COMPARATIVE PRIME-NUMBER THEORY. V

(SOME THEOREMS CONCERNING THE GENERAL CASE)

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1. In the previous papers II and III of this series (see KNAPOWSKI—TURÁN [2], [3]) we obtained rather far-reaching results on the comparison of the distribution of primes $\equiv 1$ and $\equiv l \bmod k$, or shortly in the case $(1, l)_k$. The difficulties of the general case, i. e. the case

(1. 1) $$(l_1, l_2)_k, \qquad l_1 \not\equiv l_2 \not\equiv 1 \bmod k$$

are indicated in our paper [4]; though in this paper more or less satisfactory results are given for the simplest case $k=8$, it was mentioned that already in the next difficult case $k=5$ the investigation of

$$\pi(x, 5, 2) - \pi(x, 5, 4)$$

cannot be touched at present. In the present paper we are going to investigate the general case. As explained in our paper [1] of this series, for general k practically nothing was known in this direction; hence also conditional results are of interest. Our general results are conditional; sometimes only Haselgrove's condition[1] is supposed, sometimes the „finite" RIEMANN—PILTZ conjecture, according to which no $L(s, \chi)$ mod k vanishes for a sufficiently large $c_1 \geqq 1$[2] for

(1. 2) $$\sigma > \frac{1}{2}, \quad |t| \leqq c_1 k^{10},$$

sometimes both, or what amounts to the same, no $L(s, \chi)$ with $\chi \neq \chi_0$ vanishes for

$$\sigma > \frac{1}{2}, \quad |t| \leqq c_1 k^{10}$$

(1. 3) and

$$\sigma = \frac{1}{2}, \quad |t| \leqq A(k).$$

Then putting as usual

$$\sum_{\substack{n \leqq x \\ n \equiv l \bmod k}} \Lambda(n) \overset{\text{def}}{=} \psi(x, k, l) \qquad \sum_{\substack{n \leqq x \\ n \equiv l \bmod k}} \frac{\Lambda(n)}{\log n} \overset{\text{def}}{=} \Pi(x, k, l)$$

we assert the

[1] We remind the reader that this condition means the existence of an $A=A(k)$ such that no $L(s, \chi)$ mod k vanishes in the parallelogramm $0<\sigma<1$, $|t| \leqq A(k)$ $(s=\sigma+it)$.

[2] As in the previous papers, $c_1, c_2, \dots$ are always positive numerical, explicitly calculable constants, and $(l, k)=1$, $e_\nu(x)=e_{\nu-1}(e^x)$, $\log_\nu x = \log_{\nu-1}(\log x)$.

THEOREM 1. 1. *Supposing the truth of the conjecture* (1. 3) *with sufficiently large* c_1 *and with sufficiently large* c_2

$$T > \max\left\{e_2(c_2 k^{20}),\ e_1\left(2e_1\left(\frac{1}{A(k)^3}\right)+c_2 k^{20}\right)\right\} \tag{1. 4}$$

we have for $l_1 \neq l_2$ *the inequalities*

$$\max_{T^{1/3}\leqq x\leqq T}\{\psi(x,k,l_1)-\psi(x,k,l_2)\} > \sqrt{T}\, e_1\left(-44\frac{\log T\log_3 T}{\log_2 T}\right) \tag{1. 5}$$

and

$$\max_{T^{1/3}\leqq x\leqq T}\{\Pi(x,k,l_1)-\Pi(x,k,l_2)\} > \sqrt{T}\, e_1\left(-44\frac{\log T\log_3 T}{\log_2 T}\right). \tag{1. 6}$$

Since none of l_1 and l_2 are distinguished to the other[3], they can be changed. Hence each of the functions

$$\psi(x,k,l_1) - \psi(x,k,l_2), \tag{1. 7}$$

$$\Pi(x,k,l_1) - \Pi(x,k,l_2) \tag{1. 8}$$

has a sign-change in $[T^{1/3}, T]$ whenever T satisfies (1. 4). Denoting as in the previous papers by $U_k(T, l_1, l_2)$ and $V_k(T, l_1, l_2)$ the number of sign-changes of the functions (1. 7) and (1. 8) for $0 < x \leqq T$, resp., this gives at once (like in our paper [2] of this series) the

THEOREM 1. 2. *For*

$$T > \max\left\{e_1(9e_1(2c_2k^{20})),\ e_1\left(72e_1\left(\frac{2}{A(k)^3}\right)+18c_2^2k^{40}\right)\right\}$$

the inequalities

$$U_k(T, l_1, l_2) > \frac{1}{2\log 3}\log_2 T,$$

$$V_k(T, l_1, l_2) > \frac{1}{2\log 3}\log_2 T$$

hold.

Of course this gives for the first sign-change an explicit upper bound, depending only upon k; however in the paper VII of this series we shall give such an upper bound supposing only Haselgrove's assumption, for $\psi(x,k,l_1)-\psi(x,k,l_2)$ at least.

