \varepsilon e^{1/4}.$$ (2)

Here, since $k \geq 2$, we come to the restriction $r < \log x$.

and

$$4c_{12} \log k \leq \frac{1}{8k} e^{1/2}$$

easily follow. Thus, (10.7), (10.8) and (7.1) result, using also (5.1),

$$\sum_p a_1(p, l_1, l_1) \log p e^{-\frac{1}{2} \left(\log x - \log p \right)^2} \geq \frac{1}{8k} \sqrt{\frac{\pi}{2k}} \sqrt{x} > c_1 \log x, \quad \text{q.e.d.}$$

11. The proof of Theorem IV is more difficult. We start again from integral (9.3) with $b > 100$, with integer $r \geq 2$ instead of $r/4$, $l_i = 1$ and $l_i = l$, where $(l_i, b) = 1$; then (9.6) gives

$$\left| J_4 - \frac{\varepsilon e^{1/4}}{2 r} \sum_p a_1(p, l, 1) \log p e^{-\frac{1}{2} \left(\log x - \log p \right)^2} \right| \leq \frac{c_6}{2r} \varepsilon e^{1/2} \log x.$$ (11.1)

For $J_4$ we have again the representation (10.1); nevertheless, since now the truth of the Riemann-Hilbert conjecture is not supposed, the sum regarding the $q$'s must be replaced by $\sum q$, where the prime indicates that the summation extends to the $q$'s resp. $q/2$'s right from $H$. For the integral on the right the estimation (10.5) holds again. The second sum on the right of (10.1) is trivially absolutely $\leq \varepsilon e^{\left(\log x \right)}$ and the same holds for the third sum. All in all we have (a bit roughly)

$$\left| e^{\pi i} \sum_p a_1(p, l, 1) \log p e^{-\frac{1}{2} \left(\log x - \log p \right)^2} \right| \leq \frac{2r \sqrt{\pi}}{2} \sum_{10} \left| 1 - \chi \left(\varepsilon e^{1/4} \right) \right| \sum_{10} e^{1/2} \log x.$$ (11.2)

We shall estimate roughly the contribution of the $q$'s with

$$\left(\frac{\pi}{2} > \right. \ |\arg (\varepsilon + b)| \left. > \frac{\pi}{3} \right)$$

using (10.2). This is absolutely

$$\leq c_4 \sqrt{r} \sum_{r \leq x} \log (k \varepsilon) e^{-r \left(1 + \log x \varepsilon \right)}$$

$$\leq c_4 \sqrt{r} \sum_{r \leq x} \log (k \varepsilon) e^{-r \log x} < c_4 \varepsilon e^{-ab \log k}.$$
Hence from (11.2), we get

\[(11.3) \quad \sum_{\nu} e_{\nu}(p, \nu, \lambda) \phi(p) = -\zeta'(\lambda) + \frac{\pi}{\nu} \sum_{n=1}^{\lambda} \left(1 - \frac{1}{n}\right) \sum_{m=1}^{\lambda} \left(\frac{\phi(n)}{n}\right)
- \frac{2\pi}{\nu} \sum_{n=1}^{\lambda} \left(1 - \frac{1}{n}\right) \sum_{m=1}^{\lambda} \left(\frac{\phi(m)}{m}\right)\]
\[\leq c_{\nu} \frac{\pi}{\nu} e^{\nu + \nu^{1/2} \log \nu}.
\]

12. For the sum

\[(12.1) \quad \sum_{n=1}^{\lambda} \left(1 - \frac{1}{n}\right) \sum_{m=1}^{\lambda} \left(\frac{\phi(n)}{n}\right) = Z(\nu)\]

we shall give "large positive" lower bound, resp. "large negative" upper bound choosing appropriately \(\nu\) in Lemma 1; the fullfilledness of the critical argument-condition (4.2) will be secured by a proper choice of \(\nu\). The rôle of the \(e_{\nu}\)'s will obviously be played by the numbers \(e^{\phi + \phi \nu}\); hence putting

\[\phi = \phi_{\nu} = \frac{1}{\nu} \phi'\]

we have

\[e_{\nu} = 2\nu \phi + \phi' = 2\pi e_{\nu} + \frac{1}{\nu} \phi' = \phi(\nu)\]

We apply Lemma 2 with

\[\beta = \frac{1}{\nu} \phi', \quad \alpha = \frac{\nu}{\pi}\]

Then we choose

\[\gamma = \frac{11}{10}, \quad U = \frac{1}{\pi} E(k)\]

so that

\[V = c_{\nu} \log \nu\]

