COMPARATIVE PRIME-NUMBER THEORY. VIII

(CHEBYSHEV'S PROBLEM FOR $k=8$)

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1. As mentioned in Paper I of this series, HARDY—LITTLEWOOD and LANDAU proved that[1]

$$(1.1)\qquad \lim_{x\to+0}\Big(\sum_{p\equiv 1 \bmod 4} e^{-px}-\sum_{p\equiv 3 \bmod 4} e^{-px}\Big)=-\infty$$

if and only if the function

$$(1.2)\qquad \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s},\quad s=\sigma+it$$

does not vanish for $\sigma>\frac{1}{2}$ (see HARDY—LITTLEWOOD [1], LANDAU [1], [2]). Shortly, (1.2) is a necessary and sufficient condition in order to announce that „there are more primes $\equiv 3 \bmod 4$ than $\equiv 1 \bmod 4$, in Abel-summability sense". They proved[2] also that the same holds replacing (1.1) by

$$(1.3)\qquad \lim_{x\to+0}\Big(\sum_{p\equiv 1 \bmod 4} \log p\cdot e^{-px}-\sum_{p\equiv 3 \bmod 4} \log p\cdot e^{-px}\Big)=-\infty.$$

Since the primes, apart from 2, are all $\equiv \pm 1 \bmod 4$, these two assertions cover the „rough" comparative theory for $k=4$. The corresponding results for $k=3$ could have been proved mutatis mutandis. As mentioned in Paper I of this series, in the next-difficult case $k=8$ a new phenomenon occurs; e. g. the function

$$(1.4)\qquad \sum_{p\equiv 3 \bmod 8} \log p\cdot e^{-px}-\sum_{p\equiv 5 \bmod 8} \log p\cdot e^{-px}$$

changes his sign infinitely often when $x\to+0$. In this paper we are going to get a deeper insight into the oscillatory character of the functions ($l_1\not\equiv l_2 \bmod 8$)

$$(1.5)\qquad f_{l_1 l_2}(x)\overset{\text{def}}{=}\sum_{p\equiv l_1 \bmod 8} \log p\cdot e^{-px}-\sum_{p\equiv l_2 \bmod 8} \log p\cdot e^{-px}$$

for $x\to+0$. We assert the

[1] p is throughout reserved for primes, $c_1, c_2,\ldots$ denote always positive, explicitly calculable numerical constants. $e_1(x)$ means e^x and $e_\nu(x)=e_1(e_{\nu-1}(x))$, further $\log_1 x=\log x$ and $\log_\nu x=$ $=\log(\log_{\nu-1}x)$.

[2] The necessity part does not occur explicitly but Landau's proof for the necessity of (1.2) for (1.1) can be used mutatis mutandis.

THEOREM 1. 1. *If* $0<\delta<c_1$, *then for* $l_1 \not\equiv l_2 \not\equiv 1 \bmod 8$ *the inequality*

$$\max_{\delta \leq x \leq \delta^{1/3}} \{ \sum_{p \equiv l_1 \bmod 8} \log p \cdot e^{-px} - \sum_{p \equiv l_2 \bmod 8} \log p \cdot e^{-px} \} >$$

$$> \frac{1}{\sqrt{\delta}} e_1 \left(-22 \frac{\log \frac{1}{\delta} \cdot \log_3 \frac{1}{\delta}}{\log_2 \frac{1}{\delta}} \right)$$

holds without any conjectures.

Since l_1 and l_2 can be changed, this gives automatically that the inequality

$$\min_{\delta \leq x \leq \delta^{1/3}} \{ \sum_{p \equiv l_1 \bmod 8} \log p \cdot e^{-px} - \sum_{p \equiv l_2 \bmod 8} \log p \cdot e^{-px} \} < \tag{1.6}$$

$$< -\frac{1}{\sqrt{\delta}} e_1 \cdot \left(-22 \frac{\log \frac{1}{\delta} \cdot \log_3 \frac{1}{\delta}}{\log_2 \frac{1}{\delta}} \right)$$

holds too without conjectures.

For the case $l_1 = 1$ we assert the

THEOREM 1. 2. *If for an* $l \not\equiv 1 \bmod 8$

$$\lim_{x \to +0} \{ \sum_{p \equiv 1 \bmod 8} \log p \cdot e^{-px} - \sum_{p \equiv l \bmod 8} \log p \cdot e^{-px} \} = -\infty$$

then no $L(s, \chi)$*-function* mod 8 *with* $\chi(l) \neq 1$ *can vanish for* $\sigma > \frac{1}{2}$.

Further we assert the

THEOREM 1. 3. *If no* $L(s, \chi)$ *functions* mod 8 *with* $\chi \neq \chi_0$ *vanish for* $\sigma > \frac{1}{2}$ *then for all* $l \not\equiv 1 \bmod 8$ *we have*

$$\lim_{x \to +0} \{ \sum_{p \equiv 1 \bmod 8} \log p \cdot e^{-px} - \sum_{p \equiv l \bmod 8} \log p \cdot e^{-px} \} = -\infty.$$

Theorem 1. 2 will follow by a slight modification of an argument of Landau (see LANDAU [1]) and Theorem 1. 3 by slight modification of Hardy—Littlewood—Landau's argument (see HARDY—LITTLEWOOD [1], LANDAU [2]). Theorem 1. 1 is of course deeper. Since the congruences

$$x^2 \equiv l \bmod 8 \quad l = 3, 5, 7 \tag{1.7}$$

are not solvable and evidently

$$\max_{\delta \leq x \leq \delta^{1/3}} \sum_{\substack{p, \nu \\ \nu \geq 3}} \log p \cdot e^{-p^\nu x} = O\left(\frac{1}{\delta^{1/3}} \log^2 \frac{1}{\delta} \right). \tag{1.8}$$

Theorem 1. 1 is equivalent to the inequality

$$(1.9)\qquad \max_{\delta\leqq x\leqq\delta^{1/3}}\Big\{\sum_{n\equiv l_1 \bmod 8}\Lambda(n)e^{-nx}-\sum_{n\equiv l_2 \bmod 8}\Lambda(n)e^{-nx}\Big\}>$$

$$>\frac{1}{\sqrt{\delta}}e_1\left(-22\frac{\log\frac{1}{\delta}\cdot\log_3\frac{1}{\delta}}{\log_2\frac{1}{\delta}}\right).$$

We shall prove Theorem 1. 1 in this form.

