COMPARATIVE PRIME-NUMBER THEORY. VI

(CONTINUATION OF THE GENERAL CASE)

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1. In the previous paper V of this series we discussed oscillatory properties of

$$\psi(x, k, l_1) - \psi(x, k, l_2)$$

and

$$\Pi(x, k, l_1) - \Pi(x, k, l_2)$$

for the general case

$$l_1 \not\equiv 1 \bmod k, \qquad l_2 \not\equiv 1 \bmod k \tag{1. 1}$$

or shortly, for the case $(l_1, l_2)_k$ with (1. 1). As one could except, the treatment[1] of

$$\pi(x, k, l_1) - \pi(x, k, l_2)$$

is much more difficult and we cannot cover the whole case (1. 1); what we can prove, refers to such l_1, l_2-pairs for which the number of incongruent solutions of the congruences

$$x^2 \equiv l_1 \bmod k, \quad x^2 \equiv l_2 \bmod k \tag{1. 2}$$

are equal. As easy to see, these cases make out a large part of all, in particular when k is very composite.[2] And even in these cases we cannot obtain unconditional results; we have to suppose that no $L(s, \chi)$ functions mod k with $\chi \neq \chi_0$ vanish in the domain

$$\sigma > \frac{1}{2}, \quad |t| \leqq 2c_1 k^{10} \qquad (s = \sigma + it) \tag{1. 3}$$

with a sufficiently large c_1 and moreover also for

$$\sigma = \frac{1}{2}, \quad |t| \leqq A(k) \tag{1. 4}$$

with a positive $A(k)$. More exactly we assert the

[1] We remind the reader that $\pi(x, k, l)$ denotes the number of primes not exceeding x, which are $\equiv l \bmod k$, (l, k) is always 1. As in the previous papers, $c_1, \dots$ always denote positive numerical explicitly calculable constants, further $e_1(x) = e^x$ and $e_\nu(x) = e_{\nu-1}(e_1(x))$, $\log_1 x = \log x$ and $\log_\nu x = \log_{\nu-1}(\log x)$, p always prime. Special attention must be given to the constants c_5, c_1 and c_2; c_5 must be sufficiently large, c_1 large in dependence of c_5 and c_2 large in dependence of c_5 and c_1.

[2] As remarked in Knapowski—Turán [3], for all $(l, k) = 1$ the number of solutions of $x^2 \equiv l \bmod k$ is either 0 or equal to that of $x^2 \equiv 1 \bmod k$.

THEOREM 1. 1. *If for a k the assertion* (1. 3)—(1. 4) *holds, then for*

$$T > \max\left\{e_2(c_2k^{20}),\ e_1\left(2e_1\left(\frac{1}{A(k)^3}\right)+c_2k^{20}\right)\right\} \tag{1. 5}$$

and all (l_1, l_2) *pairs with* (1. 2) *the inequalities*[3]

$$\max_{T^{1/3}\leqq x\leqq T}\{\pi(x,k,l_1)-\pi(x,k,l_2)\} > \sqrt{T}\, e_1\left(-44\frac{\log T\log_3 T}{\log_2 T}\right), \tag{1. 6}$$

$$\max_{T^{1/3}\leqq x\leqq T}\{\pi(x,k,l_2)-\pi(x,k,l_1)\} > \sqrt{T}\, e_1\left(-44\frac{\log T\log_3 T}{\log T_2}\right) \tag{1. 7}$$

hold.

The proof of this theorem will be perhaps the most complicated in this series, indeed a baroque one, the ideas being a suitable combination of those used in our papers [2] and [3]. That the difficulties are not very much to be seen on the length of the proof, is essentially due to the fact that we could take over from paper [3] of this series the rather difficult assertion (9. 13) without proof. Again it seems to be possible to deduce Theorem 1. 1 from Haselgrove-condition, i. e. that for a suitable $A=A(k)$ no $L(s,\chi)$ mod k vanishes in the parallelogramm

$$0<\sigma<1, \qquad |t|\geqq A(k). \tag{1. 8}$$

2. For the convenience of the reader we shall reproduce the necessary lemmata.

LEMMA I. *Under assumption* (1. 8) *for*

$$\tau > \max\left\{c_3,\ e_2(h),\ e_2\left(\frac{1}{A(k)^3}\right)\right\} \tag{2. 1}$$

there is a y_1 *with*

$$\frac{1}{20}\log_2\tau \leqq y_1 \leqq \frac{1}{10}\log_2\tau \tag{2. 2}$$

such that for all $\varrho=\sigma_\varrho+it_\varrho$ *non-trivial zeros of all* $L(s,\chi)$*-functions* mod k *the inequalities*

$$\pi \geqq \left|\operatorname{arc}\frac{e^{it_\varrho y_1}}{\varrho}\right| \geqq c_4\frac{A(k)^3}{k(1+|t_\varrho|)^6\log^3 k(2+|t_\varrho|)} \tag{2. 3}$$

$$\pi \geqq \left|\operatorname{arc}\frac{e^{i\frac{t_\varrho}{2}y_1}}{\varrho}\right| \geqq c_4\frac{A(k)^3}{k(1+|t_\varrho|)^6\log^3 k(2+|t_\varrho|)} \tag{2. 4}$$

hold.