2. What can be told upon

$$\pi(x,k,l_1) - \pi(x,k,l_2)$$

in the general case? If l_1 and l_2 are such that none of the congruences

$$x^2 \equiv l_1 \bmod k, \qquad x^2 \equiv l_2 \bmod k \tag{2. 1}$$

[3] In the course of the proof (see (8. 17)) we apparently make such a distinction. However, making it we shall be able to give „large positive" lower bound for max. and „large negative" upper bound for min. of $\Pi(x,k,l_1)-\Pi(x,k,l_2)$.

are solvable it follows at once from Theorem 1. 1 (replacing c_2 by a larger constant) that for

$$(2.2)\qquad T > \max\left\{e_2(c_3 k^{20}),\ e_1\left(2e_1\left(\frac{1}{A(k)^3}\right)+c_4 k^{40}\right)\right\}$$

the inequality

$$(2.3)\qquad \max_{T^{1/3}\leqq x\leqq T}\{\pi(x,k,l_1)-\pi(x,k,l_2)\} > \sqrt{T}\, e_1\left(-45\,\frac{\log T\log_3 T}{\log_2 T}\right)$$

holds. In the paper VI of this series however we shall prove (2. 3) under the weaker restriction that the number of solutions of the congruences in (2. 1) is *equal.* For the sake of orientation we remark that for odd k the number of those l-residue-classes for which the congruence $x^2\equiv l \bmod k$ is not solvable, is obviously

$$\geqq (1-2^{-\nu(k)})\varphi(k)$$

($\nu(k)$ the number of different prime-factors of k) which certainly shows that for odd k „at least 25%" of all cases are covered even by (2. 1) and the value of the number $N_k(l)$ of incongruent solutions of $x^2\equiv l \bmod k$ is for all $(l,k)=1$ either 0 or $N_k(1)$.

Does the Theorem in (2. 2)—(2. 3) make the results of our paper [4] superfluous? By no means. Putting $k=8$ the constant in (2. 2) becomes most probably so large that the truth of the corresponding RIEMANN—PILTZ conjecture cannot be verified by machines, so that in order to get in this case *unconditional* results special argument was necessary.

3. As to the race-problem we have shown in our paper [2] that for a „dense" sequence of integers

$$(3.1)\qquad \pi(x_\nu,k,1) > \frac{1}{\varphi(k)}\pi(x_\nu).$$

However plausible we cannot prove at present the corresponding inequality for general $\pi(x,k,l_0)$ instead of $\pi(x,k,1)$. What we can prove in this direction is the

THEOREM 3. 1. *Supposing the truth of* (1. 3) *we have for each* $(l,k)=1$ *and*

$$(3.2)\qquad T > \max\left\{e_2(c_2 k^{20}),\ e_1\left(2e_1\left(\frac{1}{A(k)^3}\right)+c_3 k^{20}\right)\right\}$$

the inequalities

$$(3.3)\qquad \max_{T^{1/3}\leqq x\leqq T}\left\{\Pi(x,k,l)-\frac{1}{\varphi(k)}\Pi(x)\right\} > \sqrt{T}\, e_1\left(-44\,\frac{\log T\log_3 T}{\log_2 T}\right)$$

and

$$(3.4)\qquad \min_{T^{1/3}\leqq x\leqq T}\left\{\Pi(x,k,l)-\frac{1}{\varphi(k)}\Pi(x)\right\} < -\sqrt{T}\, e_1\left(-44\,\frac{\log T\log_3 T}{\log_2 T}\right);$$

the same holds for ψ *instead of* Π *too.*

Mutatis mutandis the analogous statements hold for

$$\Pi(x, k, l) - \frac{1}{\varphi(k)} \mathrm{Li}\, x \tag{3. 5}$$

and

$$\psi(x, k, l) - \frac{1}{\varphi(k)} x \tag{3. 6}$$

too. However essentially new difficulties arise when trying to prove that for all $(l, k) = 1$ the function

$$\pi(x, k, l) - \frac{1}{\varphi(k)} \mathrm{Li}\, x \tag{3. 7}$$

changes sign infinitely often. This can be deduced supposing (1. 3) from Theorem (3. 1) mutatis mutandis if only the congruence $x^2 \equiv l \bmod k$ is not solvable; most probably this is the case generally too.

4. The proofs obviously reduce to that of Theorems 1. 1 and 3. 1; since the second follows that the first mutatis mutandis we shall confine ourselves to the proof of Theorem 1. 1. This will be rather intricate. It would be plausible to start from an integral of type

$$\frac{1}{2\pi i} \int_{(2)} \left(\frac{e^{\gamma_1 s}}{s}\right)^{\nu} \frac{1}{\varphi(k)} \left\{ \sum_{\chi} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \frac{L'}{L}(s, \chi) \right\} ds \tag{4. 1}$$

as previously. Everything goes smoothly until one comes to critical „generalized power-sum"

$$\sum_{\chi} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \sum_{\varrho(\chi)}' \left(\frac{e^{\gamma_1 \varrho}}{\varrho}\right)^{\nu}, \tag{4. 2}$$

where the last summation is to be extended over the non-trivial zeros of all $L(s, \chi)$-functions mod k in a domain, not depending upon ν. The one-sided second main theorem (stated as Lemma I in our paper [2]) as well as its generalized form (stated as Theorem 4. 1 in our paper [3]) cannot be used as previously since the „coefficients" $(\bar{\chi}(l_1) - \bar{\chi}(l_2))$ have no more a non-negative real part and not even an „essential part" of them can be singled out with non-negative real parts as in our paper [3]. As a remedy one might observe that changing the integral in (4. 1) suitably one can arrange that the critical sum should assume the form

$$\sum_{\chi} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \sum_{\varrho(\chi)}' \frac{\eta_1^{\varrho}}{\varrho} \cdot \left(\frac{e^{\gamma_1 \varrho}}{\varrho}\right)^{\nu} \tag{4. 3}$$

with an $\eta_1 > 0$ independent of ν; now the coefficients are the numbers

$$(\bar{\chi}(l_1) - \bar{\chi}(l_2)) \frac{\eta_1^{\varrho}}{\varrho} \tag{4. 4}$$

and after appropriate choice of η_1 one could obtain a (weak) positive lower bound G for a *certain* partial-sum of

$$\text{Re} \sum_{\chi} \left\{ (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \sum_{\varrho(\chi)} \frac{\eta_1^{\varrho}}{\varrho} \right\} \tag{4.5}$$

