COMPARATIVE PRIME-NUMBER THEORY. VII

(THE PROBLEM OF SIGN-CHANGES IN THE GENERAL CASE)

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1. In the previous two papers of this series we proved under the supposition that no $L(s, \chi)$ functions with $\chi \neq \chi_0 \bmod k$ $(s = \sigma + it)$ vanish for[1]

(1. 1) $$\sigma > 1/2, \quad |t| \leqq c_1 k^{10}$$

and

(1. 2) $$\sigma = 1/2, \quad |t| \leqq A(k) \quad (\leqq 1)$$

oscillation theorems for

$$\psi(x, k, l_1) - \psi(x, k, l_2)$$

and

$$\Pi(x, k, l_1) - \Pi(x, k, l_2)$$

in the general case when only $l_1 \neq l_2$ was required; here as usual

(1. 3) $$\psi(x, k, l) = \sum_{\substack{n \leqq x \\ n \equiv l \bmod k}} \Lambda(n)$$

(1. 4) $$\Pi(x, k, l) = \sum_{\substack{n \leqq x \\ n \equiv l \bmod k}} \frac{\Lambda(n)}{\log n}.$$

As mentioned l. c. these results yield an explicit D depending only upon k such that differences of the functions in (1. 3) resp. (1. 4) have for $1 \leqq x \leqq D$ at least one sign-change. The aim of the present paper is, as mentioned already in paper V of this series, to prove this result for ψ directly, requiring only what we called Haselgrove's assumption (see (1. 5) below). At the same time we shall show that the differences $\psi(x, k, l_1) - \psi(x, k, l_2)$ actually change their sign infinitely often. Our results can be formulated as follows.

THEOREM 1. 1. *If for a k no* $L(s, \chi) \bmod k$ *vanish for* $0 < \sigma < 1$, *then each function* $\psi(x, k, l_1) - \psi(x, k, l_2)$, $l_1 \neq l_2$, *changes his sign infinitely often for* $1 \leqq x < +\infty$.

As to the first sign-change we have

THEOREM 1. 2. *If for a k no* $L(s, \chi) \bmod k$ *vanish for*

$$0 < \sigma < 1, \quad |t| \leqq A(k) \quad (\leqq 1),$$

[1] We remind the reader that, as in the previous papers of this series, $c_1, c_2, \ldots$ denote positive explicitely calculable numerical constants, $(l, k) = 1$, further $e_1(x) = e^x$ and $e_\nu(x) = e_1(e_{\nu-1}(x))$, $\log_1 x = \log x$, $\log_\nu x = \log(\log_{\nu-1} x)$.

then all functions

$$\psi(x, k, l_1)-\psi(x, k, l_2), \quad l_1 \neq l_2,$$

change their sign in the interval

$$1 \leqq x \leqq \max\left(e_2(k^{c_2}),\ e_2\left(\frac{2}{A(k)^3}\right)\right)$$

with a sufficiently large c_2.

Both Theorems 1. 1 and 1. 2 are easy corollaries of the following, stronger

THEOREM. 1. 3. *If for a* k *no* $L(s, \chi) \bmod k$ *vanishes for*

$$0<\sigma<1, \quad |t| \leqq A(k) \quad (\leqq 1) \tag{1. 5}$$

then all functions

$$\psi(x, k, l_1)-\psi(x, k, l_2), \quad l_1 \neq l_2, \tag{1. 6}$$

change their sign in the interval

$$\omega \leqq x \leqq e^{2\sqrt{\omega}}$$

if only

$$\omega \geqq \max\left(e_1(k^{c_2}),\ e_1\left(\frac{2}{A(k)^3}\right)\right) \tag{1. 7}$$

with a sufficiently large c_2.

As mentioned in paper I of this series, the problem of the first sign-change was the starting point of the present investigations; this motivates in itself why a separate paper is devoted to its study. The proof of Theorem 1. 3 is less complicated than those in papers V and VI, though it needs the weakest unproved assumption. The main difficulties are analogous to those in the latter papers but owing to the special situation the use of an appropriately chosen *two-sided* theorem (lemma I below) could meet the difficulties. The present proof breaks down for $\Pi(x, k, l_1)-$ $-\Pi(x, k, l_2)$ as well as for $\pi(x, k, l_1)-\pi(x, k, l_2)$.