For the proof see our paper [1].

3. Further let m be a non-negative integer and

$$1 = |z_1| \geqq |z_2| \leqq \ldots \geqq |z_n|, \tag{3. 1}$$

[3] Since *in the course of proof* l_1 will be distinguished to l_2, it was necessary to state both (1. 6) and (1. 7).

so that with a $0 < \varkappa \leqq \frac{\pi}{2}$

$$\varkappa \leqq |\operatorname{arc} z_j| \leqq \pi \qquad (j = 1, 2, \ldots, n). \tag{3.2}$$

Let the index h be such that

$$|z_h| \geqq \frac{4n}{m + n\left(3 + \frac{\pi}{\varkappa}\right)} \tag{3.3}$$

and fixed. Further we define for given b_j numbers B and the index h_1 by

$$B = \min_{h < \xi < h_1} \operatorname{Re} \sum_{j=1}^{\xi} b_j \tag{3.4}$$

if there is an index h_1 with

$$|z_{h_1}| < |z_h| - \frac{2n}{m + n\left(3 + \frac{\pi}{\varkappa}\right)} \tag{3.5}$$

and

$$B = \min_{h \leqq \xi \leqq n} \operatorname{Re} \sum_{j=1}^{\xi} b_j \tag{3.6}$$

otherwise. Then we have the

LEMMA II. *If $B > 0$, then there are integers ν_1 and ν_2 with*

$$m + 1 \leqq \nu_1, \quad \nu_2 \leqq m + n\left(3 + \frac{\pi}{\varkappa}\right) \tag{3.7}$$

such that

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_1} \geqq \frac{B}{2n+1} \left\{ \frac{n}{24\left(m + n\left(3 + \frac{\pi}{\varkappa}\right)\right)} \right\}^{2n} \left(\frac{|z_h|}{2} \right)^{m + n\left(3 + \frac{\pi}{\varkappa}\right)} \tag{3.8}$$

and

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_2} \leqq -\frac{B}{2n+1} \left\{ \frac{n}{24\left(m + n\left(3 + \frac{\pi}{\varkappa}\right)\right)} \right\}^{2n} \left(\frac{|z_h|}{2} \right)^{m + n\left(3 + \frac{\pi}{\varkappa}\right)} \tag{3.9}$$

For the proof see Theorem 4. 1 in our paper [2]. Further we shall need the

LEMMA III. *For sufficiently large c_5 there is an ω_0 with*

$$\frac{1}{3} c_5 k^3 \leqq \omega_0 \leqq c_5 k^3 \tag{3.10}$$

and with sufficiently large c_1

$$L_1 \overset{\text{def}}{=} c_1 k^{10} \tag{3.11}$$

such that with an appropriate order of the given l_1 and l_2, and with a suitable integer ν_0 restricted by

$$\frac{L_1^2}{\log L_1} \leqq \nu_0 \leqq \frac{L_1^2}{\log L_1} + L_1^{1,16} \tag{3.12}$$

the inequality

$$\frac{1}{\varphi(k)} \operatorname{Re} \sum_{\chi} \left\{ (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \sum_{\substack{\varrho(\chi) \\ |t_\varrho| \leqq L_1}} \frac{(\omega_0 L_1^{\nu_0})^{\varrho}}{\varrho^{\nu_0+1}} \right\} \geqq e_1\left(\frac{1}{30} L_1^2\right)$$

holds.

For the proof of this lemma see our paper [3] (see the corollary (9. 13) of Lemma VI there).

Finally we shall need the

LEMMA IV. *There is a connected path V in the vertical strip $\frac{1}{5} \leqq \sigma \leqq \frac{2}{5}$, symmetrical to the real axis, consisting alternately of horizontal and vertical segments and increasing monotonically from $-\infty$ to $+\infty$ such that on it for all $L(s, \chi)$-functions* mod k *the inequality*

$$\left| \frac{L'}{L}(s, \chi) \right| \leqq c_6 k \log^3 k(2 + |t|)$$

holds.

Since the proof is mutatis mutandis identical with that of Appendix III in the book of one of us (see TURÁN [1]), we shall omit it.

4. Next we turn to the proof of Theorem 1. 1. Let ν_0, ω_0, L_1 and the order of l_1 and l_2 be defined as previously and let T satisfiy (1. 5). We define further T_1 by

$$T_1 \overset{\text{def}}{=} \frac{T}{c_5 k^3} \cdot e_1(-2L_1^2). \tag{4.1}$$

Choosing in (1. 5) c_2 sufficiently large (in dependence upon c_5 and c_1) (2. 1) is with $\tau = T_1$ fulfilled and hence from Lemma I there is an y_1 with

$$\frac{1}{20} \log_2 T_1 \leqq y_1 \leqq \frac{1}{10} \log_2 T_1 \tag{4.2}$$

such that for all ϱ-zeros of all $L(s, \chi)$-functions mod k the inequalities

$$\pi \geqq \left| \operatorname{arc} \frac{e^{it_\varrho \frac{y_1}{2}}}{\varrho} \right| \geqq c_4 \frac{A(k)^3}{k(1+|t_\varrho|)^6 \log^3 k(2+|t_\varrho|)}, \tag{4.3}$$

$$\pi \geqq \left| \operatorname{arc} \frac{e^{it_\varrho y_1}}{\varrho} \right| \geqq c_4 \frac{A(k)^3}{k(1+|t_\varrho|)^6 \log^3 k(2+|t_\varrho|)} \tag{4.4}$$

hold. Let the integer ν be temporarily restricted only by the inequality

$$\frac{\log T_1}{y_1} - \log^{0,9} T_1 \leqq \nu \leqq \frac{\log T_1}{y_1}. \tag{4.5}$$