which offers hope to the smooth applicability of Theorem 4. 1 of our paper [3]. But to the applicability we need *another* partial-sum of (4. 5) and the difference must be estimated so well that positive lower bound G should not be destroyed. In order to meet this new difficulty we modify the integral (4. 1) so that the critical sum assumes the form

$$\sum_{\chi} \left\{ (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \sum_{\varrho(\chi)} \frac{(\eta_1 \cdot \eta_2^{\nu_0})^{\varrho}}{\varrho^{\nu_0+1}} \left(\frac{e^{y_1 \varrho}}{\varrho} \right)^{\nu} \right\} \tag{4.6}$$

with a suitable $\eta_2 > 0$ and suitable „large" positive integer ν_0 so that the „coefficients" are now the numbers

$$(\bar{\chi}(l_1) - \bar{\chi}(l_2)) \frac{\eta_1^{\varrho}}{\varrho} \left(\frac{\eta_2^{\varrho}}{\varrho} \right)^{\nu_0}. \tag{4.7}$$

Using the estimation (4. 5) ν_0 can be determined by the application of Theorem 4. 1 (and even simpler) so that a *certain* partial-sum of

$$\text{Re} \sum_{\chi} \left\{ (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \sum_{\varrho(\chi)} \frac{(\eta_1 \eta_2^{\nu_0})^{\varrho}}{\varrho^{\nu_0+1}} \right\} \tag{4.8}$$

could be sufficiently estimated from below. But owing to the rapid decrease of the terms in (4. 8) the above mentioned difficulty by the application of Theorem 4. 1 to the sum (4. 6), to the determination of ν, disappears now. Hence the characteristic novelty of the proof is quite shortly the *twice* application of Theorem 4. 1.

In the proofs assumption 1. 2 is not *very* deeply used; it seems to be within the possibilities to deduce Theorem 1. 1 only from Haselgrove's conditions (1. 3).

As to a comparison of the results of the present paper to those obtainable by the classical methods see our paper [1].

5. In the proof of Theorem 1. 1 we shall rise some lemmata used also in the previous papers of this series which we shall repeat without proofs to make the paper, as told, possibly self-contained.

LEMMA I. *Under Haselgrove's condition for*

$$\tau > \max \left\{ c_6, e_2(k), e_2\left(\frac{1}{A(k)^3} \right) \right\} \tag{5.1}$$

there is a y_1 *with*

$$\frac{1}{20} \log_2 \tau \leqq y_1 \leqq \frac{1}{10} \log_2 \tau \tag{5.2}$$

such that for all $\varrho = \sigma_\varrho + it_\varrho$*-zeros of all* $L(s, \chi)$*-functions* mod k *the inequality*

$$\pi \geqq \left| \operatorname{arc} \frac{e^{it_\varrho y_1}}{\varrho} \right| \geqq c_7 \frac{A(k)^3}{k(1+|t_\varrho|)^6 \log^3 k(2+|t_\varrho|)}$$

holds.

(For the proof see our paper [2].)

Further let m be a non-negative integer and

$$1 = |z_1| \geqq |z_2| \geqq \ldots \geqq |z_n| \tag{5. 3}$$

so that with a $0 < \varkappa \leqq \frac{\pi}{2}$

$$\varkappa \leqq |\operatorname{arc} z_j| \leqq \pi \qquad (j = 1, 2, \ldots, n). \tag{5. 4}$$

Let the index h be such that

$$|z_h| \geqq \frac{4n}{m + n\left(3 + \frac{\pi}{\varkappa}\right)} \tag{5. 5}$$

and fixed. Further we define B for given b_j numbers by

$$B = \min_{h \leqq \xi \leqq n} \operatorname{Re} \sum_{j=1}^{\xi} b_j. \tag{5. 6}$$

Then we assert the

LEMMA II. *If* $B > 0$, *then there are integers* ν_1 *and* ν_2 *with*

$$m + 1 \leqq \nu_1, \quad \nu_2 \leqq m + n\left(3 + \frac{\pi}{\varkappa}\right) \tag{5. 7}$$

such that

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_1} \geqq \frac{B}{2n+1} \left\{ \frac{n}{24\left(m + n\left(3 + \frac{\pi}{\varkappa}\right)\right)} \right\}^{2n} \left(\frac{|z_h|}{2} \right)^{m + n\left(3 + \frac{\pi}{\varkappa}\right)} \tag{5. 8}$$

and

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_2} \leqq -\frac{B}{2n+1} \left\{ \frac{n}{24\left(m + n\left(3 + \frac{\pi}{\varkappa}\right)\right)} \right\}^{2n} \left(\frac{|z_h|}{2} \right)^{m + n\left(3 + \frac{\pi}{\varkappa}\right)}. \tag{5. 9}$$

(For the proof see our paper [3]).[4]

[4] Here we need a slightly weaker form of this lemma as it is proved in [3]; the index h_1 there can be chosen as n here.

Further we shall need the[5]

LEMMA III. *Let m be non-negative integer, further* $z_1, z_2, \ldots, z_n$ *with* (5. 3). *Let* h_2 *be such that*

$$|z_{h_2}| > \frac{2n}{m+n}. \tag{5. 10}$$

Finally B_1 *and the index* h_3 *be defined by*

$$B_1 = \min_{h_2 \leqq \xi < h_3} \left| \sum_{j=1}^{\xi}{}' b_j \right| \tag{5. 11}$$

if there is a z_{h_3} *with*

$$|z_{h_3}| < |z_{h_2}| - \frac{n}{m+n} \tag{5. 12}$$

and

$$B_1 = \min_{h_2 \leqq \xi \leqq n} \left| \sum_{j=1}^{\xi} b_j \right| \tag{5. 13}$$

otherwise. Then there is an integer ν_3 *with*

$$m+1 \leqq \nu_3 \leqq m+n \tag{5. 14}$$

so that

$$\left| \sum_{j=1}^{n} b_j z_j^{\nu_3} \right| \geqq \left(\frac{n}{24e(m+2n)} \right)^n B_1 \left(\frac{|z_{h_2}|}{2} \right)^{m+n}.$$