As mentioned in paper I of this series, as far as we know no previously known methods can lead to results similar in character to our theorems.

As to the c_ν's, c_2, c_3 and c_5 matter. c_3 will be chosen sufficiently large, then c_5 large in dependence upon c_3 and finally c_2 in dependence upon c_3 and c_5.

2. We shall need the following lemmata.

LEMMA I. *For* $r > e_1(k^{c_3})$ *and* $l_1 \neq l_2$ *the inequality*

$$\int_1^r \frac{|\psi(v, k, l_1)-\psi(v, k, l_2)|}{v}\, dv > r^{1/4}$$

holds.

This is a weaker form of a result found by the first of us (see KNAPOWSKI [1]).[2]

[2] Here the same lower bound was proved for

$$\int_{re_1(-\log^{0,9} r)}^{r} v^{-1}|\psi(v, k, l_1)-\psi(v, k, l_2)|\, dv.$$

Further we need the

LEMMA II. *In the vertical strip* $\frac{1}{100} \leqq \sigma \leqq \frac{1}{50}$ *there is a broken line V, symmetrical to the real axis, consisting alternately of horizontal and vertical segments, each horizontal strip of width 1 containing at most one horizontal segment of V, monotonically increasing from* $-\infty$ *to* $+\infty$, *so that on V the inequality*

$$\left|\frac{L'}{L}(s, \chi)\right| \leqq c_4 \varphi(k) \log^3 k (2+|t|) \tag{2.1}$$

holds.

Since the proof is mutatis mutandis identical with that of Appendix III in the book of one of us (see TURÁN [1]), we shall omit it.

3. We can deduce from the above lemmata the

LEMMA III. *Suppose that for a certain* (l_1^*, l_2^*)*-pair the function*

$$\psi(x, k, l_1^*) - \psi(x, k, l_2^*) \tag{3.1}$$

does not change his sign with an

$$\omega \geqq e_1(k^{c_5}) \tag{3.2}$$

in the interval $\omega \leqq x \leqq 2\omega^5$. *Then — perhaps after changing* l_1^* *and* l_2^* *— we have the inequality*

$$\frac{1}{\varphi(k)} \operatorname{Re} \sum_{\chi} \left(\bar{\chi}(l_1^*) - \bar{\chi}(l_2^*)\right) \sum\nolimits'_{\varrho(\chi)} \frac{(2\omega^5)^{\varrho}}{\varrho^{1000}} > c_6 \omega^{5/4},$$

where the dash means that the summation is to be extended over the non-trivial zeros right to V.

For the proof we start from the integral

$$H(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{(2)} \frac{(2\omega^5)^s}{s^{1000}} \frac{1}{\varphi(k)} \left\{ \sum_{\chi} \left(\bar{\chi}(l_1^*) - \bar{\chi}(l_2^*)\right) \frac{L'}{L}(s, \chi) \right\} ds. \tag{3.3}$$

Inserting the Dirichlet-series representation we get

$$H(\omega) = \frac{1}{999!} \left(\sum_{\substack{n \leqq 2\omega^5 \\ n \equiv l_2^* \bmod k}} \Lambda(n) \cdot \log^{999} \frac{2\omega^5}{n} - \sum_{\substack{n \leqq 2\omega^5 \\ n \equiv l_1^* \bmod k}} \Lambda(n) \cdot \log^{999} \frac{2\omega^5}{n} \right) =$$

$$= \frac{1}{999!} \int_1^{2\omega^5} \log^{999} \frac{2\omega^5}{x} \, d_x\left(\psi(x, k, l_2^*) - \psi(x, k, l_1^*)\right) =$$

$$= \frac{1}{999!} \int_1^{2\omega^5} \left(\psi(x, k, l_1^*) - \psi(x, k, l_2^*)\right) d_x \left(\log^{999} \frac{2\omega^5}{x}\right).$$