If none of the congruences (1. 2) are soluble, the case is settled already in our paper [3]. If all solutions of the congruences (1. 2) are

$$x \equiv \alpha_1, \alpha_2, \dots, \alpha_\mu \mod k \qquad (\mu \geqq 1) \tag{4. 6}$$

and

$$x \equiv \beta_1, \beta_2, \dots, \beta_\mu \mod k, \tag{4. 7}$$

respectively, then we start from the integral[4]

$$J(T) = \frac{1}{2\pi i}\int\limits_{(2)} \left(\frac{e^{y_1 s}}{s}\right)^{\nu} \frac{(\omega_0 L_1^{\nu_0})^s}{s^{\nu_0+1}} \cdot \frac{1}{\varphi(k)} \left\{ \sum_{\chi} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \cdot \right. \tag{4. 8}$$

$$\left. \cdot \frac{L'}{L}(s, \chi) - \sum_{j=1}^{\mu} \sum_{\chi} (\bar{\chi}(\alpha_j) - \bar{\chi}(\beta_j)) \frac{L'}{L}(2s, \chi) \right\} ds.$$

Inserting the Dirichlet-series expansions we get

$$J(T) = \sum_{\substack{n \leqq e^{\nu y_1}\omega_0 L_1^{\nu_0} \\ n = l_2 \bmod k}} \Lambda(n) \frac{\log^{\nu+\nu_0}\left(\frac{e^{\nu y_1}\omega_0 L_1^{\nu_0}}{n}\right)}{(\nu+\nu_0)!} - \tag{4. 9}$$

$$- \sum_{\substack{n \leqq e^{\nu y_1}\omega_0 L_1^{\nu_0} \\ n = l_1 \bmod k}} \Lambda(n) \frac{\log^{\nu+\nu_0}\left(\frac{e^{\nu y_1}\omega_0 L_1^{\nu_0}}{n}\right)}{(\nu+\nu_0)!} + \sum_{j=1}^{\mu} \sum_{\substack{n \leqq e^{\frac{\nu y_1}{2}} \sqrt{\omega_0 L_1^{\nu_0}} \\ n \equiv \alpha_j \bmod k}} (\Lambda) n \frac{\log^{\nu+\nu_0} \frac{e^{\nu y_1}\omega_0 L_1^{\nu_0}}{n^2}}{(\nu+\nu_0)!} -$$

$$- \sum_{j=1}^{\mu} \sum_{\substack{n \leqq e^{\frac{\nu y_1}{2}} \sqrt{\omega_0 L_1^{\nu_0}} \\ n \equiv \beta_j \bmod k}} \Lambda(n) \frac{\log^{\nu+\nu_0} \frac{e^{\nu y_1}\omega_0 L_1^{\nu_0}}{n^2}}{(\nu+\nu_0)!}.$$

Let us observe that the contribution of the prime-squares to the first sum and that of the primes to the fourth one cancel, and analogous holds for the second and third sums. Hence

$$\left| J(T) - \left\{ \sum_{\substack{p \leqq e^{\nu y_1}\omega_0 L_1^{\nu_0} \\ p \equiv l_2 \bmod k}} \log p \frac{\log^{\nu+\nu_0} \frac{e^{\nu y_1}\omega_0 L_1^{\nu_0}}{p}}{(\nu+\nu_0)!} - \right. \right. \tag{4. 10}$$

$$\left. \left. - \sum_{\substack{p \leqq e^{\nu y_1}\omega_0 L_1^{\nu_0} \\ p \equiv l_1 \bmod k}} \log p \frac{\log^{\nu+\nu_0} \frac{e^{\nu y_1}\omega_0 L_1^{\nu_0}}{p}}{(\nu+\nu_0)!} \right\} \right| <$$

$$< c_7 \frac{\log^{\nu+\nu_0+1}(e^{\nu y_1}\omega_0 L_1^{\nu_0})}{(\nu+\nu_0)!} \{(e^{\nu y_1}\omega_0 L_1^{\nu_0})^{\frac{1}{3}} + k(e^{\nu y_1}\omega_0 L_1^{\nu_0})^{\frac{1}{4}}\}.$$

[4] By $(\omega_0 L_1^{\nu_0})^s$ we mean always $e_1\{s \log(\omega_0 L_1^{\nu_0})\}$ with the real value of the logarithm.