What can be said on the Chebyshev functions

$$(1.10)\qquad \sum_{p\equiv l_1 \bmod 8}e^{-px}-\sum_{p\equiv l_2 \bmod 8}e^{-px}\,?$$

The analogon of Theorem 1. 2 follows by Landau's argument (LANDAU [1]) mutatis mutandis and can be omitted. The analogon of Theorem 1. 3 follows from Theorem 1. 3 at once, combining it with the following lemma of Hardy—Littlewood—Landau (see LANDAU [2]):

If

$$\varlimsup_{x\to+0} x^{\frac{1}{2}}\Big\{\sum_{p\equiv l \bmod 8}\log p\cdot e^{-px}-\sum_{p\equiv 1 \bmod 8}\log p\cdot e^{-px}\Big\}>0,$$

then also

$$(1.11)\qquad \varlimsup_{x\to+0} x^{\frac{1}{2}}\log\frac{1}{x}\Big\{\sum_{p\equiv l \bmod 8}e^{-px}-\sum_{p\equiv 1 \bmod 8}e^{-px}\Big\}>0.$$

Hence we shall not go into details. However the analogon of Theorem 1. 1 encounters difficulties; we shall return to it at another occasion*.

The proof of Theorem 1. 2 gives at once that if an $L(s,\chi^*)$ has a zero $\varrho^*=\sigma^*+it^*$ with $\sigma^*>\frac{1}{2}$, then for each l with $\chi^*(l)\neq 1$ each of the functions $f_{1l}(x)$ changes his sign infinitely often as $x\to+0$, even

$$\varlimsup_{x\to+0} f_{1l}(x)x^{-\sigma^*+\varepsilon}=+\infty$$

$$\varliminf_{x\to+0} f_{1l}(x)x^{-\sigma^*+\varepsilon}=-\infty.$$

However we cannot prove at present anything for this case concerning the localisation of sign-changes. But as to the function

$$(1.12)\qquad \sum_{n\equiv 1 \bmod 8}\Lambda(n)e^{-nx}-\sum_{n\equiv l \bmod 8}\Lambda(n)e^{-nx}$$

we can assert, in full analogy with (1. 9) the

* *Added in proof* (Nov. 2. 1963). Since then we proved among others the following "two-sided" theorem. If l_1 and l_2 are any two different among 3, 5 and 7, we have for $0<\delta<c$ the inequality

$$\max_{\delta\leqq x\leqq\delta^{1/3}}\Big|\sum_{p\equiv l_1 \bmod 8}e^{-px}-\sum_{p\equiv l_2 \bmod 8}e^{-px}\Big|>\frac{1}{\sqrt{\delta}}e^{-41\frac{\log\frac{1}{\delta}\log_3\frac{1}{\delta}}{\log_2\frac{1}{\delta}}}.$$

THEOREM 1. 4. *If* $0<\delta<c_1$, *then for* $l\not\equiv 1 \bmod 8$ *the inequalities*

$$\max_{\delta\le x\le\delta^{1/3}}\Big\{\sum_{n\equiv 1 \bmod 8}\Lambda(n)e^{-nx}-\sum_{n\equiv l \bmod 8}\Lambda(n)e^{-nx}\Big\}>\frac{1}{\sqrt{\delta}}e_1\left(-22\frac{\log\frac{1}{\delta}\log_3\frac{1}{\delta}}{\log_2\frac{1}{\delta}}\right)$$

and

$$\min_{\delta\le x\le\delta^{1/3}}\Big\{\sum_{n\equiv 1 \bmod 8}\Lambda(n)e^{-nx}-\sum_{n\equiv l \bmod 8}\Lambda(n)e^{-nx}\Big\}<-\frac{1}{\sqrt{\delta}}e_1\left(-22\frac{\log\frac{1}{\delta}\log_3\frac{1}{\delta}}{\log_2\frac{1}{\delta}}\right)$$

hold.

Since the proof of this theorem runs quite parallel to that of (1. 9) we shall not go into details.

2. In the proofs essential role is played by some numerical data, kindly furnished to us by Dr. P. C. HASELGROVE. Let

$$L(s,\chi_1)\overset{\text{def}}{=}\sum_{n=0}^{\infty}\left\{\frac{1}{(8n+1)^s}+\frac{1}{(8n+3)^s}-\frac{1}{(8n+5)^s}-\frac{1}{(8n+7)^s}\right\}, \tag{2. 1}$$

$$L(s,\chi_2)\overset{\text{def}}{=}\sum_{n=0}^{\infty}\left\{\frac{1}{(4n+1)^s}-\frac{1}{(4n+3)^s}\right\} \tag{2. 2}$$

$$L(s,\chi_3)\overset{\text{def}}{=}\sum_{n=0}^{\infty}\left\{\frac{1}{(8n+1)^s}-\frac{1}{(8n+3)^s}-\frac{1}{(8n+5)^s}+\frac{1}{(8n+7)^s}\right\}. \tag{2. 3}$$

Then in the domain

$$0<\sigma<1,\quad |t|\le 12$$

all zeros of $L(s,\chi_1)$ are

$$\begin{aligned}&\tfrac{1}{2}\pm i\cdot 4{,}89997399\ldots\\ &\tfrac{1}{2}\pm i\cdot 7{,}62842884\ldots\\ &\tfrac{1}{2}\pm i\cdot 10{,}80658813\ldots\end{aligned} \tag{2. 4}$$

those of $L(s,\chi_2)$

$$\begin{aligned}&\tfrac{1}{2}\pm i\cdot 6{,}02094891\ldots\\ &\tfrac{1}{2}\pm i\cdot 10{,}24377030\ldots\end{aligned} \tag{2. 5}$$

and those of $L(s,\chi_3)$

$$\begin{aligned}&\tfrac{1}{2}\pm i\cdot 3{,}57615484\ldots\\ &\tfrac{1}{2}\pm i\cdot 7{,}4344296\ldots\\ &\tfrac{1}{2}\pm i\cdot 9{,}50320196\ldots.\end{aligned} \tag{2. 6}$$

In particular, they are simple and different from each other. We shall also use the simple inequality

$$\sum_p \log p \cdot e^{-p^2 x} > 0{,}8\, x^{-\frac{1}{2}} \tag{2.7}$$

valid for $0 < x < c_2$. A number of numerical lemmata we postponed to an Appendix.