6. We shall need three more lemmata. In these and in the proof we shall have beside c_1 and c_2 also $c_8, c_9, \ldots,$ some of them immaterial but some of them must be chosen properly. These are c_1, c_2, c_{10} and c_{11}; first c_{10} must be chosen sufficiently large, then c_{11} large in dependence upon c_{10}, then c_1 large in dependence upon c_{10} and c_{11} and finally c_2 in dependence upon c_{10}, c_{11} and c_1. First we shall make some restrictions upon c_{10}; all these and the later ones are *lower* limitations. We have[6] for $x \geqq 2$

$$\left| \psi(x, k, l) - \frac{x}{\varphi(k)} + \frac{1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l) \sum_{|t_\varrho| \leqq x} {}_{\varrho(\chi)} \frac{x^\varrho}{\varrho} \right| < c_8 \log^2 kx$$

and hence if no $L(s, \chi)$ mod k vanishes for

$$\sigma > \frac{1}{2}, \quad |t| \leqq x$$

then

$$\left| \psi(x, k, l) - \frac{x}{\varphi(k)} \right| < c_9 \sqrt{x} \log^2 kx. \tag{6. 1}$$

[5] For this lemma see KNAPOWSKI [1]. This makes the so-called second main-theorem (see TURÁN [1] or SÓS—TURÁN [1]) on the same way more elastic to applications as Lemma II the one-sided second main-theorem (see TURÁN [2]).

[6] This follows at once from (4. 43) on p. 233 (with $T = x$) of PRACHAR [1] (using (1. 3) to that extent that $L(s, \chi) \neq 0$ for $0 < s < 1$).

Then we chose

$$c_{10} \geqq \max \{(8c_9)^{\frac{5}{2}}, e^{800}\} \text{ [7]} \tag{6.2}$$

and require

$$c_1 > 2^{-6} c_{10} \tag{6.3}$$

so that

$$c_1 k^{10} \geqq c_{10} k^3,$$

i. e. no $L(s, \chi) \bmod k$ vanishes for

$$\sigma > \frac{1}{2}, \quad |t| \leqq c_{10} k^3 \tag{6.4}$$

owing to (1. 2).

Next we state the simple

LEMMA IV. *Supposing only the truth of* (1. 2), *for each* $(l, k) = 1$ *there is a prime* P *with* $P \equiv l \bmod k$ *such that*

$$(2<) \frac{1}{2} c_{10} k^3 \leqq P \leqq c_{10} k^3. \tag{6.5}$$

Namely we may apply (6. 1) owing to (6. 4) with $x = c_{10} k^3$ and $x = \frac{1}{2} c_{10} k^3$; this gives owing to (6. 4) and (6. 1)

$$\left| \sum_{\substack{\frac{1}{2} c_{10} k^3 < n \leqq c_{10} k^3 \\ n \equiv l \bmod k}} \Lambda(n) - \frac{c_{10}}{2} \frac{k^3}{\varphi(k)} \right| < 2c_9 \sqrt{c_{10} k^3} \cdot \log^2 (c_{10} k^4)$$

i. e.

$$\sum_{\substack{\frac{1}{2} c_{10} k^3 < n \leqq c_{10} k^3 \\ n \equiv l \bmod k}} \Lambda(n) > \frac{c_{10}}{2} k^2 - 2c_9 \sqrt{c_{10} k^3} \log^2 (c_{10} k^4) >$$

$$> \frac{c_{10}}{2} k^2 - 2c_9 \cdot c_{10}^{\frac{3}{5}} \cdot k^{1,9} > \frac{c_{10}}{4} k^2$$

owing to (6. 2). Since further evidently

$$\sum_{\substack{p, \alpha \\ p^\alpha \leqq x \\ p^\alpha \equiv l \bmod k \\ \alpha \geqq 2}} \log p < 2\sqrt{x} \log x,$$

we have

$$\sum_{\substack{\frac{1}{2} c_{10} k^3 < n \leqq c_{10} k^3 \\ n \equiv l \bmod k}} \log p > \frac{c_{10}}{4} k^2 - 2\sqrt{c_{10} k^3} (c_{10} k^3)^{\frac{1}{20}} > 0$$

owing to (6. 2) indeed.

[7] I. e. for $x \geqq c_{10}$ $\log x < x^{\frac{1}{20}}$.

7. Further we need the

LEMMA V. *Supposing for a fixed k the truth of* (1. 2) *only, then with the above c_{10} and some $c_{11}>3$ (depending on c_{10}) for all l_1, l_2-pairs ($l_1 \not\equiv l_2$) the inequality*

$$(7.1)\qquad \max_{\frac{c_{10}}{3}k^3 \leqq \omega \leqq c_{10}k^3} \left| \frac{1}{\varphi(k)} \sum_{\chi} (\bar{\chi}(l_1)-\bar{\chi}(l_2)) \sum_{|t_\varrho| \leqq c_{11}k^4}{}_{\varrho(\chi)} \frac{\omega^\varrho}{\varrho} \right| > \frac{1}{4}\log k$$

holds (denoting again $\varrho=\sigma_\varrho+it_\varrho$).