Since from (4. 5), (4. 1), (3. 11), (3. 10) and (3. 12) we have

$$e^{\nu y_1}\omega_0 L_1^{\nu_0} \leqq T, \tag{4. 11}$$

the expression in (4. 10) is owing to (1. 5)

$$< c_7 \frac{\log^{\nu+\nu_0+1} T}{(\nu+\nu_0)!} (T^{\frac{1}{3}} + kT^{\frac{1}{4}}) < c_8 T^{\frac{1}{3}} \frac{\log^{\nu+\nu_0+1} T}{(\nu+\nu_0)!} < \tag{4. 12}$$

$$< c_8 T^{\frac{1}{3}} \left(\frac{e \log T}{\nu}\right)^{\nu+\nu_0} \log T.$$

Taking into account (4. 2), (4. 5) and (1. 5) with a sufficiently large c_2 we get

$$\nu > 5 \frac{\log T_1}{\log_2 T_1} > e \frac{\log T}{\log_2 T}$$

and hence the expression in (4. 12) is

$$< c_8 T^{\frac{1}{3}} \log T (\log_2 T)^{\nu+\nu_0}. \tag{4. 13}$$

Since from (4. 5), (4. 2), (3. 12) and (1. 5) we have

$$\nu + \nu_0 < 20 \frac{\log T_1}{\log_2 T_1} + 2L_1^2 < 21 \frac{\log T}{\log_2 T}, \tag{4. 14}$$

choosing c_2 in (1. 5) sufficiently large, (4. 13) gives that the expression on the right of (4. 10) is

$$< c_9 T^{0,4} \tag{4. 15}$$

The expression in brackets on the left of (4. 10) is

$$= \int_1^{e^{\nu y_1}\omega_0 L_1^{\nu_0}} \log x \frac{\log^{\nu+\nu_0} \dfrac{e^{\nu y_1}\omega_0 L_1^{\nu_0}}{x}}{(\nu+\nu_0)!} d_x(\pi(x, k, l_2) - \pi(x, k, l_1))$$

and hence, putting

$$e^{\nu y_1}\omega_0 L_1^{\nu_0} \overset{\text{def}}{=} Y, \tag{4. 16}$$

after partial integration

$$= \int_1^Y (\pi(x, k, l_2) - \pi(x, k, l_1)) d_x\left(-\log x \cdot \frac{\log^{\nu+\nu_0} \dfrac{Y}{X}}{(\nu+\nu_0)!}\right) = \int_1^{Y_1} + \int_{Y_1}^{Y} \overset{\text{def}}{=} J_1 + J_2 \tag{4. 17}$$

with

$$Y_1 \overset{\text{def}}{=} Y^{\frac{1}{\nu+\nu_0+1}}. \tag{4. 18}$$

5. For the (trivial) estimation of $|J_1|$ from above we remark that the function

$$\log x \cdot \log^{\nu+\nu_0} \frac{Y}{X} \tag{5.1}$$

increases for $1 \leqq x \leqq Y_1$ and decreases for $Y_1 \leqq x \leqq Y$. Hence

$$|J_1| \leqq Y_1 \log Y_1 \frac{\log^{\nu+\nu_0} \frac{Y}{Y_1}}{(\nu+\nu_0)!}. \tag{5.2}$$

Since from (4. 16), (4. 11), (4. 18), (4. 5) and (4. 2) we have choosing c_2 sufficiently large in dependence of c_1

$$Y_1 \leqq T^{\frac{1}{\nu}} < \log T$$

and hence (roughly)

$$|J_1| \leqq \log^2 T \left(\frac{e \log T}{\nu} \right)^{\nu+\nu_0}$$

we get as in (4. 12)

$$|J_1| \leqq e_1 \left(21 \frac{\log T \log_3 T}{\log_2 T} \right). \tag{5.3}$$

Using again the remark in (5. 1) we have using (4. 11)

$$J_2 \leqq \log Y_1 \cdot \frac{\log^{\nu+\nu_0} \frac{Y}{Y_1}}{(\nu+\nu_0)!} \cdot \max_{x \leqq T} \{\pi(x, k, l_2) - \pi(x, k, l_1)\}$$

and hence from this, (4. 10), (4. 15) and (5. 3) we get

$$\log Y_1 \cdot \frac{\log^{\nu+\nu_0} \frac{Y}{Y_1}}{(\nu+\nu_0)!} \max_{x \leqq T} \{\pi(x, k, l_2) - \pi(x, k, l_1)\} \geqq J(T) - c_{10} T^{0,4} \tag{5.4}$$

and analogously

$$\log Y_1 \cdot \frac{\log^{\nu+\nu_0} \frac{Y}{Y_1}}{(\nu+\nu_0)!} \min_{x \leqq T} \{\pi(x, k, l_2) - \pi(x, k, l_1)\} \leqq J(T) + c_{10} T^{0,4}. \tag{5.5}$$