Using (3. 1) this gives

$$|H(\omega)| \geqq \frac{1}{998!}\int_{\omega}^{2\omega^5} |\psi(x,k,l_1^*)-\psi(x,k,l_2^*)|\cdot\log^{998}\frac{2\omega^5}{x}\cdot\frac{dx}{x}-$$

$$-\frac{1}{998!}\int_{1}^{\omega} |\psi(x,k,l_1^*)-\psi(x,k,l_2^*)|\cdot\log^{998}\frac{2\omega^5}{x}\cdot\frac{dx}{x}>$$

$$>c_7\int_{\omega}^{\omega^5}\frac{|\psi(x,k,l_1^*)-\psi(x,k,l_2^*)|}{x}\,dx-c_8\omega\log^{999}\omega,$$

and using Lemma I (if c_5 is sufficiently large in dependence upon c_3)

$$|H(\omega)|>c_9\omega^{5/4}. \tag{3.4}$$

On the other hand Cauchy's integral theorem gives at once

$$H(\omega)=\frac{1}{\varphi(k)}\sum_{\chi}\left(\bar\chi(l_1^*)-\bar\chi(l_2^*)\right)\sum_{\substack{\varrho(\chi)\\ \varrho \text{ right to } V}}\frac{(2\omega^5)^\varrho}{\varrho^{1000}}+ \tag{3.5}$$

$$+\frac{1}{2\pi i}\int_V\frac{(2\omega^5)^s}{s^{1000}}\frac{1}{\varphi(k)}\left\{\sum_{\chi}\left(\bar\chi(l_1^*)-\bar\chi(l_2^*)\right)\frac{L'}{L}(s,\chi)\right\}ds.$$

The last integral is owing to Lemma II absolutely

$$<c_{10}\omega^{1/10}k\log^3 k$$

and owing to (3. 2)

$$<c_{11}\omega^{1/8}.$$

This, (3. 4) and (3. 5) give

$$\left|\frac{1}{\varphi(k)}\sum_{\chi}\left(\bar\chi(l_1^*)-\bar\chi(l_2^*)\right)\sum_{\substack{\varrho(\chi)\\ \varrho \text{ right to } V}}\frac{(2\omega^5)^\varrho}{\varrho^{1000}}\right|>c_{12}\omega^{5/4}, \tag{3.6}$$

choosing c_5 sufficiently large. Taking in account the symmetry of V to the real axis, the sum on the left of (3. 6) is *real* and hence the lemma follows.

4. We shall need two more lemmata.

LEMMA IV. *Supposing* (1. 5) *we have for*

$$\tau\geqq\max\left(c_{13},\,e_2(k),\,e_2\left(\frac{1}{A(k)^3}\right)\right) \tag{4.1}$$

the existence of a y_1 *with*

$$\frac{1}{20}\log_2\tau\leqq y_1\leqq\frac{1}{10}\log_2\tau \tag{4.2}$$

such that for all $\varrho \overset{\text{def}}{=} \sigma_\varrho + it_\varrho$ *non-trivial zeros of all L-functions* mod k *the inequality*

$$\pi \geqq \left|\arg \frac{e^{it_\varrho y_1}}{\varrho}\right| \geqq c_{14} \frac{A(k)^3}{k(1+|t_\varrho|)^6 \log^3 k(2+|t_\varrho|)} \tag{4.3}$$

holds.

For the proof see paper II of this series.

Further let m be a non-negative integer and

$$1 = |z_1| \geqq |z_2| \geqq \ldots \geqq |z_n| \tag{4.4}$$

so that with a $0 < \varkappa \leqq \frac{\pi}{2}$ the inequality

$$\varkappa \leqq |\arg z_j| \leqq \pi \qquad (j = 1, 2, \ldots, n) \tag{4.5}$$

holds. Let the index h be such that

$$|z_h| \geqq \frac{4n}{m + n\left(3 + \frac{\pi}{\varkappa}\right)} \tag{4.6}$$

and

$$B \overset{\text{def}}{=} \min_{h \leqq l \leqq n} \operatorname{Re} \sum_{j=1}^{l} b_j. \tag{4.7}$$