For a proof we start from the formula[8] valid for $x=[x]$, χ' primitive character mod k', $y\geqq 2$, $x\geqq 2$

$$(7.2)\qquad \left| \sum_{n\leqq x}{}' \Lambda(n)\chi'(n) - E_0 x + \sum_{|t_\varrho|\leqq y}{}_{\varrho(\chi')} \frac{x^\varrho}{\varrho} + d_0(\chi') + \nu_0(\chi')\log x \right| <$$

$$< c_{12}\frac{x}{y}(\log^2 x + \log^2 k'y),$$

where $E_0=\begin{cases}1\\0\end{cases}$ for $\chi' \begin{cases}=\chi_0'\\ \neq\chi_0'\end{cases}$ and $d_0(\chi')$, $\nu_0(\chi')$ are independent of x, y. Further if $\chi(n)$ is an arbitrary character mod k and $\chi'(n)$ is the equivalent character mod k' ($k'|k$) then it is known[9]

$$(7.3)\qquad \sum_{n\leqq x}{}' \Lambda(n)\chi(n) = \sum_{n\leqq x}{}' \Lambda(n)\chi'(n) - \sum_{p^\alpha\leqq x,\, p|k,\, p\nmid k'}{}' \chi'(p^\alpha)\log p.$$

Putting (7. 2) into (7. 3), multiplying by $\dfrac{1}{\varphi(k)}(\bar{\chi}(l_1)-\bar{\chi}(l_2))$ we sum over χ's. Then we obtain

$$(7.4)\qquad \Bigg| \sum_{\substack{n\leqq x\\ n\equiv l_1 \bmod k}}{}' \Lambda(n) - \sum_{\substack{n\leqq x\\ n\equiv l_2 \bmod k}}{}' \Lambda(n) + \frac{1}{\varphi(k)}\sum_{\chi} d_0(\chi')(\bar{\chi}(l_1)-\bar{\chi}(l_2)) +$$

$$+\frac{\log x}{\varphi(k)}\sum_{\chi} \nu_0(\chi')(\bar{\chi}(l_1)-\bar{\chi}(l_2)) +$$

$$+\frac{1}{\varphi(k)}\sum_{\chi}(\bar{\chi}(l_1)-\bar{\chi}(l_2)) \sum_{|t_\varrho|\leqq y}{}_{\varrho(\chi)} \frac{x^\varrho}{\varrho} \Bigg| < c_{12}\frac{x}{y}(\log^2 x+\log^2 ky),$$

since the set $\varrho(\chi')$ is, as well-known, identical with $\varrho(\chi)$.

Now we use (7. 4) with $x=P$ and $x=P-1$ where P is defined in Lemma IV with $l=l_1$, say. Then the contribution of the third sum on left is 0, that of the fourth

$$\frac{1}{\varphi(k)}\log\frac{P}{P-1}\sum_{\chi}\nu_0(\chi')(\bar{\chi}(l_1)-\bar{\chi}(l_2))$$

[8] See PRACHAR [1] pp. 228—229, Satz 4. 4. The term d_0 is owing to (4. 44) p. 233 of this book $O(\log k)$. Σ' means that the term corresponding to $n=x$ must be taken with coefficient $\frac{1}{2}$.

[9] See PRACHAR [1], p. 234.

and taking in account that[10] $\nu_0(\chi')$ is 0 or 1, this is absolutely

$$\leqq \log \frac{P}{P-1} \leqq \frac{1}{3}.$$

The contribution of the first sum is $\frac{1}{2}\Lambda(P)$, that of the second

$$=\begin{cases} \frac{1}{2}\log 2 & \text{if}^{11}\ P-1 \equiv l_2 \mod k \text{ and power of 2,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence choosing

$$y = c_{11}k^4$$

we obtain

$$(7.5)\qquad \left| \frac{1}{\varphi(k)} \sum_{\chi} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \left\{ \sum_{|t_\varrho| \leqq c_{11}k^4} {}_{\varrho(\chi)} \frac{P^\varrho}{\varrho} - \sum_{|t_\varrho| \leqq c_{11}k^4} {}_{\varrho(\chi)} \frac{(P-1)^\varrho}{\varrho} \right\} \right| >$$

$$> \frac{1}{2}(3\log k - \log 2 - 1) - 2c_{12}\frac{P}{c_{11}k^4}(\log^2 P + \log^2 kP) > \frac{3}{2}\log k -$$

$$- 0{,}9 - \frac{8c_{12}c_{10}}{c_{11}k}(32\log^2 k + \log^2 c_{10}) > \frac{3}{2}\log k - 1 > \frac{1}{2}\log k$$

if only

$$(7.6)\qquad c_{11} \geqq c_{10} \quad \text{and} \quad c_{11} \geqq 80 c_{12} c_{10}\left(\log^2 c_{10} + 32 \max_{r \geqq 2} \frac{\log^2 x}{x}\right).$$

From (7. 5) Lemma V follows at once.

8. Let $\omega = \omega_0$ be a value for which (7. 1) is realized and we write shortly

$$(8.1)\qquad c_1 k^{10} \overset{\text{def}}{=} L_1.$$

Then we assert the

LEMMA VI. *For* $l_1 \not\equiv l_2 \mod k$, *supposing only* (1. 2) *there is an integer* ν_0 *with*

$$(8.2)\qquad \frac{L_1^2}{\log L_1} \leqq \nu_0 \leqq \frac{L_1^2}{\log L_1} + L_1^{1,16}$$

so that

$$(8.3)\qquad \left| \frac{1}{\varphi(k)} \sum_{\chi} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \sum_{|t_\varrho| \leqq L_1} {}_{\varrho(\chi)} \frac{(\omega_0 L_1^{\nu_0})^\varrho}{\varrho^{\nu_0+1}} \right| > e_1\left(\frac{1}{30}L_1^2\right).$$

