ACTA ARITHMETICA
IX (1964)

Further developments in the comparative prime-number theory I

by

S. KNAPOWSKI (Poznań) and P. TURÁN (Budapest)

1. In a report dated from June 19, 1871, which was written to support the designation of Chebyshev as foreign member of the Academy in Berlin and signed among others by Kronecker, Kummer, and Weierstrass, one reads the following passage (see [1]).

"...Endlich ist Herr Tschebychew der erste Mathematiker, welcher für die Anzahl der Primzahlen bis zu einer hohen Grenze den Ueberschuss der Primzahlen der Form $4n+3$ über diejenigen von der Form $4n+1$ constatiert und für den asymptotischen Ausdruck $\sqrt{x}/\log x$ angegeben hat."

What was behind these lines? Chebyshev wrote in a letter in 1853, i.e. a few years after Dirichlet proved that for $(k, l_1) = (k, l_2) = 1$ in a weak sense the number of primes $\equiv l_1 (\operatorname{mod} k)$ is asymptotically equal to that of the primes $\equiv l_2 (\operatorname{mod} k)$, that he is in possession of a theorem which can be popularly expressed so that there are more primes of the form $4n+3$ than of $4n+1$. He meant by that (according to his letter, which is printed in [2]) that

$$\lim_{x\to+0} \sum_{p>2} (-1)^{(p-1)/2} e^{-px} = -\infty \tag{1.1}$$

(p denoting always primes) and stated also the existence of a sequence

$$x_1 < x_2 < \ldots \to \infty$$

such that for $\nu \to \infty$

$$\frac{\pi(x_\nu, 4, 3) - \pi(x_\nu, 4, 1)}{(\sqrt{x_\nu}/\log x_\nu)} \to 1. \tag{1.2}$$

(Here and later $\pi(x, k, l)$ stands for the number of primes not exceeding x which are $\equiv l \operatorname{mod} k$, $(k, l) = 1$, $c_1, c_2, \ldots$ positive, explicitly calculable constants.) Most probably in Germany nobody read the original letter; only rumors came to Berlin with the natural distortions of the truth, and Chebyshev's scientific prestige was very large. The report shows anyway what an interest was aroused by these announcements and that the two tendencies in the theory of primes, to find uniformities resp. discrepancies in the distribution of primes in progressions $\operatorname{mod} k$, started approximately simultaneously. Calling the second named trend comparative prime number theory one can say that this theory—in contrast to the uniformity trend—was until recently in such a shape as if we knew on the distribution of primes only that there is an infinity of primes (and even less). We made a systematic study of these questions in a series of papers (see [3]) to remove this theory from the deadlock in which it apparently was and in this new series of papers we want to continue these investigations.

2. First of all we shall give a somewhat amplified systematization of the problems of this theory (far from exhausting them), compared to the previous one, given in I. of the sequence quoted in [3]. We use the standard notation, $\Lambda(n)$ for the Dirichlet-von Mangoldt symbol, further

$$\psi(x, k, l) = \sum_{\substack{n \leqslant x \\ n \equiv l(k)}} \Lambda(n), \tag{2.1}$$

$$\Pi(x, k, l) = \sum_{\substack{n \leqslant x \\ n \equiv l(k)}} \big(\Lambda(n)/\log n\big), \tag{2.2}$$

$$\theta(x, k, l) = \sum_{\substack{p \leqslant x \\ p \equiv l(k)}} \log p, \tag{2.3}$$

$$\pi(x, k, l) = \sum_{\substack{p \leqslant x \\ p \equiv l(k)}} 1, \tag{2.4}$$

further

$$\begin{aligned} \Delta_1(x, k, l_1, l_2) &= \psi(x, k, l_1) - \psi(x, k, l_2), \\ \Delta_2(x, k, l_1, l_2) &= \Pi(x, k, l_1) - \Pi(x, k, l_2), \\ \Delta_3(x, k, l_1, l_2) &= \theta(x, k, l_1) - \theta(x, k, l_2), \\ \Delta_4(x, k, l_1, l_2) &= \pi(x, k, l_1) - \pi(x, k, l_2). \end{aligned} \tag{2.5}$$

Then the problems of the theory (in first approximation) are the following ones.

Problems 1-4. "Infinity of sign changes."

To prove that the functions $\Delta_j(x, k, l_1, l_2)$ for $j = 1, 2, 3, 4$ and $l_1 \not\equiv l_2(k)$ change sign infinitely often.

Problems 5-8. "Infinity of big sign changes."

To prove that for each of the functions $\Delta_j(x, k, l_1, l_2)$ $(j = 1, 2, 3, 4$ and $l_1 \not\equiv l_2(k))$ and arbitrarily small $\varepsilon > 0$ there is a sequence

$$x_1 < x_2 < \ldots \to +\infty \tag{2.6}$$

such that, for $v = 1, 2, \ldots$,

$$\Delta_j(x_v, k, l_1, l_2) > x_v^{1/2-\varepsilon};$$

and hence owing to the symmetry of l_1, l_2 also a sequence

$$y_1 < y_2 < \ldots \to +\infty$$

such that

$$\Delta_j(y_v, k, l_1, l_2) < -y_v^{1/2-\varepsilon}.$$

Problems 9-12. "Localised sign changes."

To prove that for $T > T_0(k, j)$ and suitable $A(T) < T$ all functions $\Delta_j(x, k, l_1, l_2)$ change sign in the interval

$$A(T) \leqslant x \leqslant T.$$

Problems 13-16. "Localised big sign changes."