6. Since obviously $J(T)$ in (4. 8) can be written in the form

$$J(T) = \frac{1}{2\pi i} \int_{(2)} \left(\frac{e^{y_1 s}}{s} \right)^{\nu} \frac{(\omega_0 L_1^{\nu_0})^s}{s^{\nu_0+1}} \cdot \frac{1}{\varphi(k)} \left\{ \sum_{\chi} ((\overline{\chi}(l_1) - \overline{\chi}(l_2)) \cdot \frac{L'}{L}(s, \chi) \right\} ds +$$

$$+ \frac{2^{\nu+\nu_0}}{2\pi i} \int_{(2)} \left(\frac{e^{\frac{y_1}{2} s}}{s} \right)^{\nu} \frac{(\omega_0 L_1^{\nu_0})^{\frac{s}{2}}}{s^{\nu_0+1}} \cdot \frac{1}{\varphi(k)} \left\{ \sum_{j=1}^{\mu} \sum_{\chi} (\overline{\chi}(\beta_j) - \overline{\chi}(\alpha_j)) \frac{L'}{L}(s, \chi) \right\} ds$$

the application of Cauchy's integral-theorem gives

$$J(T) = \frac{1}{\varphi(k)} \sum_{\chi} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \sum\nolimits'_{\varrho(\chi)} \frac{(\omega_0 L_1^{\nu_0})^{\varrho}}{\varrho^{\nu_0+1}} \cdot \left(\frac{e^{y_1\varrho}}{\varrho}\right)^{\nu} + \tag{6.1}$$

$$+ \frac{2^{\nu_0}}{\varphi(k)} \sum_{j=1}^{\mu} \sum_{\chi} (\bar{\chi}(\beta_j) - \bar{\chi}(\alpha_j)) \sum\nolimits'_{\varrho(\chi)} \frac{(\omega_0 L_1^{\nu_0})^{\frac{\varrho}{2}}}{\varrho^{\nu_0+1}} \cdot \left(2 \frac{e^{\frac{y_1}{2}\varrho}}{\varrho}\right)^{\nu} +$$

$$+ \frac{1}{2\pi i} \int_{(V)} \frac{Y^s}{s^{\nu+\nu_0+1}} \cdot \frac{1}{\varphi(k)} \left\{ \sum_{\chi} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \frac{L'}{L}(s, \chi) \right\} ds +$$

$$+ \frac{2^{\nu+\nu_0}}{2\pi i} \int_{(V)} \frac{Y^{\frac{s}{2}}}{s^{\nu+\nu_0+1}} \cdot \frac{1}{\varphi(k)} \left\{ \sum_{j=1}^{\mu} \sum_{\chi} (\bar{\chi}(\beta_j) - \bar{\chi}(\alpha_j)) \frac{L'}{L}(s, \chi) \right\} ds,$$

where V is given by Lemma IV and the dash means that the summation refers to the non-trivial zeros right[5] to V. Using Lemma IV the two last integrals are absolutely

$$< c_{11} \{ Y^{\frac{2}{5}} \cdot 5^{\nu+\nu_0} k \log^3 k + Y^{\frac{1}{5}} \cdot 10^{\nu+\nu_0} k^2 \log^3 k \}$$

and hence using (4. 11), (4. 16) and (1. 5)

$$< c_{12} T^{0,41} \cdot 10^{\nu+\nu_0} \tag{6.2}$$

if only c_2 in (1. 5) is sufficiently large. (4. 14) gives further from (6. 2) the upper bound

$$c_{13} T^{0,42}. \tag{6.3}$$

We estimate roughly the contribution of the non-trivial zeros with

$$|t_\varrho| > \log^{\frac{1}{10}} T_1 \tag{6.4}$$

to the sums in (6. 1). Using the well-known[6] fact that the total-number of non-trivial zeros of L-functions mod k with imaginary parts between r and $r+1$ (r real) is

$$\leqq c_{14} \varphi(k) \log k \, (2 + |r|) \tag{6.5}$$

this contribution is absolutely

$$\leqq 2c_{14} \sum_{n \geqq [\log^{\frac{1}{10}} T_1]} \log kn \left(\frac{Y}{n^{\nu+\nu_0+1}} + 2^{\nu+\nu_0} \frac{\sqrt{Y}}{n^{\nu+\nu_0+1}} \right)$$

and hence from (4. 14), (4. 16) and (4. 11)

$$< c_{15} \frac{T \log (k \log^{\frac{1}{10}} T_1)}{[\log T_1]^{\frac{\nu}{10}}}$$

[5] We remark that owing to the symmetry of V the sums occuring in (6. 1) are *real*.
[6] See PRACHAR [1].

and from (4. 2), (4. 5) and (1. 5) roughly

(6. 6) $$< c_{15} \frac{T}{T_1} e_1 (2 \log^{0,9} T \cdot \log_1 T) < c_{16} T^{0,42}$$

if only c_2 in (1. 5) is sufficiently large. If $\varrho_1 = \sigma_1 + it_1$ is a non-trivial zero of $L(s, \chi)$ with $\chi(l_1) \neq \chi(l_2)$ such that

(6. 7) $$|t_\varrho| \leqq \log^{\frac{1}{10}} T_1$$

and

$$\left| \frac{e^{y_1 \varrho}}{\varrho} \right|$$

is maximal, (6. 1), (6. 3), (6. 6) and (5. 4) give the inequality[7]