The hypothesis (1. 2) and Siegel's theorem (see SIEGEL [1]) according to which each $L(s, \chi)$ mod k has a zero in the domain

$$0 < \sigma < 1$$

$$(8.4)\qquad |t| < \frac{c_{13}}{\log_3 (k + e_3(1))},$$

[10] See PRACHAR [1], p. 224.
[11] $P-1$ is even.

result that the non-trivial zero ϱ_1 with the minimal positive imaginary part of all $L(s,\chi)$-functions mod k with $\overline{\chi}(l_1)\neq\overline{\chi}(l_2)$ has the real part $\frac{1}{2}$, i. e.

$$\varrho_1 = \frac{1}{2}+it_1. \tag{8.5}$$

Let further

$$\varrho_2 = \sigma_2+it_2$$

be the non-trivial zero with the greatest imaginary part

$$\leqq c_{11}k^4 \tag{8.6}$$

among all non-trivial zeros of all L-functions mod k with $\chi(l_1)\neq\chi(l_2)$ and

$$\varrho_3 = \sigma_3+it_3 \tag{8.7}$$

the non-trivial zero with the smallest imaginary part $\geqq (c_{11}+1)k^4$ among the above mentioned zeros. If

$$c_1 > 2^{-6}(c_{11}+1) \tag{8.8}$$

(which is meant as a lower bound for c_1 as told in 5) then

$$\sigma_2=\sigma_3=\frac{1}{2}. \tag{8.9}$$

We write the sum under consideration in the form

$$S \overset{\text{def}}{=} \frac{1}{\varphi(k)}\frac{(L_1^{\nu_0})^{\varrho_1}}{\varrho_1^{\nu_0}}\sum_{\chi}\left\{(\overline{\chi}(l_1)-\overline{\chi}(l_2))\cdot\sum_{|t_\varrho|\leqq L_1}{}_{\varrho(\chi)}\frac{\omega_0^{\varrho}}{\varrho}\left(\frac{L_1^{\varrho-\varrho_1}}{\varrho}\varrho_1\right)^{\nu_0}\right\}. \tag{8.10}$$

We shall apply Lemma III with

$$z_j=\frac{L_1^{\varrho-\varrho_1}}{\varrho}\varrho_1$$
$$b_j=\frac{\omega_0^{\varrho}}{\varrho}(\overline{\chi}(l_1)-\overline{\chi}(l_2))\frac{1}{\varphi(k)}; \tag{8.11}$$

then assumption (1. 2) assures that $\max_j |z_j|=1$ is fulfilled. Let further be

$$m=\left[\frac{L_1^2}{\log L_1}\right] \tag{8.12}$$

and

$$z_{h_2}=\frac{L_1^{\varrho_2-\varrho_1}}{\varrho_2}\varrho_1 \tag{8.13}$$

$$z_{h_3}=\frac{L_1^{\varrho_3-\varrho_1}}{\varrho_3}\varrho_1. \tag{8.14}$$

9. We have to verify (5. 10) and (5. 12). First we remark that

$$n < c_{14} k L_1 \log (k L_1) < c_{15} (k L_1)^{1,05} = c_{15} \left(\frac{L_1^{1,1}}{c_1^{1/10}} \right)^{1,05} < L_1^{1,16} \tag{9. 1}$$

if only $c_1 > c_{15}^{10}$ and hence

$$\frac{2n}{m+n} < \frac{2n}{m} < \frac{4 L_1^{1,16}}{L_1^2} \log L_1 < c_{16} L_1^{-4/5}, \tag{9. 2}$$

whereas from (8. 5), (8. 9) and (8. 6)

$$|z_{h_2}| = \left| \frac{\varrho_1}{\varrho_2} \right| \geqq \frac{\frac{1}{2}}{c_{11} k^4 + \frac{1}{2}} \geqq \frac{c_1^{2/5}}{3 c_{11}} L_1^{-2/5} \tag{9. 3}$$

and hence if

$$\frac{c_1^{2/5}}{3 c_{11}} > c_{16} \tag{9. 4}$$

(which again is to be interpreted as a lower bound for c_1 as told in 5) then (5. 10) is verified indeed. As to (5. 12) we remark that from (1. 2), (8. 13) and (8. 14) it follows taking in account (8. 9)

$$|z_{h_2}| - |z_{h_3}| = |\varrho_1| \left(\frac{1}{|\varrho_2|} - \frac{1}{|\varrho_3|} \right) \geqq \frac{1}{2} \left(\frac{1}{c_{11} k^4 + 1} - \frac{1}{(c_{11}+1) k^4} \right) >$$

$$> \frac{1}{4 c_{11}^2} \cdot \frac{1}{k^4} = \frac{c_1^{2/5}}{4 c_{11}^2} \cdot L_1^{-2/5}$$

whereas from (9. 2)

$$\frac{n}{m+n} < \frac{c_{16}}{2} L_1^{-4/5}$$

and hence if

$$c_1^{2/5} > 2 c_{16} c_{11}^2 \tag{9. 5}$$

(again a lower bound for c_1) then (5. 12) is verified too. Before applying Lemma III we need a lower bound for B_1. Owing to the definition of B_1 is

$$B_1 = \frac{1}{\varphi(k)} \sum_{\chi} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \sum_{|t_\varrho| \leqq H(\chi)} {}_{\varrho(\chi)} \frac{\omega_0^\varrho}{\varrho}, \tag{9. 6}$$

where all $H(\chi)$'s are between $(c_{11}-1) k^4$ and $(c_{11}+2) k^4$ if only

$$c_{11} > c_{17}.$$

Replacing in (9. 6) however the inner sum by

$$\sum_{|t_\varrho| \leqq c_{11} k^4} {}_{\varrho(\chi)} \frac{\omega_0^\varrho}{\varrho}$$

the total-error cannot exceed absolutely

$$\frac{1}{\varphi(k)}\sum_{\chi}\sum_{(c_{11}-1)k^4\leqq|t_\varrho|\leqq(c_{11}+2)k^4}^{\varrho(\chi)}\left|\frac{\omega_0^\varrho}{\varrho}\right|\leqq c_{18}\log(c_{11}k^4)\frac{\omega_0^{1/2}}{(c_{11}-1)k^4}$$

i. e. using $\omega_0\leqq c_{10}k^3$

$$(9.7)\qquad <c_{19}(4\log k+\log c_{11})\frac{\sqrt{c_{10}}}{c_{11}k^{5/2}}<4c_{19}\frac{\sqrt{c_{10}}\log c_{11}}{c_{11}}\cdot\frac{\log k}{k^{5/2}}.$$