To prove that for $T > T_1(k, j)$ and suitable $A(T) < T$ for each function $\Delta_j(x, k, l_1, l_2)$ the inequalities

$$\max_{A(T) \leqslant x \leqslant T} \Delta_j(x, k, l_1, l_2 > \frac{T^{1/2}}{\Phi(T)}$$

and hence also

$$\min_{A(T) \leqslant x \leqslant T} \Delta_j(x, k, l_1, l_2) < -\frac{T^{1/2}}{\Phi(T)}$$

hold, with a $\Phi(x) > 0$, satisfying also

$$\lim_{x \to +\infty} \frac{\log \Phi(x)}{\log x} = 0. \tag{2.7}$$

Problems 17-20. "First sign change."

To determine for $j = 1, 2, 3, 4$ functions $A_j(k)$ (which are generally better than those given at the previous problems) such that for $1 \leqslant x \leqslant$ $\leqslant A_j(k)$ all

$$\Delta_j(x, k, l_1, l_2), \quad l_1 \not\equiv l_2(k), \ k \text{ fixed},$$

functions change sign at least once.

Problems 21-24. "Asymptotic estimation of the number of sign changes."

Clear.

PROBLEMS 25-28. "Average preponderance problems."

To mention a typical one, the results of Hardy-Landau-Littlewood indicate that the inequality

$$\pi(n,4,1)-\pi(n,4,3)<0 \tag{2.8}$$

is true "much more often" than the inequality

$$\pi(n,4,1)-\pi(n,4,3)\geqslant 0. \tag{2.9}$$

Hence denoting by $N(x)$ the number of indices $n\leqslant x$ with the property (2.9) probably the relation

$$\lim_{x\to+\infty}\frac{N(x)}{x}=0 \tag{2.10}$$

holds.

PROBLEMS 29-32. "Strongly localized accumulation problems."

In the previous problems in various ways the number of *all* primes $\leqslant x$ in a fixed progression occured. One can imagine that one can much better localize relatively small intervals where the primes of some progressions preponderate. Again instead of writing out generally the pertaining problems we confine ourselves to indicating the character of them by mentioning just one. Is it true for $T>c_1$ (c_1 numerically positive constant) that for suitable $T\leqslant U_1<U_2\leqslant 2T$ we have

$$\sum_{\substack{U_1\leqslant p\leqslant U_2\\ p\equiv 1(4)}}1-\sum_{\substack{U_1\leqslant p\leqslant U_2\\ p\equiv 3(4)}}1>\frac{\sqrt{T}}{\Phi(T)}$$

in the sense of (2.7)?

PROBLEMS 33-36. "Littlewood-generalizations."

A typical problem of this type would be the existence of a sequence $x_1<x_2<\ldots\to+\infty$ such that simultaneously the inequalities

$$\pi(x_\nu,4,1)\geqslant\tfrac{1}{2}\operatorname{Li}x_\nu=\tfrac{1}{2}\int_2^{x_\nu}\frac{du}{\log u} \tag{2.11}$$

and

$$\pi(x_\nu,4,3)\geqslant\tfrac{1}{2}\operatorname{Li}x_\nu \tag{2.12}$$

hold. This would constitute an obvious generalization of Littlewood's classical theorem that for a suitable sequence $y_1<y_2<\ldots\to+\infty$ the inequality

$$\pi(y_\nu)\geqslant\operatorname{Li}y_\nu$$

holds. Some further natural questions on the localization of the x_ν's in (2.11)-(2.12), we did not take up into this provisional list of problems.

PROBLEMS 37-40. "Racing problems."

Again only a sample of these problems: if $l_1,l_2,\ldots,l_{\varphi(k)}$ is any prescribed order of the reduced residue systems mod k then for a suitable sequence $x_1<x_2<\ldots\to+\infty$ the inequalities

$$\pi(x_\nu,k,l_1)\geqslant\pi(x_\nu,k,l_2)\geqslant\ldots\geqslant\pi(x_\nu,k,l_{\Phi(k)})$$

hold.

G. G. Lorentz called our attention to the fact that comparison of the primes of any two arithmetical progressions mod k_1 and k_2 ($k_1\neq k_2$) is not trivial in the case when

$$\varphi(k_1)=\varphi(k_2),$$

and analogous problems occur for moduli $k_1,k_2,\ldots,k_r$ with

$$\varphi(k_1)=\varphi(k_2)=\ldots=\varphi(k_r).$$

This leads straight to the following problems

PROBLEMS 41-44. "Union-problems".

A typical problem of this kind is the following: For a given modulus k do there exist two disjoint subsets A and B, consisting of the same number of residue-classes, such that

$$\sum_{p\in A,\,p\leqslant x}1>\sum_{p\in B,\,p\leqslant x}1 \tag{2.13}$$

for all sufficiently large x's?

Chebyshev's conjecture and the subsequent investigations of Hardy, Littlewood, and Landau (see [4] and [5]) give an importance to all these questions considered "in the Abelian sense." This gives the problems 45-88 replacing in the respective ones

$$\psi(x,k,l)\quad\text{by}\quad\sum_{n\equiv l(k)}\Lambda(n)e^{-nr},$$

$$\Pi(x,k,l)\quad\text{by}\quad\sum_{n\equiv l(k)}\frac{\Lambda(n)}{\log n}e^{-nr},$$

$$\theta(x,k,l)\quad\text{by}\quad\sum_{p\equiv l(k)}\log p\cdot e^{-pr},$$

$$\pi(x,k,l)\quad\text{by}\quad\sum_{p\equiv l(k)}e^{-pr},$$

and

$$\operatorname{Li}x\quad\text{by}\quad\int_2^\infty\frac{e^{-yr}}{\log y}\,dy.$$

Obviously comparing the primes represented by two quadratic forms f and g, respectively, we encounter problems of (2.13)-type again. Another set of interesting problems arises by comparing the primes represented by a fixed quadratic form in various angles (theorem of Dirichlet-Hecke).

All these problems have their natural analoga in algebraic number fields, replacing residue-classes by ideal-classes in various senses. The only result in this direction, as far as we know, is contained in a paper of Landau in Mathematische Zeitschrift 2 (1918), pp. 52-154. In all these problems our methods have still more definite advantages compared to the older ones, since they give more explicit dependence of the estimations in terms of the parameters of the field concerned.