For the \(T\)'s in (5.5) we define \(\tau\) of Lemma 2 by

\[(12.2) \quad T = e^{\nu \tau}, \quad \tau = \log^{1/2} T\]

and choose \(A = \tau^{1/2}\). The restriction \((8.7)\) of this lemma is owing to (5.5) satisfied. Hence we may choose \(b = b_{\nu} = \nu\) of \(\xi\) of this lemma; thus for all \(j\)'s

\[2\pi e_{\nu} \log \nu \leq e_{\nu} \mod 2\pi = 2\pi e_{\nu} \log \nu \leq \frac{2\pi}{24\log \nu} \leq \frac{1}{1 + \frac{\nu}{\pi}}\]

Since from (11.3)

\[|e_{\nu}| \leq (\nu + \nu^{1/2})\]

and from (12.2) and (5.5)

\[(12.3) \quad \tau > e^{\nu^{1/2}},\]

we get, using also

\[(14.2) \quad \tau \leq b_{\nu} \leq \tau + \nu^{1/2},\]

for all \(e_{\nu}\)'s in \(Z(\nu)\) the estimation

\[\left(\pi \gg \frac{\pi}{2\nu} \right) \quad \frac{2\pi}{24\log \nu} \leq \frac{1}{\tau + \nu^{1/2}} \geq \nu^{1/2} \geq \nu^{1/2}\]

if \(\nu\) in (5.5) is sufficiently large. Hence we can choose as \(\kappa\) of Lemma 1

\[(12.5) \quad \kappa = \nu^{1/2}\]

13. Now we apply Lemma 1. The rôle of \(d_{\nu}\)'s will obviously be played by the numbers \(1 - \frac{1}{\nu} \phi\) and hence as \(D\) we can choose

\[(13.1) \quad D = 8k^{-1}\]

Owing to (10.2), (12.3) and (12.4), we have

\[n < c_{\nu} k \log (k \log \nu) < c_{\nu} \log^{2} \nu\]

and hence \(N\) can be chosen as

\[(13.2) \quad N = c_{\nu} \log^{2} \nu\]

As \(m\) of Lemma 1 we choose

\[(13.3) \quad m = \frac{e_{\nu}}{2b_{\nu}}\]

Then Lemma 1 gives the existence of integers \(r_{1}\) and \(r_{2}\) with

\[(13.4) \quad \frac{e_{\nu}}{2b_{\nu}} \leq r_{1}, \quad r_{2} \leq \frac{e_{\nu}}{2b_{\nu}} + c_{\nu} \log^{2} \nu\]

so that

\[(13.5) \quad Z(\nu) = \frac{c_{\nu}}{e_{\nu}} \log^{2} \nu \leq \frac{c_{\nu}}{e_{\nu}} \log^{2} \nu \leq e^{-\nu \log^{2} \nu}\]

choosing \(e_{\nu}\) in (5.5) sufficiently large) and

\[(13.6) \quad Z(\nu) < e^{-\nu \log^{2} \nu}\]
14. We have to give lower bounds for $|s_1|^{t_1}$ and $|s_1|^{t_2}$. The first is owing to (13.4) and (12.2)

\[ |s_1|^{t_1} > e^{t_1^2/2 \log b_{t_1}} > e^{t_1^2/2} > T^{t_1^2/2} > T^{t_1^2/2 \log b_{t_1}}. \]

and the same for $|s_1|^{t_2}$. This, (13.5), (12.2) and (11.3) give, putting

\[ e^{t_1^2/2} = x, \quad r = 1, 2, \]

the inequality

\[
\sum_p a_p(p, l, 1) \log p \cdot e^{-\frac{1}{8t_1} \log^2 \frac{p}{s_1}} > \frac{V_{t_1}}{k} \left( T^{t_1^2/2 + \frac{1}{8t_1} \log b_{t_1}} \cdot 2V_{t_1} - o_{\log^2 \frac{1}{8t_1} \log b_{t_1}} \right).
\]

(13.4), (12.3), (12.4) and (12.2) give

\[ c_9 e^{t_1^2/2} x_1^2 \log b_{t_1} < c_9 \log^2 x \cdot e^{t_1^2/2} \cdot e^{t_1^2/2 + \frac{1}{8t_1} \log b_{t_1}} < \sqrt{T} e^{2t_1^2/2 + \frac{1}{8t_1} \log b_{t_1}} \]

if $c_9$ is sufficiently large, i.e.