Choosing now c_{11} so that

$$(9.8)\qquad 4\frac{c_{19}\log c_{11}}{c_{11}}<\frac{1}{8\sqrt{c_{10}}}$$

we get from (9. 8), (9. 7) and Lemma V

$$(9.9)\qquad |B_1|>\frac{1}{8}\log k.$$

Hence the application of Lemma III gives the existence of an integer ν_0 with (8. 2) so that

$$(9.10)\qquad \left|\frac{1}{\varphi(k)}\sum_{\chi}(\bar\chi(l_1)-\bar\chi(l_2))\sum_{|t_\varrho|\leqq L_1}^{\varrho(\chi)}\frac{(\omega_0L_1^{\nu_0})^\varrho}{\varrho^{\nu_0+1}}\right|>$$

$$>\left(\frac{\sqrt{L_1}}{|\varrho_1|}\right)^{\nu_0}\frac{1}{8}\left(\frac{n}{24e(m+2n)}\right)^n\left(\frac{|z_{h_2}|}{2}\right)^{m+n}.$$

Owing to (8. 12) and (9. 1) we have

$$\frac{n}{24e(m+2n)}\geqq\frac{1}{24e(L_1^2+2L_1^{1,16})}>\frac{1}{200}L_1^{-2}$$

and hence the last but first factor in (9. 10) is

$$(9.11)\qquad >\left(\frac{1}{200}L_1^{-2}\right)^{L_1^{1,16}}>e_1(-3L_1^{1,16}\log L_1)$$

if only $c_1>c_{20}$. Further from (8. 4), (8. 5) and (8. 2) we get

$$(9.12)\qquad \left(\frac{\sqrt{L_1}}{|\varrho_1|}\right)^{\nu_0}\geqq\left(\frac{1}{1+c_{13}}\sqrt{L_1}\right)^{\frac{L_1^2}{\log L_1}}>e_1(0{,}45L_1^2)$$

if only $c_1>_{21}$.

Finally using (9. 3) and (9. 4) we get $c_1>c_{22}$

$$\left(\frac{|z_{h_2}|}{2}\right)^{m+n}>\left(\frac{c_1^{2/5}}{6c_{11}}L_1^{-2/5}\right)^{\frac{L_1^2}{\log L_1}+L_1^{1,16}}>\left(\frac{c_{16}}{2}L_1^{-2/5}\right)^{\frac{L_1^2}{\log L_1}+L_1^{1,16}}>e_1(-0{,}41L_1^2).$$

Collecting this, (9. 14), (9. 13) and (9. 11) we get for our sum in (8. 3) the lower bound

$$\frac{1}{8}e_1\left(\frac{L_1^2}{25}-3c_{15}L_1^{1,16}\log L_1\right)>e_1\left(\frac{1}{30}L_1^2\right)$$

indeed if only $c_1>c_{23}$.

We shall use Lemma VI in a bit different form. We may observe that the sum in (8. 3) is *real*; hence changing l_1 and l_2 if necessary we obtain that for an integer ν_0 satisfying (8. 2) we have

$$\frac{1}{\varphi(k)} \operatorname{Re} \sum_{\chi}{}' (\overline{\chi}(l_1) - \overline{\chi}(l_2)) \sum_{|t_\varrho| \leqq c_1 k^{10}} {}_{\varrho(\chi)} \frac{(\omega_0 L_1^{\nu_0})^\varrho}{\varrho^{\nu_0+1}} \geqq e_1\left(\frac{1}{30} L_1^2\right). \tag{9. 13}$$

Finally we shall need the

LEMMA VIII. *There is a connected path V in the vertical strip $\frac{1}{5} \leqq \sigma \leqq \frac{2}{5}$ symmetrical to the real axis, consisting alternately of horizontal and vertical segments and monotonically increasing from $-\infty$ to $+\infty$, on which for all $L(s, \chi)$-functions* mod k *the inequality*

$$\left| \frac{L'}{L}(s, \chi) \right| < c_{24} k \log^3 k (2 + |t|)$$

holds.

Since the proof follows mutatis mutandis that of the Appendix III of the book of one of us (see TURAN [1]) we omit it.

10. Now we can turn to the proof of Theorem 1. 1; it will suffice again to prove (1. 6). Let ω_0, ν_0, L_1 and the order of l_1 and l_2 defined as previously and T satisfy (1. 4). We define further T_1 by

$$T_1 \overset{\text{def}}{=} \frac{T}{c_{10} k^3} e_1(-2L_1^2) \tag{10. 1}$$

with the previously mentioned c_{10}. Choosing c_2 (in dependence upon c_{10} as told in 5) sufficiently large it follows from (1. 4) that $\tau = T_1$ satisfies 5. 1 and hence Lemma I is applicable. This gives the existience of a y_1 with