3. In our previous papers we proved several theorems concerning problems 1-24; we had no results at the time for accumulation problems 29-32. In the first few papers of this series we shall be engaged with these problems; as far as we know they are the first results of their species. Contrary to our first series [3], which was written out in the possession of the whole material, this time we have it only for the first three or four papers of this new series.

First we state the following theorem (c_1 and later $c_2, \ldots$ stand for explicitly calculable positive constants).

If c_1 is sufficiently large, $3 \leqslant k \leqslant 10$ and $T > c_1$, then for all $(l, k) = 1$, $l \not\equiv 1(k)$, there are suitable U_1, U_2, U_3, U_4 numbers with

$$Te^{-\log^{11/12}T} \leqslant U_1 < U_2 \leqslant T$$

and

$$Te^{-\log^{11/12}T} \leqslant U_3 < U_4 \leqslant T$$

so that the inequalities

$$\sum_{\substack{n\equiv 1(k)\\ U_1\leqslant n\leqslant U_2}} \Lambda(n) - \sum_{\substack{n\equiv l(k)\\ U_1\leqslant n\leqslant U_2}} \Lambda(n) > \sqrt{T}e^{-\log^{11/12}T} \tag{3.1}$$

and

$$\sum_{\substack{n\equiv 1(k)\\ U_3\leqslant n\leqslant U_4}} \Lambda(n) - \sum_{\substack{n\equiv l(k)\\ U_3\leqslant n\leqslant U_4}} \Lambda(n) < -\sqrt{T}e^{-\log^{11/12}T} \tag{3.2}$$

hold.

In the case of a general modulus k, we can prove an analogous theorem only for such k's, for which the so-called Haselgrove condition holds (which is the case certainly for $3 \leqslant k \leqslant 10$). This condition requires the explicit value of an $E = E(k)$ such that no $L(s, \chi)$ functions belonging to $\bmod k$ and $s = \sigma + it$, can vanish in the domain

$$0 < \sigma < 1, \quad |t| \leqslant E(k). \tag{3.3}$$

That this condition is intrinsically connected with our subject, was already mentioned at the end of our first paper in series [3]; hence it would be a problem of high interest to prove that the number of such k's is infinite or even the number of those k's is infinite for which $\prod_{\chi} L(s, \chi)$ has no positive zeros. For the moduli k we can state the following theorem (1).

If c_2 is sufficiently large and

$$T > \max\big(c_2, e_2(k), e_2(1/E(k))\big) \tag{3.4}$$

then for all $(l, k) = 1$ there are suitable U_1, U_2, U_3, U_4 numbers with

$$Te^{-\log^{11/12}T} \leqslant U_1 < U_2 \leqslant T$$

and

$$Te^{-\log^{11/12}T} \leqslant U_3 < U_4 \leqslant T$$

so that the inequalities

$$\sum_{\substack{n\equiv 1(k)\\ U_1\leqslant n\leqslant U_2}} \Lambda(n) - \sum_{\substack{n\equiv l(k)\\ U_1\leqslant n\leqslant U_2}} \Lambda(n) > \sqrt{T}e^{-\log^{11/12}T}$$

and

$$\sum_{\substack{n\equiv 1(k)\\ U_3\leqslant n\leqslant U_4}} \Lambda(n) - \sum_{\substack{n\equiv l(k)\\ U_3\leqslant n\leqslant U_4}} \Lambda(n) < -\sqrt{T}e^{-\log^{11/12}T}$$

hold.

4. What we shall actually prove is a bit more. We assert the following theorem.

THEOREM. *Let k fulfill the Haselgrove condition, $(l, k) = 1$, and let*

$$\varrho_0 = \beta_0 + i\gamma_0,$$

be a zero of an $L(s, \chi^*)$ with $\chi^*(l) \neq 1$ and $\beta_0 \geqslant \frac{1}{2}$. Then with a sufficiently large c_3 for

$$T > \max\big(c_3, e_2(k), e_2(1/E(k)), e_2(|\varrho_0|)\big) \tag{4.1}$$

with suitable U_1, U_2, U_3, U_4 satisfying

$$Te^{-\log^{11/12}T} \leqslant U_1 < U_2 \leqslant T, \tag{4.2}$$

$$Te^{-\log^{11/12}T} \leqslant U_3 < U_4 \leqslant T \tag{4.3}$$

the inequalities

$$\sum_{\substack{n\equiv 1(k)\\ U_1\leqslant n\leqslant U_2}} \Lambda(n) - \sum_{\substack{n\equiv l(k)\\ U_1\leqslant n\leqslant U_2}} \Lambda(n) \geqslant T^{\beta_0}e^{-\log^{11/12}T} \tag{4.4}$$

(1) In what follows is $e_1(x) = e^x$, $e_\nu(x) = e_1(e_{\nu-1}(x))$, $\log_1 x = \log x$, $\log_{\nu+1} x = \log(\log_\nu x)$.

and

$$\sum_{\substack{n\equiv l(k)\\ U_3\leqslant n\leqslant U_4}} \Lambda(n) - \sum_{\substack{n\equiv l(k)\\ U_3\leqslant n\leqslant U_4}} \Lambda(n) \leqslant -T^{\beta_0} e^{-\log^{11/12} T} \tag{4.5}$$

hold.

This theorem contains the one given in section 3, e.g. owing to the following theorem of C. L. Siegel (see [6]).