3. To sketch the proof of Theorem 1. 2 after the pattern of Landau we remark that Euler's integral representation for $\Gamma(s)$ gives for $\sigma > 1$, for $l = 3$ say, the formula

$$\int_0^\infty x^{s-1} f_{13}(x)\, dx = \Gamma(s) \left\{ \sum_{p \equiv 1 \bmod 8} \frac{\log p}{p^s} - \sum_{p \equiv 3 \bmod 8} \frac{\log p}{p^s} \right\} = \tag{3.1}$$
$$= -\frac{\Gamma(s)}{2} \left\{ \frac{L'}{L}(s, \chi_2) + \frac{L'}{L}(s, \chi_3) + \varphi(s) \right\}$$

where $\varphi(s)$ is regular for $\sigma > \frac{1}{2}$. From the hypothesis of the Theorem it follows the existence of a $q > 0$ such that for $0 < x \leqq q$ the function $f_{13}(x)$ is of constant sign and obviously

$$\int_0^q x^{s-1} f_{13}(x)\, dx = -\frac{\Gamma(s)}{2} \left\{ \frac{L'}{L}(s, \chi_2) + \frac{L'}{L}(s, \chi_3) + \varphi_1(s) \right\} \tag{3.2}$$

where $\varphi_1(s)$ is regular for $\sigma > \frac{1}{2}$. But from (2. 5) and (2. 6) (and even simpler) the function on the right of (3. 2) is regular on the segment $\frac{1}{2} < s \leqq 1$; hence the classical theorem of Landau (see e. g. LANDAU [1]) gives at once that

$$\frac{L'}{L}(s, \chi_2) + \frac{L'}{L}(s, \chi_3)$$

is regular for $\sigma > \frac{1}{2}$ and thus obviously both $L(s, \chi_2)$ and $L(s, \chi_3)$ do not vanish for $\sigma > \frac{1}{2}$. For $l = 5$ and $l = 7$ the proof is analogous.[3]

To sketch the proof of Theorem 1. 3 we start with Hardy—Littlewood from Mellin's integral which gives for $x > 0$ the formula (for $l = 3$ say)

$$\sum_{n \equiv 1 \bmod 8} \Lambda(n) e^{-nx} - \sum_{n \equiv 3 \bmod 8} \Lambda(n) e^{-nx} =$$
$$- = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(s)}{x^s} \cdot \frac{1}{2} \cdot \left(\frac{L'}{L}(s, \chi_2) + \frac{L'}{L}(s, \chi_3) \right) ds.$$

Applying Cauchy's integral-theorem routine-estimations give

$$\sum_{n \equiv 1 \bmod 8} \Lambda(n) e^{-nx} - \sum_{n \equiv 3 \bmod 8} \Lambda(n) e^{-nx} =$$
$$= -\frac{1}{2} \left(\sum_{\varrho(\chi_2)} \Gamma(\varrho) x^{-\varrho} + \sum_{\varrho(\chi_3)} \Gamma(\varrho) x^{-\varrho} \right) + O\left(x^{-\frac{1}{4}}\right)$$

[3] Obviously the same theorem holds for a general modulus k, supposed that no L-functions mod k vanish on the segment $\frac{1}{2} < s \leqq 1$.

or owing to (1. 7) and (1. 8), if $0<x<c_3$,

$$\sum_{p\equiv 1 \bmod 8} \log p\cdot e^{-px} - \sum_{p\equiv 3 \bmod 8} \log p\cdot e^{-px} = -\sum_{p>2} \log p\cdot e^{-p^2x} -$$

$$-\frac{1}{2}\left(\sum\nolimits_{\varrho(\chi_2)} \Gamma(\varrho)x^{-\varrho} + \sum\nolimits_{\varrho(\chi_3)} \Gamma(\varrho)x^{-\varrho}\right) + O\left(x^{-\frac{1}{3}}\log^2\frac{1}{x}\right).$$

Hence, using also (2. 7) this gives

$$f_{13}(x) \leqq x^{-\frac{1}{2}}\left\{-0{,}8 + \frac{1}{2}\sum\nolimits_{\varrho(\chi_2)}|\Gamma(\varrho)| + \right. \tag{3. 3}$$

$$\left. + \frac{1}{2}\sum\nolimits_{\varrho(\chi_3)}|\Gamma(\varrho)|\right\} + O\left(x^{-\frac{1}{3}}\log^2\frac{1}{x}\right).$$

Since from Lemma VI (in Appendix) we have

$$\sum\nolimits_{\varrho(\chi_2)}|\Gamma(\varrho)| < \frac{1}{10}, \quad \sum\nolimits_{\varrho(\chi_3)}|\Gamma(\varrho)| < \frac{1}{10},$$

Theorem 1. 3 follows from (3. 3) for $l=3$ at once. So for the other l-values.

4. For the proof of Theorem 1. 1 we shall need some lemmata.

LEMMA I. *For* $0<\delta<c_4$ *there is a* y_1 *with*

$$\frac{1}{20}\log_2\frac{1}{\delta} \leqq y_1 \leqq \frac{1}{10}\log_2\frac{1}{\delta} \tag{4. 1}$$

so that for all non-trivial zeros $\varrho = \sigma_\varrho + it_\varrho$ *of all L-functions* mod 8 *we have*

$$\pi \geqq \left|\operatorname{arc}\frac{e^{it_\varrho y_1}}{\varrho}\right| \geqq \frac{c_5}{t_\varrho^6 \log^3|t_\varrho|}. \tag{4. 2}$$

For the proof see Paper II of this series.

As to Lemma II let m be a positive integer and

$$1 = |z_1| \geqq |z_2| \geqq \ldots \geqq |z_n| \tag{4. 3}$$

and with a $0<\varkappa\leqq\frac{\pi}{2}$

$$\varkappa \leqq |\operatorname{arc} z_j| \leqq \pi. \tag{4. 4}$$

Let the index h be such that

$$|z_h| > \frac{4n}{m+n\left(3+\frac{\pi}{\varkappa}\right)} \tag{4. 5}$$

and fixed. Further let

$$B \overset{\text{def}}{=} \min_{h\leqq\xi\leqq n} \operatorname{Re}\sum_{j=1}^{\xi} b_j. \tag{4. 6}$$

Then we have the

LEMMA II. *If $B>0$, then there are integers ν_1 and ν_2 with*

$$m+1 \leqq \nu_1, \quad \nu_2 \leqq m+n\left(3+\frac{\pi}{\varkappa}\right) \tag{4.7}$$

such that

$$\mathrm{Re}\sum_{j=1}^{n} b_j z_j^{\nu_1} \geqq \frac{B}{2n+1}\left\{\frac{n}{24\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)}\right\}^{2n} \cdot \left(\frac{|z_h|}{2}\right)^{m+n\left(3+\frac{\pi}{\varkappa}\right)}$$

and

$$\mathrm{Re}\sum_{j=1}^{n} b_j z_j^{\nu_2} \leqq -\frac{B}{2n+1}\left\{\frac{n}{24\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)}\right\}^{2n} \cdot \left(\frac{|z_h|}{2}\right)^{m+n\left(3+\frac{\pi}{\varkappa}\right)}.$$