(6. 8) $$\log Y_1 \cdot \frac{\log^{\nu+\nu_0} (Y)^{\frac{\nu+\nu_0}{\nu+\nu_0+1}}}{(\nu+\nu_0)!} \cdot \max_{x \leqq T} \{\pi(x, k, l_2) - \pi(x, k, l_1)\} \geqq$$

$$\geqq \left(\frac{e^{y_1 \sigma_1}}{|\varrho_1|} \right)^\nu \left\{ \frac{1}{\varphi(k)} \sum_\chi (\bar\chi(l_1) - \bar\chi(l_2)) \sum\nolimits''_{\varrho(\chi)} \frac{(\omega_0 L_1^{\nu_0})^\varrho}{\varrho^{\nu_0+1}} \cdot \left(\frac{e^{y_1(\varrho - \sigma_1)}}{\varrho} |\varrho_1| \right)^\nu + \right.$$

$$\left. + \frac{2^{\nu_0}}{\varphi(k)} \sum_{j=1}^{\mu} \sum_\chi (\bar\chi(\beta_j) - \bar\chi(\alpha_j)) \sum\nolimits''_{\varrho(\chi)} \frac{(\omega_0 L_1^{\nu_0})^{\frac{\varrho}{2}}}{\varrho^{\nu_0+1}} \left(2 \frac{e^{y_1 \left(\frac{\varrho}{2} - \sigma_1\right)}}{\varrho} |\varrho_1| \right)^\nu \right\} - c_{17} T^{0,42}$$

and analogously from (5. 5) a reverse inequality for

$$\min_{x \leqq T} \{\pi(x, k, l_2) - \pi(x, k, l_1)\}.$$

We remark that the definition of ϱ_1 gives owing to the functional-equation

(6. 9) $$\sigma_1 \geqq \frac{1}{2}.$$

Further we shall use the known fact[8] that no $L(s, \chi) \bmod k$ vanishes for

(6. 10) $$\sigma > 1 - \frac{c_{18}}{\max \{\log k, \log^{4/5} k (2 + |t|)\}} \qquad (t \neq 0).$$

This means owing to (1. 5), choosing c_2 sufficiently large that for our ϱ's

(6. 11) $$\sigma_\varrho \leqq 1 - \frac{c_{19}}{(\log_2 T_1)^{4/5}}$$

and owing to the functional-equation

(6. 12) $$\sigma_\varrho \geqq \frac{c_{19}}{(\log_2 T_1)^{4/5}}.$$

[7] The double dash means that the summation refers to the non-trivial zeros, right to V, satisfying the inequality (6. 7).

[8] See e. g. PRACHAR [1], p. 295.

7. The integer ν was restricted so far only by (4. 5); we shall determine it exactly by Lemma II. Let us choose

$$m=\left[\frac{\log T_1}{y_1}-\log^{0,9} T_1\right]. \tag{7. 1}$$

We shall distinguish among the z_j's as „first class z_j's" the numbers

$$\frac{e^{y_1(\varrho-\sigma_1}}{\varrho}|\varrho_1|,$$

and as „second class z_j's" the numbers

$$2\frac{e^{y_1\left(\frac{\varrho}{2}-\sigma_1\right)}}{\varrho}|\varrho_1|.$$

Correspondingly we call „first class b_j's" the numbers

$$\frac{1}{\varphi(k)}(\bar\chi(l_1)-\bar\chi(l_2))\frac{(\omega_0 L_1^{\nu_0})^\varrho}{\varrho^{\nu_0+1}}$$

and „second class b_j's" the numbers

$$\frac{2^{\nu_0}}{\varphi(k)}(\bar\chi(\beta_j)-\bar\chi(\alpha_j))\frac{(\omega_0 L_1^{\nu_0})^{\frac{\varrho}{2}}}{\varrho^{\nu_0+1}}.$$

It is trivial that $\max_j |z_j|=1$ among the z_j's of the first class; since

$$\left|2\frac{e^{y_1\left(\frac{\varrho}{2}-\sigma_1\right)}}{\varrho}|\varrho_1|\right|=2e^{-\frac{y_1\sigma_\varrho}{2}}\left|\frac{e^{y_1(\varrho-\sigma_1)}}{\varrho}|\varrho_1|\right|\leqq 2e^{-\frac{y_1\sigma_\varrho}{2}}$$

we get, using (6. 12), (4. 2) and (1. 5)

$$\left|2\frac{e^{y_1\left(\frac{\varrho}{2}-\sigma_1\right)}}{\varrho}|\varrho_1|\right|<2e_1\left\{-\frac{c_{19}}{40}(\log_2 T_1)^{\frac{1}{5}}\right\}<1$$

if only c_2 in (1. 5) is sufficiently large. Hence

$$\max_j |z_j|=1 \tag{7. 2}$$

is fulfilled. Owing to (6. 5) the total number n of our z_j's cannot exceed

$$c_{20}k\log^{\frac{1}{10}} T_1\cdot\log(2k\log^{\frac{1}{10}} T_1)$$

and hence owing to (1. 5), choosing c_2 sufficiently large, the inequality

$$n<\log^{\frac{1}{10}} T_1\cdot(\log_2 T_1)^3 \tag{7. 3}$$

holds. Owing to (4. 3)—(4. 4), choosing in (1. 5) c_2 sufficiently large we obtain at once that in (3. 2) we may choose