If χ_1 is a primitive character belonging to the modulus k, then $L(s, \chi_1)$ has a zero $\varrho' = \sigma' + it'$ in

$$\tfrac{1}{2} \leqslant \sigma' < 1, \quad |t'| \leqslant c_4/\log_3(k+e^{100}). \tag{4.6}$$

5. In the series [3] the starting points were integrals of the form

$$\frac{1}{2\pi i}\int_{(2)} \frac{e^{xs}}{s^\nu} F(s)\,ds \tag{5.1}$$

with

$$F(s) = -\frac{1}{\varphi(k)}\sum_{\chi} \bar{\chi}(l) \frac{L'}{L}(s, \chi). \tag{5.2}$$

The essential novelty is that now we investigate integrals of the form

$$\frac{1}{2\pi i}\int_{(2)} e^{As}\left(\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^{\nu} F(s)\,ds \tag{5.3}$$

with suitable positive A, B and integer ν. The advantages of the kernel

$$e^{As}\left(\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^{\nu} \tag{5.4}$$

over the kernel

$$e^{xs}/s^\nu \tag{5.5}$$

were first realized for other aims by the second of us (see [7]). But for *one-sided* theorems it could be applied only recently, through a lemma, well known in calculus of probability as mentioned to us by Mr. Rényi and Mr. Kendall. This is the

Lemma I. *If $k \geqslant 2$ is an integer, then the function*

$$f_k(x) \overset{\text{def}}{=} \int_0^\infty \left(\frac{\sin t}{t}\right)^k \cos xt\,dt$$

decreases monotonically for $x \geqslant 0$.

We prefer to give for this lemma our first independent proof, based on induction with respect to k. For $k = 2$ this is a well-known property of the Fejér kernel. Suppose that it is true for a $k \geqslant 2$. Owing to the formula

$$f'_{k+1}(x) = -\tfrac{1}{2}\big(f_k(x-1) - f_k(x+1)\big) = -\tfrac{1}{2}\big(f_k(1-x) - f_k(1+x)\big)$$

applied for $x \geqslant 1$, resp. for $0 \leqslant x \leqslant 1$, the assertion follows indeed.

6. The basic tool for the proofs is again the following one-sided theorem which we state as Lemma II(2).

Lemma II. *Let $n \leqslant N$,*

$$|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n|,$$

and with an $0 < \varkappa \leqslant \frac{1}{2}\pi$

$$\varkappa \leqslant |\operatorname{arc} z_j| \leqslant \pi, \quad j = 1, 2, \ldots, n, \tag{6.1}$$

further for the b_j-numbers the restriction

$$D \overset{\text{def}}{=} \min_{\mu} \operatorname{Re}(b_1 + \ldots + b_\mu) > 0 \tag{6.2}$$

should hold. Then to each non-negative m there are integers ν_1 and ν_2 with

$$m \leqslant \nu_1, \nu_2 \leqslant m + N(3+\pi/\varkappa) \tag{6.3}$$

such that the inequalities

$$\operatorname{Re}\sum_{j=1}^{n} b_j z_j^{\nu_1} \geqslant D\frac{|z_1|^{\nu_1}}{2N+1}\cdot\left\{\frac{N}{24e\big(m+N(3+\pi/\varkappa)\big)}\right\}^{2N} \tag{6.4}$$

and

$$\operatorname{Re}\sum_{j=1}^{n} b_j z_j^{\nu_2} \leqslant -D\frac{|z_1|^{\nu_2}}{2N+1}\cdot\left\{\frac{N}{24e\big(m+N(3+\pi/\varkappa)\big)}\right\}^{2N} \tag{6.5}$$

hold.

7. We shall need two further lemmas(3). Let $\{x\}$ denote the fractional part of x, $\beta_1, \beta_2, \ldots$ arbitrary real numbers, further let $\alpha_1, \alpha_2, \ldots$ be real numbers for which

$$|\alpha_\nu| \geqslant \lambda, \quad 0 < \lambda \leqslant 1 \tag{7.1}$$

---

(2) For the proof of this lemma, see the paper [8]. We remark that Mr. S. Uchiyama succeeded in reducing the constant $24e$ to $8e$ and shortened the proof by one step using an ingenious remark (in a letter). (In (4.8) and (4.9) on p. 34f. l.c. the constant 24 is to be replaced by $(24e.)$

(3) In a special case and in a slightly different and weaker form this was proved in [9].

and for all real h-values we have

(7.2) $$\sum_{h\leqslant \alpha_\nu\leqslant h+1} 1 \leqslant \omega(|h|)$$

such that

(7.3) $$S \overset{\mathrm{def}}{=} \sum_{\nu=0}^{\infty} \omega(\nu)/(\nu^2+1) < \infty.$$

Then we assert the

LEMMA III. *For any* $c_5 < c_6$ *and*

(7.4) $$\tau > 1/(c_6-c_5)$$

here is a ξ_0 *with*

(7.5) $$c_5\tau \leqslant \xi_0 \leqslant c_6\tau$$

such that for all indices ν *the inequality*

(7.6) $$\frac{1}{100}\cdot\frac{\lambda}{S}\cdot\frac{1}{1+\alpha_\nu^2} \leqslant \{\alpha_\nu\xi_0+\beta_\nu\} \leqslant 1-\frac{1}{100}\cdot\frac{\lambda}{S}\cdot\frac{1}{1+\alpha_\nu^2}.$$

holds.