This is a special case of Theorem 4. 1 from the Paper III of this series. We need further the

LEMMA III. *For $x>0$ (taking the real value of $\log x$) and a positive integer ν we have*

$$\frac{1}{2\pi i}\int_{(2)} \frac{\Gamma(s)}{s^\nu} x^{-s}\,dx = \frac{1}{(\nu-1)!}\int_x^\infty \frac{\log^{\nu-1}\frac{r}{x}}{r} e^{-r}\,dr = \frac{1}{(\nu-1)!}\int_0^\infty e^{-xe^y} y^{\nu-1}\,dy.$$

Namely Cahen—Mellin's formula gives for $r>0$

$$\frac{1}{2\pi i}\int_{(2)} \Gamma(s) r^{-s}\,ds = e^{-r}.$$

Multiplying on both sides by

$$\frac{1}{(\nu-1)!}\cdot\frac{1}{r}\log^{\nu-1}\frac{r}{x}$$

and integrating with respect to r over (x, ∞) we can obviously change on the left the order of integration and we get

$$\frac{1}{2\pi i}\int_{(2)} \frac{\Gamma(s)}{(\nu-1)!}\left\{\int_x^\infty r^{-s-1}\log^{\nu-1}\frac{r}{x}\,dr\right\} ds.$$

Replacing in the inner integral r by xe^v, this is

$$x^{-s}\int_0^\infty e^{-sv} v^{\nu-1}\,dv = (\nu-1)!\,x^{-s}s^{-\nu}$$

as well-known.

Finally we need the

LEMMA IV. *There is in the vertical strip* $\frac{1}{10} \leqq \sigma \leqq \frac{1}{5}$ *a broken line* V, *symmetrical to the real axis, consisting alternately of horizontal and vertical segments, increasing from* $-\infty$ *to* $+\infty$, *each horizontal strip of width* 1 *containing at most one horizontal segment of* V *so that on* V *for all* L-*functions* mod 8 *the inequality*

$$\left| \frac{L'}{L}(s, \chi) \right| < c_6 \log^3(2+|t|)$$

holds.

For the (simple) proof see Appendix III of first named authors book (TURÁN [1]).

5. Now we turn to the proof of Theorem 1. 1. It will suffice to consider the case

(5. 1) $$l_1=5, \quad l_2=3 \quad \text{say.}^4$$

Let be $0<\delta<c_4$ (as in Lemma I); further y_1 as in Lemma I and we stipulate the integer ν at present only by

(5. 2) $$\frac{1}{y_1}\log\frac{1}{\delta} - \log^{0,9}\frac{1}{\delta} \leqq \nu \leqq \frac{1}{y_1}\log\frac{1}{\delta}$$

and start from the integral

(5.3) $$J_\nu(\delta) \overset{\text{def}}{=} \frac{1}{2\pi i}\int\limits_{(2)} \left(\frac{e^{y_1 s}}{s}\right)^\nu \Gamma(s)\left\{\frac{L'}{L}(s,\chi_1) - \frac{L'}{L}(s,\chi_2)\right\} ds.$$

Inserting the Dirichlet-series we get owing to Lemma III with $x=ne^{-\nu y_1}$

$$J_\nu(\delta) = \frac{1}{(\nu-1)!}\sum_{n=2}^{\infty} \Lambda(n)(\chi_2(n)-\chi_1(n)) \int\limits_0^\infty y^{\nu-1}e^{-(ne^{-\nu y_1})e^y}\,dy.$$

As easy to see the order of summation and integration can be changed and hence

$$J_\nu(\delta) = \frac{1}{(\nu-1)!}\int\limits_0^\infty y^{\nu-1}\{\sum_n \Lambda(n)(\chi_2(n)-\chi_1(n))\cdot e^{-ne_1(y-\nu y_1)}\}\,dy.$$

Owing to (2. 1) and (2. 2) this relation assumes the form

(5. 4) $$J_\nu(\delta) = \frac{2}{(\nu-1)!}\int\limits_0^\infty y^{\nu-1}\{\sum_{n\equiv 5 \bmod 8} \Lambda(n)e^{-ne_1(y-\nu y_1)} - \\ - \sum_{n\equiv 3 \bmod 8} \Lambda(n)e^{-ne_1(y-\nu y_1)}\}\,dy = \\ = \frac{2}{(\nu-1)!}\int\limits_{-\nu y_1}^\infty (r+\nu y_1)^{\nu-1} g_{5,3}(e^r)\,dr$$

[4] The slight changes, necessary in other cases will be indicated.

where

$$(5.5)\qquad g_{5,3}(y)=\sum_{n\equiv 5 \bmod 8}\Lambda(n)e^{-ny}-\sum_{n\equiv 3 \bmod 8}\Lambda(n)e^{-ny}.$$

Since evidently for $y>0$

$$|g_{5,3}(y)|<\sum_{n}\Lambda(n)e^{-ny}<c_7(1+y^{-1})e^{-2y}$$

we have

$$\left|\frac{2}{(\nu-1)!}\int\limits_{\nu y_1}^{\infty}(r+\nu y_1)^{\nu-1}g_{5,3}(e^r)dr\right|<$$

$$<\frac{c_8}{(\nu-1)!}\int\limits_{\nu y_1}^{\infty}e^{-e_1(r)}(r+\nu y_1)^{\nu-1}dr<\frac{c_8}{(\nu-1)!}\int\limits_{\nu y_1}^{\infty}e^{-e_1\left(\frac{r+\nu y_1}{2}\right)}(r+\nu y_1)^{\nu-1}dr=$$

$$=\frac{c_8}{(\nu-1)!}\int\limits_{2\nu y_1}^{\infty}e^{-e_1\left(\frac{r}{2}\right)}r^{\nu-1}dr<\frac{c_8}{(\nu-1)!}\int\limits_{2\nu y_1}^{\infty}e^{-\frac{r}{2}}r^{\nu}dr\leqq$$

$$\leqq\frac{c_8}{(\nu-1)!}\max_{r\geqq 2\nu y_1}\left(r^{\nu}e^{-\frac{r}{4}}\right)\leqq c_9\cdot 5^{\nu}.$$

Hence

$$(5.6)\qquad \left|J_\nu(\delta)-\frac{2}{(\nu-1)!}\int\limits_{-\nu\delta_1}^{\nu\delta_1}(r+\nu y_1)^{\nu-1}g_{5,3}(e^r)dr\right|<c_9\cdot 5^{\nu}<e_1\left(21\frac{\log\frac{1}{\delta}}{\log_2\frac{1}{\delta}}\right)$$

owing to (5. 2) and Lemma I if c_4 is sufficiently small.