$$\varkappa = \log^{-\frac{2}{3}} T_1 . \tag{7.4}$$

Next we have to choose the indices R and R_1. Let

$$\varrho_2 = \sigma_2 + it_2 \tag{7.5}$$

be one of the non-trivial zeros of all $L(s, \chi)$-functions mod k with $\chi(l_1) \neq \chi(l_2)$ with the absolutely maximal imaginary part not exceeding L_1 and

$$\varrho_3 = \sigma_3 + it_3 \tag{7.6}$$

that with the absolutely minimal t_ϱ for which[9]

$$(2L_1 \geqq) |t_3| \geqq L_1 + 1. \tag{7.7}$$

Owing to (1. 3) we have

$$\sigma_2 = \sigma_3 = \frac{1}{2} . \tag{7.8}$$

Let then be

$$z_h = \frac{e^{y_1(\varrho_2 - \sigma_1)}}{\varrho_2} |\varrho_1| \tag{7.9}$$

$$z_{h_1} = \frac{e^{y_1(\varrho_3 - \sigma_1)}}{\varrho_3} |\varrho_1| . \tag{7.10}$$

8. We assert that all z_j's of the second class are absolutely

$$\leqq |z_{h_1}| . \tag{8.1}$$

Replacing their values, (8. 1) is equivalent with

$$2\left|\frac{\varrho_3}{\varrho}\right| e^{y_1\left(\frac{\sigma_\varrho}{2} - \sigma_3\right)} \leqq 1 \tag{8.2}$$

for all of our ϱ's. In order to prove it we remark that choosing c_2 in (1. 5) sufficiently large we have from (1. 5)

$$10 L_1 e_1 \left\{ -\frac{c_{18}}{40} (\log_2 T_1)^{\frac{1}{5}} \right\} < 1$$

and hence from (7. 7), (1. 5), (4. 2), (6. 7) and $|\varrho| \geqq \frac{1}{2}$

$$2\left|\frac{\varrho_3}{\varrho}\right| e_1\left(-\frac{c_{18}}{2} \cdot \frac{y_1}{\max\{\log k, \log^{4/5}(2 + |t_\varrho|)\}} \right) < 1$$

or owing to (7. 8)

$$2\left|\frac{\varrho_3}{\varrho_1}\right| e_1\left(y_1 \left\{ \frac{1}{2}\left(1 - \frac{c_{18}}{\max\{\log k, \log^{4/5}(2 + |t_\varrho|)\}}\right) - \sigma_3 \right\} \right) < 1,$$

[9] The first, if c_1 in (3. 11) is sufficiently large.

what proves owing to (6. 10) the assertion (8. 2). *Hence all z_j's with index $\leqq h_1$ belong to the first class and B is defined by* (3. 4) (*not by* (3. 6)).

We have further to verify (3. 3) and (3. 5) for our choice (7. 9)—(7. 10). Since from (7. 4)

$$\frac{4n}{m+n\left(3+\frac{\pi}{\varkappa}\right)} < 2\varkappa = 2\log^{-\frac{2}{3}} T_1,$$

it suffices to show owing to (4. 2), (6. 9) and (7. 8) that

$$\left|\frac{\varrho_1}{\varrho_2}\right| e_1 \left\{\frac{1}{10}\log_2 T_1\left(\frac{1}{2}-\sigma_1\right)\right\} > 2\log^{-\frac{2}{3}} T_1$$

and owing to $\sigma_1 \leqq 1$, that

(8. 3) $$\left|\frac{\varrho_1}{\varrho_2}\right| > 2\log^{-\frac{1}{2}} T_1.$$

But since (roughly)

$$|\varrho_1| \geqq \frac{1}{2}, \quad |\varrho_2| \leqq 1+\log^{\frac{1}{10}} T_1,$$

(8. 3) follows from (1. 5) if c_2 is sufficiently large. Hence (3. 3) is verified in our case.

In order to verify (3. 5) we write it in the form (taking in account (7. 9)—(7. 10) and (7. 8))

$$|\varrho_1|\left(\frac{1}{|\varrho_2|}-\frac{1}{|\varrho_3|}\right) > \frac{2n}{m+n\left(3+\frac{\pi}{\varkappa}\right)} e_1\left\{y_1\left(\sigma_1-\frac{1}{2}\right)\right\}$$

what is certainly true owing to (7. 4) and (4. 2) if

$$|\varrho_1|\left(\frac{1}{|\varrho_2|}-\frac{1}{|\varrho_3|}\right) > \log^{-\frac{1}{2}} T_1$$

or

$$|\varrho_1|\frac{|\varrho_3|-|\varrho_2|}{|\varrho_2|\,|\varrho_3|} > \log^{-\frac{1}{2}} T_1.$$

But this is true owing to (1. 5), choosing c_2 sufficiently large, since

$$|\varrho_3|-|\varrho_2| \geqq 1$$

$$\frac{1}{|\varrho_2|\,|\varrho_3|} \geqq \frac{1}{4L_1^2}, \quad |\varrho_1| \geqq \frac{1}{2}.$$