For the proof, first we fix the index ν and consider the linear form $f_\nu = \alpha_\nu x+\beta_\nu$. If x runs over the interval in (7.5) then f_ν passes at most

(7.7) $$1+(c_6-c_5)|\alpha_\nu|\tau$$

integers; add two more integers: the one immediately greater and the one immediately less, then call all them q's. Fixing any q, the x-values with

$$|\alpha_\nu x+\beta_\nu-q| < \frac{1}{100}\cdot\frac{\lambda}{S}\cdot\frac{1}{1+\alpha_\nu^2},$$

form an interval of length

$$\frac{1}{50}\cdot\frac{\lambda}{S}\cdot\frac{1}{1+\alpha_\nu^2}\cdot\frac{1}{|\alpha_\nu|};$$

the total length of these "bad"-intervals at a fixed ν cannot exceed, owing to (7.7),

$$\big(3+(c_6-c_5)|\alpha_\nu|\tau\big)\frac{1}{50}\cdot\frac{\lambda}{S}\cdot\frac{1}{1+\alpha_\nu^2}\cdot\frac{1}{|\alpha_\nu|}$$

and if ν runs over all positive integers the total measure of the set of "bad" x-values cannot exceed

(7.8) $$\frac{\lambda}{50S}\sum_{\nu=1}^{\infty}\frac{3+(c_6-c_5)|\alpha_\nu|\tau}{(1+\alpha_\nu^2)|\alpha_\nu|} = \frac{\lambda}{50S}\Big(\sum_{|\alpha_\nu|\leqslant 1}+\sum_{|\alpha_\nu|>1}\Big) \overset{\mathrm{def}}{=} \frac{\lambda}{50S}(S_1+S_2).$$

For S_1 we have from (7.1), (7.2), and (7.4)

(7.9) $$|S_1| \leqslant 3\,\frac{\omega(0)+\omega(1)}{\lambda}+(c_6-c_5)\tau\big(\omega(0)+\omega(1)\big)$$
$$< \frac{4}{\lambda}(c_6-c_5)\tau\big(\omega(0)+\omega(1)\big) < \frac{8}{\lambda}S(c_6-c_5)\tau.$$

For S_2 we have from (7.4) and (7.3)

$$S_2 \leqslant \sum_{|\alpha_\nu|\geqslant 1}\frac{3}{1+\alpha_\nu^2}+(c_6-c_5)\tau\sum_{|\alpha_\nu|\geqslant 1}\frac{1}{1+\alpha_\nu^2} < 4(c_6-c_5)\tau\sum_{|\alpha_\nu|\geqslant 1}\frac{1}{1+\alpha_\nu^2}$$
$$\leqslant 4(c_6-c_5)\tau\sum_{\nu=0}^{\infty}\frac{\omega(\nu)+\omega(\nu+1)}{1+\nu^2} < 16(c_6-c_5)\tau S \leqslant \frac{16}{\lambda}S(c_6-c_5)\tau.$$

From this (7.9) and (7.8), the measure of the "bad" set cannot exceed

$$\frac{\lambda}{50S}\cdot\frac{24}{\lambda}S(c_6-c_5)\tau < \frac{1}{2}(c_6-c_5)\tau$$

which completes the proof of Lemma III.

8. Finally we shall need the

LEMMA IV. *There exists a broken line* W *contained in the vertical strip* $\frac{1}{200}\leqslant\sigma\leqslant\frac{1}{100}$, *say, consisting alternately of horizontal and vertical segments such that*

a) *any horizontal strip of width* 1 *contains at most one of the horizontal segments,*

b) *on* W *the inequality*

$$\left|\frac{L'}{L}(s,\chi)\right| \leqslant c_7 k\log^2 k(2+|t|),$$

holds for all characters χ.

The routine-proof of this lemma follows exactly the lines of one given in Appendix III of the book [10] and can be omitted. Introducing the function

(8.1) $$F(s) = -\frac{1}{\varphi(k)}\sum_{\chi}\big(1-\bar\chi(l)\big)\frac{L'}{L}(s,\chi)$$

this lemma gives that on W the inequality

(8.2) $$|F(s)| \leqslant c_8 k\log^2 k(2+|t|)$$

holds.

9. Now we can turn to the proof of our theorem. Let

$$(9.1)\qquad \vartheta_1 = \frac{15}{113}, \quad \vartheta_4 = \frac{98}{339}\left(=\frac{1-\vartheta_1}{3}\right),$$

further ϑ_3 such that

$$(9.2)\qquad \frac{1-4\vartheta_1}{3} > \frac{51}{98}\vartheta_3 > \frac{51}{98}\cdot\frac{1}{0.47}\vartheta_1$$

(that goes) and

$$(9.3)\qquad \vartheta_2 = \vartheta_3 - 10^{-100}.$$

With these notations, let

$$(9.4)\qquad B \overset{\text{def}}{=} \log^{-\vartheta_1} T,$$

the positive A and integer r restricted in this minute only by

$$(9.5)\qquad \vartheta_2 \log_2 T \leqslant A \leqslant \vartheta_3 \log_2 T,$$

$$(9.6)\qquad (3 \leqslant)\ 0.99\frac{\log T}{A+B} \leqslant r \leqslant \frac{\log T}{A+B}.$$

We start from the integral

$$(9.7)\qquad J_r = \frac{1}{2\pi i}\int_{(2)} F(s)\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r ds.$$

Replacing $F(s)$ by its Dirichlet-series the reasoning of [7] gives

$$(9.8)\qquad J_r = \sum_{n\equiv 1(k)}{}' \Lambda(n) f(n) - \sum_{n\equiv l(k)}{}' \Lambda(n) f(n)$$

where the summation refers on both places to

$$(9.9)\qquad e_1\big((A-B)r\big) \leqslant n \leqslant e_1\big((A+B)r\big)$$

and

$$f(n) = \frac{1}{2\pi i}\int_{(2)}\left(\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r e^{(Ar-\log n)s} ds.$$

Shifting the line of integration to the line $\sigma = 0$ we get also

$$(9.10)\qquad f(n) = \frac{1}{\pi}\int_0^\infty \left(\frac{\sin Bt}{Bt}\right)^r \cos(Ar - \log n)t\, dt.$$

This makes sure that J_r is *real*. Introducing $\psi_l(x)$ by

$$(9.11)\qquad \psi_l(x) = \sum_{\substack{n\equiv 1(k)\\ n\leqslant x}} \Lambda(n) - \sum_{\substack{n\equiv l(k)\\ n\leqslant x}} \Lambda(n)$$