6. Applying Cauchy's integral-theorem to $J_\nu(\delta)$ we get

$$(6.1)\qquad J_\nu(\delta)=\sum\nolimits'_{\varrho(\chi_1)}\Gamma(\varrho)\left(\frac{e^{y_1\varrho}}{\varrho}\right)^{\nu}-\sum\nolimits'_{\varrho(\chi_2)}\Gamma(\varrho)\left(\frac{e^{y_1\varrho}}{\varrho}\right)^{\nu}+$$

$$+\frac{1}{2\pi i}\int\limits_{V}\left(\frac{e^{y_1 s}}{s}\right)^{\nu}\Gamma(s)\left(\frac{L'}{L}(s,\chi_1)-\frac{L'}{L}(s,\chi_2)\right)ds$$

where the dash means that the summation refers to the non-trivial zeros of $L(s,\chi_1)$ and $L(s,\chi_2)$, respectively, right from V. Owing to Lemma IV the last integral is absolutely

$$(6.2)\qquad <c_{10}e^{\frac{\nu y_1}{5}}\cdot 10^{\nu}<\left(\frac{1}{\delta}\right)^{\frac{1}{4}},$$

using (5. 2) (c_4 again sufficiently small). Let us consider the contribution of the non-trivial zeros with

$$|t_\varrho| > \log^{\frac{1}{10}} \frac{1}{\delta}$$

to the sums in (6. 1). Since the number of zeros of $L(s, \chi_1)$ and $L(s, \chi_2)$ with $r \leqq t_\varrho < r+1$ (r real) is at most

$$c_{11} \log(2+|r|), \tag{6. 3}$$

this contribution is absolutely

$$\leqq c_{12} \sum_{d \geqq \left[\log^{\frac{1}{10}} \frac{1}{\delta}\right]} e^{-\frac{\pi}{2}d} d^{\frac{1}{2}} \frac{e^{\nu y_1}}{d^\nu} \log d$$

and using (5. 2) and Lemma I

$$< c_{13} e_1\left(\log^{0,91} \frac{1}{\delta}\right). \tag{6. 4}$$

If $\varrho_1 = \sigma_1 + it_1$ stands for one of the non-trivial zeros of $L(s, \chi_1)$ and $L(s, \chi_2)$ for which

$$\left|\frac{e^{\varrho y_1}}{\varrho}\right| \tag{6. 5}$$

is maximal among those right of V with

$$|t_\varrho| \leqq \log^{\frac{1}{10}} \frac{1}{\delta}, \tag{6. 6}$$

further collecting (5. 6), (6. 1), (6. 2) and (6. 4) we get

$$\left|\frac{2}{(\nu-1)!} \int_{-\nu y_1}^{\nu y_1} (r+\nu y_1)^{\nu-1} g_{3,5}(e^r)\, dr - \right. \tag{6. 7}$$

$$\left. -\left(\frac{e^{y_1\sigma_1}}{|\varrho_1|}\right)^\nu \left\{\sum\nolimits''_{\varrho(\chi_2)} \Gamma(\varrho)\left(\frac{e^{y_1(\varrho-\sigma_1)}}{\varrho}|\varrho_1|\right)^\nu - \sum\nolimits''_{\varrho(\chi_1)} \Gamma(\varrho)\left(\frac{e^{y_1(\varrho-\sigma_1)}}{\varrho}|\varrho_1|\right)^\nu\right\}\right| < \delta^{-0,3},$$

if only c_4 is sufficiently small. Here Σ'' means of course that the summation is to be extended to non-trivial zeros right to V, satisfying (6. 6); owing to the symmetry of V the sum in brackets is *real*.

7. So far ν was restricted only by (5. 2); now we shall determine a ν_1 and a ν_2 by Lemma II. The role of z_j's is played of course by the numbers

$$\frac{e^{y_1(\varrho-\sigma_1)}}{\varrho}|\varrho_1|,$$

that of the b_j's by the numbers

$$\pm\Gamma(\varrho);$$

the condition $\max\limits_j |z_j| = 1$ is obviously satisfied. Let be

$$m = \left[\frac{1}{y_1}\log\frac{1}{\delta} - \log^{0,91}\frac{1}{\delta}\right]. \tag{7. 1}$$

As to n of the Lemma II we have owing to (6. 3)

$$n < 2c_{11}\log^{\frac{1}{10}}\frac{1}{\delta}\cdot\log\left(2+\log^{\frac{1}{10}}\frac{1}{\delta}\right) < \log^{\frac{1}{10}}\frac{1}{\delta}\cdot\left(\log_2\frac{1}{\delta}\right)^2, \tag{7. 2}$$

if c_4 is sufficiently small. Owing to Lemma I we have

$$\pi \geqq |\arg z_j| \geqq \varkappa$$

if only

$$\varkappa = \log^{-\frac{2}{3}}\frac{1}{\delta}. \tag{7. 3}$$

As to the index h let be[5]

$$\varrho_2 = \tfrac{1}{2} + i\cdot 4{,}89997399\ldots \tag{7. 4}$$

and

$$z_{h-1} = \frac{e^{y_1(\varrho_2-\sigma_1)}}{\varrho_2}|\varrho_1|, \quad z_h = \frac{e^{y_1(\bar\varrho_2-\sigma_1)}}{\varrho_2}|\varrho_1|. \tag{7. 5}$$