Then we have the

LEMMA V. *If* $B > 0$, *we have integer* ν_1 *and* ν_2 *numbers with*

$$m + 1 \leqq \nu_1, \quad \nu_2 \leqq m + n\left(3 + \frac{\pi}{\varkappa}\right)$$

so that

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_1} \geqq \frac{B}{2n+1} \left(\frac{n}{24\left(m + n\left(3 + \frac{\pi}{\varkappa}\right)\right)} \right)^{2n} \cdot \left(\frac{|z_h|}{2}\right)^{m+n\left(3+\frac{\pi}{\varkappa}\right)}$$

and

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_2} \leqq -\frac{B}{2n+1} \left(\frac{n}{24\left(m + n\left(3 + \frac{\pi}{\varkappa}\right)\right)} \right)^{2n} \cdot \left(\frac{|z_h|}{2}\right)^{m+n\left(3+\frac{\pi}{\varkappa}\right)}$$

For the proof see our paper [5], Theorem 4. 1.

5. Next we turn to the proof of Theorem 1. 3. Suppose that for a certain (l_1^*, l_2^*)-pair $\psi(x, k, l_1^*) - \psi(x, k, l_2^*)$ does not change his sign in the interval

$$\omega \leqq x \leqq 2\omega^5 e^{\sqrt{\omega}}, \tag{5.1}$$

where

$$\omega \geqq \max\left(e_1(k^{c_2}), e_1\left(\frac{2}{A(k)^3}\right)\right). \tag{5.2}$$

Lemma IV is applicable with $\tau = e^{\sqrt{\omega}}$ (the condition (4. 1) being amply satisfied owing to (5. 2)) and hence there is a y_1 with

$$\frac{1}{40}\log\omega \leqq y_1 \leqq \frac{1}{20}\log\omega \tag{5.3}$$

such that (4. 3) holds. Fixing this y_1 let the integer ν be momentaneously restricted only by

$$\frac{\sqrt{\omega}}{y_1} - \omega^{0,4} \leqq \nu \leqq \frac{\sqrt{\omega}}{y_1}. \tag{5.4}$$

Applying Lemma III we obtain with a well-determined order of l_1^* and l_2^*

$$S \overset{\text{def}}{=} \frac{1}{\varphi(k)} \operatorname{Re} \sum_{\chi} \left(\bar\chi(l_1^*) - \bar\chi(l_2^*)\right) \cdot \sum\nolimits'_{\varrho(\chi)} \frac{(2\omega^5)^\varrho}{\varrho^{1000}} > c_6\omega^{5/4}, \tag{5.5}$$

where the dash means that the summation is to be extended over the ϱ's right to V. Then we start from the integral

$$J_\nu \overset{\text{def}}{=} \frac{1}{2\pi i} \int\limits_{(2)} \frac{(2\omega^5)^s}{s^{1000}} \left(\frac{e^{y_1 s}}{s}\right)^\nu \cdot \frac{1}{\varphi(k)} \sum_\chi \left\{(\bar\chi(l_1^*) - \bar\chi(l_2^*)) \frac{L'}{L}(s,\chi)\right\} ds. \tag{5.6}$$

Inserting the Dirichlet-series we get

$$J_\nu = \left(\sum_{\substack{n \leqq e^{\nu y_1}\cdot 2\omega^5 \\ n \equiv l_2^* \bmod k}} \Lambda(n) \log^{\nu+999} \frac{e^{\nu y_1}\cdot 2\omega^5}{n} - \right.$$

$$\left. - \sum_{\substack{n \leqq e^{\nu y_1}\cdot 2\omega^5 \\ n \equiv l_1^* \bmod k}} \Lambda(n) \log^{\nu+999} \frac{e^{\nu y_1}\cdot 2\omega^5}{n} \right) \frac{1}{(\nu+999)!} =$$

$$= \frac{1}{(\nu+999)!} \int\limits_1^{e^{\nu y_1}\cdot 2\omega^5} \log^{\nu+999} \frac{e^{\nu y_1}\cdot 2\omega^5}{x}\, d_x\left(\psi(x,k,l_2^*) - \psi(x,k,l_1^*)\right) = \tag{5.7}$$

$$= \frac{1}{(\nu+999)!} \int\limits_\omega^{e^{\nu y_1}\cdot 2\omega^5} \left(\psi(x,k,l_1^*) - \psi(x,k,l_2^*)\right) d_x \left(\log^{\nu+999} \frac{e^{\nu y_1}\cdot 2\omega^5}{x}\right) +$$

$$+ O\left(e_1\left(41 \frac{\sqrt{\omega}}{\log\omega} \log_2\omega\right)\right) \overset{\text{def}}{=} J_\nu' + O\left(e_1\left(41\frac{\sqrt{\omega}}{\log\omega}\log_2\omega\right)\right).$$