(9.8) can be written as

$$J_r = \int_{e_1((A-B)r)}^{e_1((A+B)r)} f(x)\, d\psi_l(x) = -\int_{e_1((A-B)r)}^{e_1((A+B)r)} \psi_l(x) f'(x)\, dx.$$

Introducing the new variable

$$x = e_1\big((A-y)r\big)$$

this gives

$$(9.12)\qquad J_r = \int_{-B}^{B} \psi_l(e^{(A-y)r})\frac{df(e^{(A-y)r})}{dy}\, dy = \int_{-B}^{B} \psi_l(e^{(A-y)r})\, V'(y)\, dy$$

where

$$V(y) = \frac{1}{\pi}\int_0^\infty \left(\frac{\sin Bt}{Bt}\right)^r \cos ryt\, dt = \frac{1}{\pi B}\int_0^\infty \left(\frac{\sin t}{t}\right)^r \cos\frac{ry}{B} t\, dt.$$

Since $V'(y)$ is obviously an odd function of y, (9.9) gives

$$J_r = \int_0^B \{\psi_l(e^{(A+y)r}) - \psi_l(e^{(A-y)r})\}\big(-V'(y)\big)\, dy$$

or, owing to Lemma I,

$$(9.13)\qquad J_r = \int_0^B \{\psi_l(e^{(A+y)r}) - \psi_l(e^{(A-y)r})\}\, |V'(y)|\, dy.$$

Denoting shortly

$$(9.14)\qquad \begin{aligned} &\max_{e^{(A-B)r}\leqslant U_1 < U_2 \leqslant e^{(A+B)r}} \{\psi_l(U_2) - \psi_l(U_1)\} \quad \text{by} \quad M_1,\\ &\min_{e^{(A-B)r}\leqslant U_3 < U_4 \leqslant e^{(A+B)r}} \{\psi_l(U_4) - \psi_l(U_3)\} \quad \text{by} \quad M_2 \end{aligned}$$

we get from (9.13) and Lemma I

$$(9.15)\qquad J_r \leqslant M_1 \int_0^B |V'(y)|\, dy = M_1 V(0) = M_1 \frac{1}{\pi B}\int_0^\infty \left(\frac{\sin t}{t}\right)^r dt$$

and analogously,

$$(9.16)\qquad J_r \geqslant M_2 \frac{1}{\pi B}\int_0^\infty \left(\frac{\sin t}{t}\right)^r dt.$$

10. Now let us consider J_r in the form (9.7). Shifting the line of integration to the broken line of Lemma IV we obtain, since J_r is *real*,

$$J_r = -\frac{1}{\varphi(k)}\operatorname{Re}\sum_{\chi}(1-\overline{\chi}(l))\sum_{\varrho(\chi)}{}'\left(e^{A\varrho}\frac{e^{B\varrho}-e^{-B\varrho}}{2B\varrho}\right)^r + \tag{10.1}$$

$$+\operatorname{Re}\left(\frac{1}{2\pi i}\int_W F(s)\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r ds\right),$$

where the dash means that the summation has to be extended only to such zeros of the L-functions which are right to W (and, of course, are zeros of such L-functions for which $\chi(l)\neq 1$). The last integral is owing to (8.2) absolutely

$$< c_9 k\log^2 k\cdot e^{r(A+B)/100}(200/B)^r. \tag{10.2}$$

The last factor is, owing to (9.4), (9.6), (9.5), (9.2) and (9.3),

$$< (200)^{\log T/\vartheta_2\log_2 T}\, T^{\vartheta_1/\vartheta_2} < T^{0.48},$$

if c_3 in (4.1) is sufficiently large. Further, using (9.6) resp. (4.1), we have

$$e^{r(A+B)/100} \leqslant T^{0.01},$$

resp.

$$k\log^2 k < (\log_2 T)^2,$$

and hence, if c_3 in (4.1) is sufficiently large, the expression in (10.2) is

$$< T^{0.491}. \tag{10.3}$$

Next, writing the ϱ-zeros in the form

$$\varrho = \sigma_\varrho + it_\varrho$$

we consider the contribution of zeros with

$$|t_\varrho| \geqslant \log^{\vartheta_4} T. \tag{10.4}$$

Since the total number of zeros with $\lambda\leqslant t_\varrho\leqslant\lambda+1$ cannot exceed as well known

$$c_{10}k\log k(2+|\lambda|), \tag{10.5}$$

this contribution cannot exceed

$$c_{11}k\log kT\frac{e^{(A+B)r}}{(B\log^{\vartheta_4}T)^r} \leqslant \frac{T(\log T)^2}{(\log^{\vartheta_4-\vartheta_1}T)^r} \tag{10.6}$$

if c_3 is sufficiently large, owing to (4.1), (9.4) and (9.6). Using the first half of (9.6) and also (9.5), the quantity in (10.6) is in turn for sufficiently large c_3

$$< T^{(1-(\vartheta_4-\vartheta_1))0.98/\vartheta_3}(\log T)^2 < T^{0.495}, \tag{10.7}$$

owing to (9.2) and (9.1).