Then we have to verify (4. 5). We have from (7. 3)

$$\frac{4n}{m+n\left(3+\dfrac{\pi}{\varkappa}\right)} < \frac{4}{\pi}\varkappa < 2\log^{-\frac{2}{3}}\frac{1}{\delta}$$

and from (6. 6) and (4. 1)

$$|z_h| \geqq \frac{e^{-\frac{1}{2}y_1}}{2\log^{\frac{1}{10}}\dfrac{1}{\delta}}\cdot\frac{1}{10} > \frac{1}{20}\log^{-\frac{1}{5}}\frac{1}{\delta} > 2\log^{-\frac{2}{3}}\frac{1}{\delta},$$

if only c_4 is sufficiently small; hence (4. 5) is fulfilled indeed. As to B we have owing to (7. 5), (2. 4), (2. 5) and (2. 6)

$$\begin{aligned} B \geqq & -2\,\mathrm{Re}\,\Gamma(\tfrac{1}{2}+i\cdot 4{,}89\ldots) - 2|\Gamma(\tfrac{1}{2}+i\cdot 6{,}02\ldots)| - \\ & - 2|\Gamma(\tfrac{1}{2}+i\cdot 7{,}62\ldots)| - 2|\Gamma(\tfrac{1}{2}+i\cdot 10{,}243\ldots)| - \\ & - 2|\Gamma(\tfrac{1}{2}+i\cdot 10{,}80\ldots)| - \sum_{\substack{\varrho(\chi_1)\\ |t_\varrho|\geqq 12}}|\Gamma(\varrho)| - \sum_{\substack{\varrho(\chi_2)\\ |t_\varrho|\geqq 12}}|\Gamma(\varrho)|. \end{aligned}$$

[5] I. e. ϱ_2 is one of the non-trivial zeros of $L(s,\chi_1)L(s,\chi_2)$ with absolutely minimal imaginary part. In the cases, different from that in (5.1) we have to put according to (2.6) $\varrho_2 = \frac{1}{2} + i.\ 3{,}57615484\ldots$.

Using (10. 2), (10. 7) and Lemma IX in Appendix we get indeed

$$B \geqq c_{14}. \tag{7. 6}$$

Further we remark that the interval $\left[m+1, m+n\left(3+\frac{\pi}{\varkappa}\right)\right]$ is owing to (7. 1) contained in the interval (5. 2), using also (7. 2) and (7. 3). Hence, denoting the sum in curly brackets in (6. 7) by $Z(\nu)$, Lemma II is applicable and we obtain the existence of integer ν_1 and ν_2 satisfying (5. 2) and — using (7. 2), (7. 1), (7. 3) and (7. 5) —

$$Z(\nu_1) = \operatorname{Re} Z(\nu_1) > \frac{c_{14}}{3\log^{\frac{1}{10}}\frac{1}{\delta}\left(\log_2\frac{1}{\delta}\right)^2} \cdot \left(\frac{y_1}{25\log\frac{1}{\delta}}\right)^{2\log^{\frac{1}{10}}\frac{1}{\delta}\cdot\left(\log_2\frac{1}{\delta}\right)^2} 2^{-\frac{1}{y_1}\log\frac{1}{\delta}}.$$

$$\cdot\left(\frac{e^{y_1(\frac{1}{2}-\sigma_1)}}{|\varrho_2|}|\varrho_1|\right)^{\nu_1} \cdot \left(\frac{e^{y_1(\frac{1}{2}-\sigma_1)}}{|\varrho_2|}|\varrho_1|\right)^{n\left(3+\frac{\pi}{\varkappa}\right)} > e_1\left(-21\frac{\log\frac{1}{\delta}}{\log_2\frac{1}{\delta}}\right)\cdot\left(\frac{e^{y_1(\frac{1}{2}-\sigma_1)}}{|\varrho_2|}|\varrho_1|\right)^{\nu_1}$$

and hence from (6. 7)

$$\frac{2}{(\nu_1-1)!}\int\limits_{-\nu y_1}^{\nu y_1}(r+\nu_1 y_1)^{\nu_1-1} g_{3,5}(e^r)\,dr > \frac{e^{\frac{\nu_1 y_1}{2}}}{(|\varrho_2|)^{\nu_1}} e_1\left(-21\frac{\log\frac{1}{\delta}}{\log_2\frac{1}{\delta}}\right) - \delta^{-0,3} \tag{7. 7}$$

and

$$\frac{2}{(\nu_2-1)!}\int\limits_{-\nu_2 y_1}^{\nu_2 y_1}(r+\nu_2 y_1)^{\nu_2-1} g_{3,5}(e^r)\,dr < -\frac{e^{\nu_2 y_1\frac{1}{2}}}{(|\varrho_2|)^{\nu_2}}\cdot e_1\left(-21\frac{\log\frac{1}{\delta}}{\log_2\frac{1}{\delta}}\right) + \delta^{-0,3}. \tag{7. 8}$$

The right member of (7. 7) is owing to (5. 2) and (4. 1)

$$> \delta^{-\frac{1}{2}} e_1\left(-22\frac{\log\frac{1}{\delta}}{\log_2\frac{1}{\delta}}\right)\cdot 10^{-\nu_1} - \delta^{-0,3} > \delta^{-\frac{1}{2}} e_1\left(-120\frac{\log\frac{1}{\delta}}{\log_2\frac{1}{\delta}}\right), \tag{7. 9}$$

if only c_4 is sufficiently small. Since the left member of (7. 7) is owing to (5. 2)

$$\leqq \frac{2}{\nu_1!}(2\nu_1 y_1)^{\nu_1} \max_{y \geqq e_1(-\nu_1 y_1)} g_{3,5}(y) \leqq 2(2ey_1)^{\nu_1} \max_{y \geqq \delta} g_{3,5}(y) \leqq$$

$$\leqq e_1\left(21\frac{\log\frac{1}{\delta}\log_3\frac{1}{\delta}}{\log_2\frac{1}{\delta}}\right)\max_{y\geqq\delta} g_{3,5}(y),$$

the first assertion of our Theorem follows from this, (7. 9) and (7. 7) at once. The second follows analogously from (7. 8) and (6. 7).

Appendix

Numerical lemmas

8. LEMMA V. *With the notation*

$$S(\chi)=\sum_{\varrho(\chi)}\frac{1}{\varrho(1-\varrho)}$$

the sum being extended to the non-trivial zeros of $L(s,\chi)$ mod 8, $\chi\neq\chi_0$, *we have*

$$(0<)\,S(\chi)<3.$$

For $\chi=\chi_2$ this has been proved by Landau (see LANDAU [2], p. 216). In order to prove it for $\chi=\chi_1$ we start from

$$\xi(1-s,\chi_1)=\xi(s,\chi_1) \tag{8. 1}$$

where

$$\xi(s,\chi_1)\overset{\text{def}}{=}\left(\frac{8}{\pi}\right)^{\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi_1). \tag{8. 2}$$