Owing to (5. 4) and (5. 2) we have

$$e^{\nu y_1}\cdot 2\omega^5 \leqq 2\omega^5 \cdot e^{\sqrt{\omega}} < e^{2\sqrt{\omega}},$$

whence owing to (5. 1) the representation (5. 7) gives at once that the sign of J'_ν would be independent of the choice of ν (satisfying of course (5. 4)). Hence if we can show the existence of such ν_1 and ν_2 with

$$\text{(5. 8)} \qquad \operatorname{sign} J'_{\nu_1} = -\operatorname{sign} J'_{\nu_2}, \quad |J'_{\nu_1}| > e^{\frac{1}{100}\sqrt{\omega}}, \quad |J_{\nu_2}| > e^{\frac{1}{100}\sqrt{\omega}}$$

then our theorem will be proved.

6. In order to do so we apply Cauchy's integral-theorem to J_ν. Then we get

$$\text{(6. 1)} \qquad J_\nu = \frac{1}{\varphi(k)} \sum_\chi (\bar\chi(l_1^*) - \bar\chi(l_2^*)) \sum\nolimits'_{\varrho(\chi)} \frac{(2\omega^5)^\varrho}{\varrho^{1000}} \cdot \left(\frac{e^{y_1\varrho}}{\varrho}\right)^\nu +$$

$$+ \frac{1}{2\pi i} \int\limits_V \frac{(e^{y_1\nu} \cdot 2\omega^5)^s}{s^{\nu+1000}} \cdot \frac{1}{\varphi(k)} \sum_\chi \left\{ (\bar\chi(l_1^*) - \bar\chi(l_2^*)) \frac{L'}{L}(s, \chi) \right\} ds,$$

where the dash means again that only ϱ's right to V are counted. The last integral is owing to (2. 1), (5. 4) and (5. 2) absolutely

$$\text{(6. 2)} \qquad \begin{aligned} &< c_{15} \omega^{1/10} (e^{\nu y_1})^{1/50} 100^\nu \cdot k \log^3 k \\ &< c_{16} e^{1/40 \sqrt{\omega}} . \end{aligned}$$

Using the fact that for a real r the number of the non-trivial zeros of all L-functions mod k with imaginary parts between r and $(r+1)$ is

$$\text{(6. 3)} \qquad < c_{17}\, \varphi(k) \log k(2 + |r|),$$

the contribution of the ϱ's with

$$|t_\varrho| > \omega^{1/20} + 1$$

to the sum in (6. 1) is absolutely

$$< c_{18} \sum_{n \geqq \omega^{1/20}} \frac{\log kn}{n^{\nu+1000}} \omega^5 e^{\sqrt{\omega}}.$$

Owing to (5. 2), (5. 3) and (5. 4) this is

$$\text{(6. 4)} \qquad < c_{19} \omega^5 e^{\sqrt{\omega}} \frac{\log \omega}{\omega^{\nu/20}} < c_{20} e_1(\omega^{0,41}).$$

(6. 2), (6. 4), (5. 7) and (6. 1) give, using also the definition (5. 7) of J'_ν,

$$\text{(6. 5)} \qquad \left| J'_\nu - \frac{1}{\varphi(k)} \sum_\chi (\bar\chi(l_1^*) - \bar\chi(l_2^*)) \sum\nolimits''_{\varrho(\chi)} \frac{(2\omega^5)^\varrho}{\varrho^{1000}} \cdot \left(\frac{e^{y_1\varrho}}{\varrho}\right)^\nu \right| < c_{21} e_1\left(\frac{1}{40}\sqrt{\omega}\right)$$

if only c_2 is sufficiently large; here Σ'' means that the summation is to be extended over those ϱ's right to V for which