With the ϱ_0 of the theorem we may write the remaining sum in (10.1) as

$$-e^{Ar\beta_0}\left|\frac{e^{B\varrho_0}-e^{-B\varrho_0}}{2\beta\varrho_0}\right|^r\operatorname{Re}\left\{\sum_{\substack{\chi\\ \chi(l)\neq 1}}\frac{1-\overline{\chi}(l)}{\varphi(k)}\sum_{\varrho(\chi)}{}''\left(e^{A(\varrho-\beta_0)}\frac{e^{B\varrho}-e^{-B\varrho}}{|e^{B\varrho_0}-e^{-B\varrho_0}|}\cdot\frac{|\varrho_0|}{\varrho}\right)^r\right\}$$

$$\stackrel{\text{def}}{=} -e^{Ar\beta_0}\left|\frac{e^{B\varrho_0}-e^{-B\varrho_0}}{2B\varrho_0}\right|^r\operatorname{Re}Z(r),$$

where $\sum''$ means that the summation is extended to those ϱ's, right to W, for which the restriction

$$|t_\varrho| < \log^{\vartheta_4} T \tag{10.8}$$

holds. Then collecting all these we have

$$\left|J_r + e^{Ar\beta_0}\left|\frac{e^{B\varrho_0}-e^{-B\varrho_0}}{2B\varrho_0}\right|^r\operatorname{Re}Z(r)\right| \leqslant 2T^{0.495}. \tag{10.9}$$

11. $Z(r)$ is obviously of generalized power-sum-type and we shall try to apply Lemma II. The role of the b_j's are obviously played by the quantities $\dfrac{1-\chi(l)}{\varphi(k)}$ and hence for the D in this lemma we have

$$D \geqslant \frac{1-\cos(2\pi/\varphi(k))}{\varphi(k)} > \frac{1}{k^3} > (\log_2 T)^{-3} \tag{11.1}$$

owing to (4.1). Owing to (4.1) we have

$$|\gamma_0| \leqslant |\varrho_0| < \log^{\vartheta_4} T;$$

hence $\varrho=\varrho_0$ occurs among our ϱ's and thus

$$|z_1| \geqslant 1 \tag{11.2}$$

(the role of z_j's are obviously played by the

$$e^{A(\varrho-\beta_0)}\frac{e^{B\varrho}-e^{-B\varrho}}{|e^{B\varrho_0}-e^{-B\varrho_0}|}\cdot\frac{|\varrho_0|}{\varrho}$$

numbers). The arguments of these numbers are obviously given by

$$t_\varrho A + \arg\left(\frac{e^{B\varrho}-e^{-B\varrho}}{\varrho}\right) = 2\pi\left\{\frac{t_\varrho}{2\pi}A + \frac{1}{2\pi}\arg\left(\frac{e^{B\varrho}-e^{-B\varrho}}{\varrho}\right)\right\},$$

A was so far restricted only by (9.5); now we shall determine it by applying Lemma III. The role of the a's in this lemma is obviously played by the $\frac{t\varrho}{2\pi}$'s, that of the β's by the

$$\frac{1}{2\pi}\arg\left(\frac{e^{B\varrho}-e^{-B\varrho}}{\varrho}\right)$$

numbers; as λ we can choose $\frac{1}{2\pi}E(k)$, as τ the number $\log_2 T$, further $c_5=\vartheta_2$, $c_6=\vartheta_3$, and as A we choose the ξ_0 of this lemma. As $\omega(x)$ we can choose, owing to (10.5), the function

$$7c_{10}k\log k(8+2\pi x),$$

and hence as S of this lemma we can choose $c_{11}k\log k$. Then Lemma III assures the choice of A such that for all z_j's

$$|\operatorname{arc} z_j|\geqslant 2\pi\cdot\frac{1}{100}\cdot\frac{E(k)}{2\pi}\cdot\frac{1}{c_{11}k\log k}\cdot\frac{1}{1+t_\varrho^2},$$

and hence owing to (10.8)

$$\varkappa\geqslant c_{12}\frac{E(k)}{k\log k}\log^{-2\vartheta_4}T.$$

Taking in account (4.1), we obtain that as $\varkappa$ we might choose

$$\log^{-2\vartheta_4}T(\log_2 T)^{-3}. \tag{11.3}$$

As N we can choose (if c_3 in (4.1) is sufficiently large)

$$\log^{\vartheta_4}T(\log_2 T)^3 \tag{11.4}$$

owing to (10.5), (10.8) and (4.1). Hence (if c_3 is large enough)

$$N\left(3+\frac{\pi}{\varkappa}\right)<\log^{3\vartheta_4}T(\log_2 T)^7; \tag{11.5}$$

remarking that

$$3\vartheta_4=\tfrac{98}{113}<1,$$

we choose

$$m=\frac{\log T}{A+B}-\log^{3\vartheta_4}T(\log_2 T)^7 \tag{11.6}$$

(this is large positive owing to (9.5)). Choosing r as ν_1 and ν_2 of Lemma II, the requirement (9.6) is certainly fulfilled; hence for sufficiently large c_3

$$\operatorname{Re} Z(\nu_1)>(\log_2 T)^{-3}\frac{1}{\log T}\left(\frac{A+B}{24e\log T}\right)^{2\log^{\vartheta_4}T(\log_2 T)^3}>e^{-\log^{\vartheta_4}T(\log_2 T)^5}, \tag{11.7}$$

and analogously

$$\operatorname{Re} Z(\nu_2)<-e^{-\log^{\vartheta_4}T(\log_2 T)^5}. \tag{11.8}$$

12. Next give a lower bound for

$$e^{Ar\beta_0}\left|\frac{e^{B\varrho_0}-e^{-B\varrho_0}}{2B\varrho_0}\right|^r. \tag{12.1}$$

Writing the first factor in the form

$$(e^{(A+B)r})^{\beta_0}e^{-Br\beta_0},$$

and since, for r, Lemma II gives

$$r\geqslant\frac{\log T}{A+B}-\log^{3\vartheta_4}T(\log_2 T)^7, \tag{12.2}$$

we obtain for it the lower bound

$$T^{\beta_0}e^{-\log^{3\vartheta_4}T(\log_2 T)^8-\log^{1-\vartheta_1}T}, \tag{12.3}$$

using also (9.4). As to the second factor in (12.1), we have

$$\left|\frac{e^{B\varrho_0}-e^{-B\varrho_0}}{2B\varrho_0}\right|^r\geqslant\left(1-\sum_{\nu=1}^{\infty}\frac{(B|\varrho_0|)^{2\nu}}{(2\nu+1)!}\right)^r$$

and, since from (4.1)

$$B|\varrho_0|=\frac{|\varrho_0|}{\log^{\vartheta_1}T}\leqslant\frac{\log_2 T}{\log^{\vartheta_1}T},$$

for sufficiently large c_3, this is

$$\geqslant e^{-c_{13}\log^{1-\vartheta_1}T}.$$

This and (12.3) give for the quantity in (12.1) for sufficiently large c_3 the lower bound

$$T^{\beta_0}e^{-c_{14}\{\log^{3\vartheta_4}T(\log_2 T)^7+\log^{1-\vartheta_1}T\}}.$$