\[
\sum_p a_p(p, l, 1) \log p \cdot e^{-\frac{1}{8t_1} \log^2 \frac{p}{s_1}} > \frac{V_{t_1}}{k} \left( T^{t_1^2/2 + \frac{1}{8t_1} \log b_{t_1}} \right) \cdot 
\]

But owing to (5.5)

\[ (\beta_1 - \frac{1}{2}) \log T > (4 + \gamma_1^2) \log b_{t_1}, \]

taking also in account that

\[ V_{t_1} > \left( \frac{t_1^2}{\tau + \frac{1}{2}} \right)^{1/2} > \tau > k, \quad \text{i.e.} \quad \frac{V_{t_1}}{k} > 1, \]

we have

\[ \sum_p a_p(p, l, 1) \log p \cdot e^{-\frac{1}{8t_1} \log^2 \frac{p}{s_1}} > T^{t_1^2/2 + \frac{1}{8t_1} \log b_{t_1}} \]

and analogously

\[ \sum_p a_p(p, l, 1) \log p \cdot e^{-\frac{1}{8t_1} \log^2 \frac{p}{s_1}} < -T^{t_1^2/2 + \frac{1}{8t_1} \log b_{t_1}}. \]

Further, from (14.2), (13.4) and (12.2) we have

\[ x_1 > e^{t_1^2/2} = T \]

and

\[ x_1 < e^{t_1^2/2 + \frac{1}{8t_1} \log^2 b_{t_1}} \leq T e^{2t_1^2/2 + \frac{1}{8t_1} \log b_{t_1}} \]

indeed. Finally

\[ r_1 > \frac{t_1^2}{2t_1} > \frac{t_1^2}{2} \left( 1 - \frac{1}{\tau + r_1^2} \right) > \frac{1}{2} \tau - r_1^2 = \frac{1}{2} \log^2 T - \log^2 T \]

and

\[ r_1 < \frac{t_1^2}{2t_1} + r_1^2 < \frac{1}{2} \tau + r_1^2 = \frac{1}{2} \log^2 T + \log^2 T, \]

(5.6) is obviously shown too and the proof is complete.

Appendix

As remarked by G. G. Lorentz, the comparison of primes in the progressions $x = 1 \mod k$ and $x = 1 \mod k$ with $\varphi(k_1) = \varphi(k_2)$ leads to still more difficult problems. Here we shall make only such remarks which are immediate corollaries of our previous work. The simplest case is obviously

\[ k_1 = 3, \quad k_2 = 4. \]

We want to compare first the progressions $(3x + 1)$ and $(4x + 1)$. As easy to see we have

\[ \pi(x, 3, 1) - \pi(x, 4, 1) = \pi(x, 12, 7) - \pi(x, 12, 5) \]

and both 7 and 5 are quadratic non-residues mod 12. Since for the modulus 12 the Haselgrove-condition is satisfied, Theorem 1.1 of ours (see Knappowski-Turán [1]) gives massata matadis

Corollary 1. For $T > c_9$ we have

\[ x_1 > \frac{1}{T} \sqrt{\log T} \log T \]

and

\[ \min x_1 < x_1 < \frac{1}{T} \sqrt{\log T} \log T. \]

Comparing the progressions $(3x + 2)$ and $(4x + 1)$, we have evidently

\[ \pi(x, 3, 2) - \pi(x, 4, 1) = \pi(x, 12, 11) - \pi(x, 12, 1) + 1; \]

and

\[ \pi(x, 3, 1) - \pi(x, 4, 1) = \pi(x, 12, 7) - \pi(x, 12, 5). \]
since 1 resp. 11 are quadratic residues, resp. non-residues mod 12, this
case is different from the previous one. Nevertheless Theorem 5 of
our paper [2] leads to the following

**Corollary 2.** For \( T > \xi_0 \) we have

\[
\max_{\nu \in \mathbb{N}, \nu < T} \log \nu = (\pi(x, 3, 3) - \pi(x, 4, 1)) > 1
\]

and

\[
\min_{\nu \in \mathbb{N}, \nu < T} \log \nu = (\pi(x, 3, 2) - \pi(x, 4, 1)) < -1.
\]

For \( \pi(x, 4, 3) - \pi(x, 3, 2) \),
resp.
\( \pi(x, 4, 3) - \pi(x, 3, 1) \)
we have evidently the same behaviour.
These remarks settle the case with
\( \psi(k_1) = \psi(k_2) = 2 \).
The next case, when
\( \psi(k_1) = \psi(k_2) = 4 \)
(which is essentially only the case of \( k_1 = 5, k_2 = 8 \)) seems to be more
difficult; we hope to return to it later.

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