Hence Lemma II is applicable to our sum if the interval $\left(m+1, m+n\left(3+\frac{\pi}{\varkappa}\right)\right)$ is contained in the interval

$$\left(\frac{\log T_1}{y_1}-\log^{0,9} T_1, \frac{\log T_1}{y_1}\right)$$

(so that the requirement (4. 5) be fulfilled). Obviously it is enough to show that

$$n\left(3+\frac{\pi}{\varkappa}\right) \leqq \log^{0,9} T_1$$

and this follows at once from (7. 3) and (7. 4) if only c_2 in (1. 5) is sufficiently large. Thus we have to estimate B in (3. 4) from below. This is owing to (8. 1) a sum of the form

$$\frac{1}{\varphi(k)} \operatorname{Re} \sum_{\chi} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \sum_{\varrho(\chi)}''' \frac{(\omega_0 L_1^{\nu_0})^{\varrho}}{\varrho^{\nu_0+1}}$$

where the summation is extended to the non-trivial zeros of $L(s, \chi)$ right to V with $|t_\varrho| \leqq L_1$, and also to *some* with

$$L_1 < |t_\varrho| \leqq \log^{\frac{1}{10}} T_1 .$$

Replacing this sum by

$$\frac{1}{\varphi(k)} \operatorname{Re} \sum_{\chi} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \sum_{\substack{\varrho(\chi) \\ |t_\varrho| \leqq L_1}} \frac{(\omega_0 L_1^{\nu_0})^{\varrho}}{\varrho^{\nu_0+1}}$$

and taking in account (6. 5), the error is absolutely less than

$$c_{21} \log (kL_1) \omega_0 \cdot 2^{\nu_0} < e_1\left(2 \frac{L_1^2}{\log L_1}\right)$$

owing to the definition of L_1, ω_0 and ν_0, if only c_1 is sufficiently large. But then Lemma III gives the estimation

$$B \geqq e_1\left(\frac{1}{30} L_1^2\right) - e_1\left(2 \frac{L_1^2}{\log L_1}\right) > 1, \tag{8. 4}$$

if only c_1 is sufificiently large. Hence Lemma II gives the existence of integers ν_1 and ν_2 with

$$\frac{\log T_1}{y_1} - \log^{0,9} T_1 \leqq \nu_1 \leqq, \quad \nu_2 \leqq \frac{\log T_1}{y_1} \tag{8. 5}$$

such that the expression $D(\nu)$ in curly bracket on the right of (6. 8) (taking in account its *reality* and (7. 3), (7. 4), (1. 5)) is for $\nu = \nu_1$

$$> \frac{1}{3} \log^{-\frac{1}{10}} T_1 \cdot (\log_2 T_1)^{-3} \left\{ \frac{1}{24\left(\log T_1 + n\left(3+\frac{\pi}{\varkappa}\right)\right)} \right\}^{\log^{\frac{1}{10}} T_1 \cdot (\log_2 T_1)^3} \cdot$$

$$\cdot \left(\frac{|z_h|}{2}\right)^{m+n\left(3+\frac{\pi}{\varkappa}\right)} .$$

Hence owing to the previous estimation

$$|z_h| > 2 \log^{-\frac{2}{3}} T_1$$

we get using also (7. 9), (7. 1) and (1. 5)

$$D(\nu_1) > e_1(-\log^{\frac{1}{2}} T_1) \left(\frac{|z_h|}{2}\right)^{\nu_1} \cdot (\log^{-\frac{2}{3}} T_1)^{n\left(3+\frac{\pi}{\varkappa}\right)} >$$

$$> e_1(-\log^{\frac{4}{5}} T_1) \left(\frac{e^{\left(\frac{1}{2}-\sigma_1\right)y_1}}{2|\varrho_2|} |\varrho_1|\right)^{\nu_1}.$$

Putting it into (6. 8) we get for the right side the lower bound

$$\frac{e^{\frac{\nu_1 y_1}{2}}}{(2|\varrho_2|)^{\nu_1}} \cdot e_1(-\log^{\frac{4}{5}} T_1) - c_{17} T^{0,42} > \sqrt{T_1}\, e_1\left(-21 \frac{\log T_1 \log_3 T_1}{\log_2 T_1}\right) - c_{17} T^{0,42}.$$

Taking in account that from (4. 18), (4. 16), (4. 11), (4. 14), (4. 5), (4. 2) and alsc (1. 5) with sufficiently large c_2

$$\log Y_1 \frac{\log^{\nu_1+\nu_0}\left(Y^{\frac{\nu_1+\nu_0}{\nu_1+\nu_0+1}}\right)}{(\nu_1+\nu_0)!} < c_{22} \log T_1 \cdot \left(\frac{e}{\nu_1} \log T_1\right)^{\nu_1+\nu_0} < e_1\left(22 \frac{\log T_1 \log_3 T_1}{\log_2 T_1}\right)$$

and also from (4. 1) and (1. 5)

$$T_1 > T e_1(-\log^{\frac{1}{2}} T)$$

the first part of the Theorem follows at once. Analogously the second part, choosing $\nu = \nu_2$.

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