Collecting this, (11.7), (11.8), and (10.9) we get, for sufficiently large c_3,

$$J_{\nu_2}>T^{\beta_0}e^{-c_{15}(\log^{1-\vartheta_1}T+\log^{3\vartheta_4}T(\log_2 T)^8)}>T^{\beta_0}e^{-\frac{1}{2}\log^{11/12}T} \tag{12.4}$$

and

$$J_{\nu_1}<-T^{\beta_0}e^{-\frac{1}{2}\log^{11/12}T}, \tag{12.5}$$

using (9.1).

Now we shall use (9.15) with $r = \nu_2$ remarking that

$$\int_0^{\infty} \left(\frac{\sin t}{t}\right)^{\nu_2} dt < 1 + \int_1^{\infty} \frac{dt}{t^{\nu_2}} < \pi;$$

using also (9.15), we get

$$M_1 \geqslant J_{\nu_2} B > T^{\beta_0} e_1(-\log^{11/12} T) \tag{12.6}$$

and analogously

$$M_2 \leqslant -T^{\beta_0} e_1(-\log^{11/12} T). \tag{12.7}$$

What can be said about the interval which contains (U_1, U_2), resp. (U_3, U_4)? Since

$$e^{(A+B)r} \leqslant T$$

and, from (12.2),

$$e^{(A-B)r} = e^{(A+B)r} e^{-2Br} \geqslant Te^{-2\vartheta_3 \log^{3\vartheta_4} T (\log_2 T)^8 - 2\log^{1-\vartheta_1} T} > Te^{-\log^{11/12} T}. \quad \text{Q.E.D.}$$

References

[1] Kurt R. Biermann, *Vorschlage zur Wahl von Mathematikern in die Berliner Akad*, Abhandl. der deutschen Akad. der Wiss. Berlin, Jahrg. 1960, No. 3, (1960), p. 43.

[2] *Lettre de M. le professeur Tchébychev a M. Fuss sur un nouveau théoreme relatif aus nombres premiers contenus dans les formes* $4n+1$ *et* $4n+3$, Bull. de la classe phys.-math. de l'Acad. Imp. St. Petersburg (1853), p. 208.

[3] S. Knapowski and P. Turán, *Comparative prime number theory I-VIII*, Partly appeared in Acta Math. Hung. P. XIII. Fasc. 3-4 (1962), pp. 299-364 and partly in press here.

[4] G. H. Hardy and F. E. Littlewood, *Contributions to the theory of Riemann zeta-function and the theory of the distribution of primes*, Acta Math. (1918), pp. 119-196.

[5] E. Landau, *Über einige ältere Vermutungen und Behauptungen in der Primzahltheorie, I-II*, Math. Zeitschr. (1918), pp. 1-24, resp. 213-219.

[6] C. L. Siegel, *On the zeros of the Dirichlet L-functions*, Annals of Math. (1945), pp. 409-422.

[7] P. Turán, *On the so-called density-hypothesis in the theory of zeta-function of Riemann*, Acta Arith. 4 (1958), pp. 31-56.

[8] — *On some further one-sides theorems of new type in the theory of diophantine approximations*, Acta Math. Hung. 12 (1961), pp. 455-468.

[9] S. Knapowski, *On the sign-changes in the remainder term in the prime-number formula*, Journ. of Lond. Math. Soc. (1961), pp. 451-460.

[10] P. Turán, *Eine neue Methode in der Analysis und deren Anwendungen*, Akad. Kiado, 1953, Budapest. A completely rewritten new edition will appear in the Interscience Tracts series.

Reçu par la Rédaction le 1. 4. 1963

ACTA ARITHMETICA
IX (1964)

The star number of coverings of space with convex bodies

by

P. Erdös and C. A. Rogers (London)

In honour of Professor L. J. Mordell

1. When a system of sets covers a space, the star number of the covering is the supremum over the sets of the system of the cardinals of the numbers of sets of the system meeting a set of the system. The standard Lebesgue 'brick-laying' construction provides an example, for each positive integer n, of a lattice covering of R^n by closed rectangular parallelepipeds with star number $2^{n+1}-1$. In view of the results of dimension theory, it is natural to conjecture that any covering of R^n by closed sets of uniformly bounded diameter has star number at least $2^{n+1}-1$; and this has been proved by V. Boltyanskiĭ [1] in the special case $n = 2$.

In this paper we consider only coverings of R^n by translates of a fixed convex body. We first give a simple proof (the idea of which comes from the work of Minkowski and Voronoï) of

THEOREM 1. *The star number of a lattice covering of* R^n *by translates of a closed symmetrical convex body is at least* $2^{n+1}-1$.

Then we consider the problem of constructing coverings of R^n by translates of a given closed convex body K with as small a star number as possible. By a minor modification of method we used in [2] we prove

THEOREM 2. *Provided* n *is sufficiently large, if* K *is a closed convex body in* R^n *with difference body* DK, *there is a covering of* R^n *by translates of* K *with star number less than*

$$\frac{V(DK)}{V(K)} \{n \log n + n \log\log n + 4n + 1\}.$$

Here the ratio of the volumes $V(DK)/V(K)$ *is at most* $\binom{2n}{n}$, *in general, and is equal to* 2^n *if* K *is symmetric.*

We can neither prove that general coverings by translates of a closed convex body must have a large star number, nor show that lattice