$$\text{(6. 6)} \qquad |t_\varrho| \leqq \omega^{1/20} + 1.$$

Let

$$\varrho_1 = \sigma_1 + it_1$$

be one of the non-trivial zeros with (6. 6) for which

$$\left|\frac{e^{y_1\varrho}}{\varrho}\right| = \text{maximal}. \tag{6.7}$$

Then putting

$$\frac{1}{\varphi(k)} \sum_{\chi} (\bar{\chi}(l_1^*) - \bar{\chi}(l_2^*)) \sum\nolimits''_{\varrho(\chi)} \frac{(2\omega^5)^{\varrho}}{\varrho^{1000}} \cdot \left(\frac{e^{y_1(\varrho-\sigma_1)}}{\varrho} \cdot |\varrho_1|\right)^{\nu} \stackrel{\text{def}}{=} Z(\nu). \tag{6.8}$$

(6. 5) assumes the form

$$\left| J'_\nu - \frac{(e^{\nu y_1})^{\sigma_1}}{|\varrho_1|^\nu} Z(\nu) \right| < c_{22} e_1\left(\frac{1}{40}\sqrt{\omega}\right)$$

or owing to the reality of $Z(\nu)$

$$\left| J'_\nu - \frac{(e^{\nu y_1})^{\sigma_1}}{|\varrho_1|^\nu} \operatorname{Re} Z(\nu) \right| < c_{22} e_1\left(\frac{1}{40}\sqrt{\omega}\right). \tag{6.9}$$

7. In order to prove (5. 8) we shall use Lemma V. The role of the z_j-vectors will be played by the numbers

$$\frac{e^{y_1(\varrho-\sigma_1)}}{\varrho}|\varrho_1|$$

with ϱ's satisfying (6. 6), that of b_j's by the numbers

$$\frac{1}{\varphi(k)}(\bar{\chi}(l_1^*) - \bar{\chi}(l_2^*)) \frac{(2\omega^5)^{\varrho}}{\varrho^{1000}}.$$

The condition $\max_j |z_j| = 1$ is owing to (6. 7) satisfied. Since (4. 3) is fulfilled, for the ϱ's with (6. 6) we have

$$\pi \geqq |\arg z_j| \geqq c_{23} \frac{A(k)^3}{k\omega^{0,3}\log^3(k\omega)},$$

whence, taking in account (5. 2), we may choose

$$\varkappa = \omega^{-1/3} \tag{7.1}$$

if only c_2 is sufficiently large. Let

$$m = \left[\frac{\sqrt{\omega}}{y_1} - \omega^{0,4}\right]. \tag{7.2}$$

From (6. 3) we get for the number n of Lemma V the upper bound

$$2c_{17}k\log\{k(3+\omega^{1/20})\}\cdot(\omega^{1/20}+2)$$

and owing to (5. 2)

$$n<\omega^{1/20}(\log\omega)^3 \tag{7. 3}$$

if only c_2 is sufficiently large. Let z_h of Lemma V be defined by $\dfrac{e^{y_1(\varrho_2-\sigma_1)}}{\varrho_2}|\varrho_1|$ where $\varrho_2 = \sigma_2+it_2$ minimizes

$$\left|\frac{e^{y_1\varrho}}{\varrho}\right| \tag{7. 4}$$

among the ϱ's in the domain

$$\sigma\geqq 1/5, \quad |t|\leqq\omega^{1/250}. \tag{7. 5}$$

Then using (7. 4), (7. 1) and (5. 3) we have

$$|z_h|\geqq\frac{e^{-\frac{4}{5}y_1}}{2\omega^{1/250}}\cdot\frac{1}{100}\geqq\frac{1}{200}\omega^{-\frac{1}{25}-\frac{1}{250}}>\frac{4n}{m+n\left(3+\dfrac{\pi}{\varkappa}\right)} \tag{7. 6}$$

i. e. (4. 6) holds in our case. Hence from (4. 7), using the minimum property in the definition of z_h we have

$$B=\frac{1}{\varphi(k)}\operatorname{Re}\sum_{\chi}(\bar\chi(l_1^*)-\bar\chi(l_2^*))\sum\nolimits_{\varrho(\chi)}'''\frac{(2\omega^5)^\varrho}{\varrho^{1000}},$$