(8. 1) gives

$$\frac{\xi'}{\xi}(s,\chi_1)=-\frac{\xi'}{\xi}(1-s,\chi_1). \tag{8. 3}$$

But (see e. g. PRACHAR [1], p. 217)

$$\xi(s,\chi_1)=e^{c_{15}s+c_{16}}\prod_{\varrho(\chi_1)}\left(1-\frac{s}{\varrho}\right)e^{\frac{s}{\varrho}}$$

and hence

$$\frac{\xi'}{\xi}(s,\chi_1)=c_{15}+\sum_{\varrho(\chi_1)}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right);$$

putting $s=1$ we get

$$s(\chi_1)=-c_{15}+\frac{\xi'}{\xi}(1,\chi_1)=\frac{\xi'}{\xi}(1,\chi_1)-\frac{\xi'}{\xi}(0,\chi_1).$$

and using (8. 3), resp. (8. 2)

$$=2\frac{\xi'}{\xi}(1,\chi_1)=\log\frac{8}{\pi}+\frac{\Gamma'}{\Gamma}(1)+2\frac{L'}{L}(1,\chi_1). \tag{8. 4}$$

Now

$$\frac{L'}{L}(1,\chi_1)=\frac{-\frac{\log 3}{3}+\frac{\log 5}{5}+\frac{\log 7}{7}-\dots}{\frac{1}{1}+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\dots}<$$

$$<\frac{-\frac{\log 3}{3}+\frac{\log 5}{5}+\frac{\log 7}{7}}{\frac{1}{1}-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}}<\frac{\frac{2}{7}}{\frac{104}{105}}<\frac{1}{3}$$

and hence from (8. 4)

$$S(\chi_1)<\log\frac{8}{\pi}+\frac{\Gamma'}{\Gamma}(1)+\frac{2}{3}<1-0{,}57+\frac{2}{3}<2.$$

For $\chi=\chi_3$ we start from

$$\xi(1-s,\chi_3)=\xi(s,\chi_3) \tag{8. 5}$$

where

$$\xi(s,\chi_3)\overset{\text{def}}{=}\left(\frac{8}{\pi}\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi_3).$$

(8. 5) gives again

$$\frac{\xi'}{\xi}(s,\chi_3)=-\frac{\xi'}{\xi}(1-s,\chi_3).$$

As before, we get

$$S(\chi_3)=2\frac{\xi'}{\xi}(1,\chi_3)=\log\frac{8}{\pi}+\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)+2\frac{L'}{L}(1,\chi_3). \tag{8. 6}$$

But

$$\frac{L'}{L}(1,\chi_3)=\frac{\frac{\log 3}{3}+\frac{\log 5}{5}-\frac{\log 7}{7}+\dots}{\left(\frac{1}{1}-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}\right)+\dots}<\frac{\frac{\log 3}{3}+\frac{\log 5}{5}}{\frac{1}{1}-\frac{1}{3}-\frac{1}{5}}<\frac{\frac{1{,}2}{3}+\frac{2}{5}}{\frac{7}{15}}=\frac{12}{7}$$

and from the partial-fraction representation of $\frac{\Gamma'}{\Gamma}$

$$\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)=-2-0{,}57+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{1}{2}}\right)<-\frac{5}{2}+\frac{1}{2}\sum_{1}^{\infty}\frac{1}{n^2}<-\frac{3}{2}$$

and thus from (8. 6)

$$S(\chi_3)<1-\frac{3}{2}+\frac{24}{7}<3.$$

9. We need further the

LEMMA VI. *If for an* $1 \leqq \nu \leqq 3$ *the non-trivial zeros* $\varrho(\chi_\nu)$ *of* $L(s, \chi_\nu)$ *are on the line* $\sigma = \frac{1}{2}$, *then*

$$\sum_{\varrho(\chi_\nu)} |\Gamma(\varrho)| < \frac{1}{10}.$$

Since in the rectangle

$$0 < \sigma < 1, \quad |t| \leqq 7$$

there are only two zeros of $L(s, \chi_\nu)$, owing to (2. 4), (2. 5), and (2. 6), the simple inequality[6]

$$\left|\Gamma\left(\frac{1}{2} + i\gamma\right)\right| < 3e^{-\frac{\pi}{2}|\gamma|} \tag{9. 1}$$

gives at once

$$\sum_{|I\varrho(\chi_\nu)| \leqq 7} |\Gamma(\varrho)| < 6e^{-5}. \tag{9. 2}$$

Again (9. 1) gives for $|\gamma| > 7$ $\left(\text{since the function } \left(\frac{1}{4} + \mu^2\right) e^{-\frac{\pi}{2}\mu} \text{ decreases for } \mu \geqq \frac{4}{\pi}\right)$

$$\frac{\left|\Gamma\left(\frac{1}{2} + i\gamma\right)\right|}{\left(\dfrac{1}{\frac{1}{4} + \gamma^2}\right)} < \frac{3\left(\frac{1}{4} + \gamma^2\right)}{e^{\frac{\pi}{2}|\gamma|}} < 3e^{-6}.$$

Hence this and Lemma V give

$$\sum_{|I\varrho(\chi_2)| > 7} |\Gamma(\varrho)| < 3e^{-6} \sum_{\varrho(\chi_\nu)} \frac{1}{\frac{1}{4} + \gamma^2} = 3e^{-6} \left| \sum_{\varrho(\chi_\nu)} \frac{1}{\varrho(1-\varrho)} \right| < 9e^{-6}.$$

This and (9. 2) prove the Lemma VI.

LEMMA VII. *With the notation*

$$S_1(\chi) = \sum_{\varrho(\chi)} \frac{1}{|\varrho|^2}$$

the sum being extended to the non-trivial zeros of $L(s, \chi)$ mod 8, $\chi \neq \chi_0$ *we have*

$$(0 <) S_1(\chi) < 4.$$

[6] See e. g. E. LANDAU [1], p. 218.