where Σ''' means the summation *certainly* extended over all ϱ's in the domain (7. 5). In order to obtain a positive lower bound for B we shall use (5. 5). This gives namely (since $\sigma_\varrho\geqq 1/100$)

$$B\geqq S-\frac{2}{\varphi(k)}\sum_{\chi}\sum_{\frac{1}{100}\leqq\sigma_\varrho\leqq\frac{1}{5}}\nolimits_{\varrho(\chi)}\frac{(2\omega^5)^{1/5}}{|\varrho|^{1000}}-\frac{2}{\varphi(k)}\sum_{\chi}\sum_{|t_\varrho|\geqq\omega^{1/250}}\nolimits_{\varrho(\chi)}\frac{2\omega^5}{|\varrho|^{1000}},$$

and hence

$$B>c_6\omega^{5/4}-c_{24}\left(\omega\log k+\frac{\omega^5\log k\omega}{\omega^{999/250}}\right)>c_{25}\omega^{5/4},$$

if only c_2 is sufficiently large. Before applying Lemma V we have to show that

$$\left(m+1, m+n\left(3+\frac{\pi}{\varkappa}\right)\right)\subset\left(\frac{\sqrt{\omega}}{y_1}-\omega^{0,4},\frac{\sqrt{\omega}}{y_1}\right);$$

but this follows readily from (7. 2), (7. 1) and (7. 3). Hence we get by applying Lemma V for sufficiently large c_2

$$\operatorname{Re}Z(v_1)>\frac{c_{25}\omega^{5/4}}{3\omega^{1/20}\log^3\omega}\left(\frac{y_1}{25\sqrt{\omega}}\right)^{2\omega^{1/20}\log^3\omega}\cdot|z_h|^{v_1}\cdot|z_h|^{n\left(3+\frac{\pi}{\varkappa}\right)}\cdot 2^{-m-n\left(3+\frac{\pi}{\varkappa}\right)},$$

i. e. from (7. 4), (7. 1), (7. 2) and (7. 3)

$$\operatorname{Re} Z(\nu_1) > e_1\left(-41\frac{\sqrt{\omega}}{\log\omega}\right)\cdot\left(\frac{e^{y_1(1/5-\sigma_1)}}{|\varrho_2|}\cdot|\varrho_1|\right)^{\nu_1}\cdot\left(\frac{e^{-\frac{4}{5}y_1}}{|\varrho_2|}\cdot|\varrho_1|\right)^{\omega^{0,4}}$$

for sufficiently large c_2. Owing to (7. 6) this implies

$$\operatorname{Re} Z(\nu_1) > \left(\frac{e^{y_1(1/5-\sigma_1)}\cdot|\varrho_1|}{|\varrho_2|}\right)^{\nu_1}\cdot e_1\left(-42\frac{\sqrt{\omega}}{\log\omega}\right).$$

Thus from (6. 9)

$$J'_{\nu_1} \geqq \frac{e^{\frac{\nu_1 y_1}{5}}}{|\varrho_2|^{\nu_1}}\, e_1\left(-42\frac{\sqrt{\omega}}{\log\omega}\right) - c_{22}e_1\left(\frac{1}{40}\sqrt{\omega}\right);$$

hence from (5. 4), (5. 3) and (7. 5)

$$J'_{\nu_1} \geqq \frac{e^{\frac{1}{5}\sqrt{\omega}-\omega^{0,41}}}{(2\omega^{1/250})^{40\sqrt{\omega}/\log\omega}}\cdot e_1\left(-42\frac{\sqrt{\omega}}{\log\omega}\right) - c_{22}e_1\left(\frac{1}{40}\sqrt{\omega}\right) >$$

$$> e_1\left(\left(\frac{1}{5}-\frac{4}{25}\right)\sqrt{\omega} - 80\frac{\sqrt{\omega}}{\log\omega}\right) - c_{22}e_1\left(\frac{1}{40}\sqrt{\omega}\right) > 0$$

if c_2 is sufficiently large. Similarly we get

$$J'_{\nu_2} < 0.$$

Hence (5. 8) is proved if only (5. 2) is satisfied which finishes tne proof.

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