Lemma V gives namely

$$3>\sum_{\varrho(\chi)}\frac{1}{\varrho(1-\varrho)}=\sum_{t_\varrho>0}^{\varrho(\chi)}\left\{\frac{1}{(\sigma_\varrho+it_\varrho)(1-\sigma_\varrho-it_\varrho)}+\frac{1}{(\sigma_\varrho-it_\varrho)(1-\sigma_\varrho+it_\varrho)}\right\}=$$

$$=2\sum_{t_\varrho>0}^{\varrho(\chi)}\frac{\sigma_\varrho(1-\sigma_\varrho)+t_\varrho^2}{(\sigma_\varrho^2+t_\varrho^2)\{(1-\sigma_\varrho)^2+t_\varrho^2\}}>2\sum_{t_\varrho>0}^{\varrho(\chi)}\frac{t_\varrho^2}{(\sigma_\varrho^2+t_\varrho^2)\{(1-\sigma_\varrho)^2+t_\varrho^2\}}.$$

Hence

$$\sum_{t_\varrho>0}^{\varrho(\chi)}\frac{t_\varrho^2}{(\sigma_\varrho^2+t_\varrho^2)\{1-\sigma_\varrho)^2+t_\varrho^2\}}<\frac{3}{2}, \tag{9.3}$$

and thus from the functional-equation

$$S_1(\chi)=\sum_{t_\varrho>0}^{\varrho(\chi)}\left\{\frac{1}{\sigma_\varrho^2+t_\varrho^2}+\frac{1}{(1-\sigma_\varrho)^2+t_\varrho^2}\right\}= \tag{9.4}$$

$$=\sum_{t_\varrho>0}^{\varrho(\chi)}\frac{\{\sigma_\varrho^2+(1-\sigma_\varrho)^2\}+2t_\varrho^2}{(\sigma_\varrho^2+t_\varrho^2)\{(1-\sigma_\varrho)^2+t_\varrho^2\}}\leqq 2\sum_{t_\varrho>0}^{\varrho(\chi)}\frac{\frac{1}{2}+t_\varrho^2}{(\sigma_\varrho^2+t_\varrho^2)\{(1-\sigma_\varrho)^2+t_\varrho^2\}}.$$

Since from (2. 4), (2. 5) and (2. 6)

$$\frac{1}{2}<\frac{1}{12}t_\varrho^2$$

(9. 3) and (9. 4) give

$$S_1(\chi)<\left(2+\frac{1}{12}\right)\sum_{t_\varrho>0}^{\varrho(\chi)}\frac{t_\varrho^2}{(\sigma_\varrho^2+t_\varrho^2)\{(1-\sigma_\varrho)^2+t_\varrho^2\}}<4.$$

10. LEMMA VIII. *We have*

$$2\,\mathrm{Re}\,\Gamma\left(\frac{1}{2}+i\cdot 3{,}57615\ldots\right)>\frac{\sqrt{2\pi}}{e^6}\cdot 0{,}3 \tag{10.1}$$

and

$$2\,\mathrm{Re}\,\Gamma\left(\frac{1}{2}+i\cdot 4{,}899973\ldots\right)<-\frac{\sqrt{2\pi}}{e^8}\cdot 0{,}3. \tag{10.2}$$

For the proof we shall use the slightly modified formula (see e. g. GELFOND [1], p. 138)

$$\log\Gamma(s)=\left(s-\frac{1}{2}\right)\log s-s+\frac{1}{2\pi}\log 2\pi+\frac{\vartheta}{|s|}, \tag{10.3}$$

valid for $|s|\geqq 3{,}5$ with $|\vartheta|\leqq 1$. This gives

$$\operatorname{arc}\Gamma(s)=\mathrm{Im}\left\{\left(s-\frac{1}{2}\right)\log s-s\right\}+\frac{\vartheta}{|s|} \tag{10.4}$$

and hence

$$0 < \operatorname{arc} \Gamma\left(\frac{1}{2} + i\cdot 3{,}57615\ldots\right) < 79^\circ$$

and

$$\left(\frac{3}{4}\pi <\right) 2{,}594 < \operatorname{arc} \Gamma\left(\frac{1}{2} + i\cdot 4{,}899973\ldots\right) < 3{,}216 \left(< \frac{5}{4}\pi\right).$$

These give

$$2 \operatorname{Re} \Gamma\left(\frac{1}{2} + i\cdot 3{,}5761\ldots\right) > 2 \sin 11^\circ \cdot \left|\Gamma\left(\frac{1}{2} + i\cdot 3{,}5761\ldots\right)\right| \tag{10. 5}$$

and

$$2 \operatorname{Re} \Gamma\left(\frac{1}{2} + i\cdot 4{,}899973\ldots\right) < -\sqrt{2} \left|\Gamma\left(\frac{1}{2} + i\cdot 4{,}89\ldots\right)\right|, \tag{10. 6}$$

respectively. Using the well-known formula

$$\left|\Gamma\left(\frac{1}{2} + i\vartheta\right)\right| = \sqrt{\frac{2\pi}{e^{\pi\vartheta} + e^{-\pi\vartheta}}} \tag{10. 7}$$

(10. 1) and (10. 2) follow from (10. 5) and (10. 6) at once.

Finally we need the

LEMMA IX. *With the notation* (χ mod 8, $\chi \neq \chi_0$)

$$S_2(\chi) \overset{\text{def}}{=} \sum_{\substack{\varrho(\chi) \\ |\operatorname{Im} \varrho| \geqq 1/2}} |\Gamma(\varrho)|$$

we have

$$|S_2(\chi)| < \sqrt{2\pi}\cdot 0{,}14 \cdot e^{-8}.$$

The well-known formula

$$|\Gamma(s)| < \frac{\sqrt{1+t^2}\,\Gamma(1+\sigma)}{\sqrt{\sigma^2+t^2}} \sqrt{\frac{2\pi t}{e^{\pi t} - e^{-\pi t}}} \qquad (0 < \sigma < 1)$$

(see e. g. N. NIELSEN [1], p. 25) gives for $|t| \geqq 12$ the inequality

$$|\Gamma(s)| < \sqrt{2\pi}\cdot 2\sqrt{|t|}\, e^{-\frac{\pi}{2}|t|} \qquad (0 < \sigma < 1). \tag{10. 8}$$

Since $\sqrt{t}\,(1+t^2) e^{-\frac{\pi}{2}t}$ decreases for $t > 3$, (10. 8) gives for $|t| \geqq 12$

$$\frac{|\Gamma(s)|}{\left(\dfrac{1}{1+t^2}\right)} < \sqrt{2\pi}\cdot 2\sqrt{12}\cdot 145 \cdot e^{-18{,}8} < \sqrt{2\pi}\cdot 1015 \cdot e^{-18{,}8}$$

i. e. using Lemma VII

$$S_2(\chi) < \sqrt{2\pi} \cdot 1015 \cdot e^{-18,8} \sum_{\varrho(\chi)} \frac{1}{1+t_\varrho^2} <$$

$$< \sqrt{2\pi} \cdot 1015 \cdot e^{-18,8} \sum_{\varrho(\chi)} \frac{1}{|\varrho|^2} <$$

$$< \sqrt{2\pi} \cdot 4060 \cdot e^{-18,8} < \sqrt{2\pi} \cdot 0,14 \cdot e^{-8}$$

indeed.

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