$$\frac{1}{20} \log_2 T_1 \leqq y_1 \leqq \frac{1}{10} \log_2 T_1 \tag{10. 2}$$

so that for all ϱ's

$$\pi \geqq \left| \operatorname{arc} \frac{e^{i t_\varrho y_1}}{\varrho} \right| \geqq c_7 \frac{A(k)^3}{k(1 + |t_\varrho|)^6 \log^3 k (2 + |t_\varrho|)} \tag{10. 3}$$

holds. If finally the integer ν is at present restricted only by

$$\frac{\log T_1}{y_1} - \log^{0,9} T_1 \leqq \nu \leqq \frac{\log T_1}{y_1} \tag{10. 4}$$

we consider the integral

$$J(T) = -\frac{1}{2\pi i} \int_{(2)} \left(\frac{e^{y_1 s}}{s}\right)^\nu \frac{(\omega_0 L_1^{\nu_0})^s}{s^{\nu_0+1}} \cdot \frac{1}{\varphi(k)} \left\{ \sum_{\chi(l_1) \neq \chi(l_2)} {}_{\chi} (\overline{\chi}(l_1) - \overline{\chi}(l_2)) \frac{L'}{L}(s, \chi) \right\} ds. \tag{10. 5}$$

Replacing $\frac{L'}{L}(s, \chi)$ by its Dirichlet-series, the well-known integral-formula (d pos.

int.)

$$\frac{1}{2\pi i}\int_{(2)}\frac{\xi^s}{s^{d+1}}\,ds=\begin{cases}\frac{1}{d!}\log^d\xi & \text{for}\quad \xi\geqq 1,\\ 0 & \text{otherwise}\end{cases}$$

gives

$$J(T)=\sum_{\substack{n\leqq\omega_0L_1^{\nu_0}e^{\nu y_1}\\ n\equiv l_2 \bmod k}}\Lambda(n)\frac{\log^{\nu+\nu_0}\frac{\omega_0L_1^{\nu_0}e^{\nu y_1}}{n}}{(\nu+\nu_0)!}-\sum_{\substack{n\leqq\omega_0L_1^{\nu_0}e^{\nu y_1}\\ n\equiv l_2 \bmod k}}\Lambda(n)\frac{\log^{\nu+\nu_0}\frac{\omega_0L_1^{\nu_0}e^{\nu y_1}}{n}}{(\nu+\nu_0)!}$$

or putting

$$\omega_0L_1^{\nu_0}e^{\nu y_1}\stackrel{\text{def}}{=}Y_1 \tag{10.6}$$

also

$$J(T)=\frac{1}{(\nu+\nu_0)!}\int_1^{Y_1}\left(\log x\log^{\nu+\nu_0}\frac{Y_1}{x}\right)d\big(\Pi(x,k,l_1)-\Pi(x,k,l_2)\big)= \tag{10.7}$$

$$=\frac{1}{(\nu+\nu_0)!}\int_1^{Y_1}\{\Pi(x,k,l_2)-\Pi(x,k,l_1)\}\cdot\left(\log x\log^{\nu+\nu_0}\frac{Y_1}{x}\right)'dx.$$

11. Since the function $\log x\log^{\nu+\nu_0}\frac{Y_1}{x}$ increases for $1\leqq x\leqq Y_1^{\frac{1}{\nu+\nu_0+1}}$ and then decreases, we split the last integral into

$$\frac{1}{(\nu+\nu_0)!}\int_1^{Y_1^{\frac{1}{\nu+\nu_0+1}}}\stackrel{\text{def}}{=}J_1 \tag{11.1}$$

$$\frac{1}{(\nu+\nu_0)!}\int_{Y_1^{\frac{1}{\nu+\nu_0+1}}}^{Y_1}\stackrel{\text{def}}{=}J_2. \tag{11.2}$$

Evidently

$$|J_1|<\frac{1}{(\nu+\nu_0+1)!}\left(\frac{\nu+\nu_0}{\nu+\nu_0+1}\right)^{\nu+\nu_0}Y_1^{\frac{1}{\nu+\nu_0+1}}\log^{\nu+\nu_0+1}Y_1.$$

From (10. 4), (8. 2), (10. 1) we have for $c_1>c_{25}$ the estimation

$$Y_1\leqq c_{10}k^3L_1^{2\frac{L_1^2}{\log L_1}}T_1=T_1\cdot c_{10}k^3\cdot e^{2L_1^2}=T, \tag{11.3}$$

from (1. 4), (8. 1) and (10. 1) for

$$c_1^2>\max_{x\geqq 2}\frac{\log(c_{10}x^3)}{x^{20}},\quad c_2>3c_1^2$$

the inequality

$$(11.4) \qquad T_1 > \frac{T}{\log T},$$

further from (10. 4), (10, 2), (1. 4) and (11. 7)

$$(11.5) \qquad \nu > \frac{10 \log T_1}{\log_2 T_1} - \log^{0,9} T_1 > 2\frac{\log T_1}{\log_2 T_1} > \frac{\log T}{\log_2 T}.$$

Hence

$$Y_1^{\frac{1}{\nu+\nu_0+1}} < Y_1^{\frac{1}{\nu}} < \log T$$

and

$$\frac{1}{(\nu+\nu_0+1)!}\left(\frac{\nu+\nu_0}{\nu+\nu_0+1}\right)^{\nu+\nu_0} \log^{\nu+\nu_0+1} Y_1 < 2\left(\frac{e}{\nu}\log Y_1\right)^{\nu+\nu_0+1} <$$

$$< 2(e\log_2 T)^{\nu+\nu_0+1} < 2e_1\left\{2\log_3 T\left(20\frac{\log T_1}{\log_2 T_1} + 2c_{10}^2 k^{20}\right)\right\} <$$

$$< e_1\left(41\frac{\log T \log_3 T}{\log_2 T}\right)$$

and thus

$$(11.6) \qquad |J_1| < e_1\left(42\frac{\log T \log_3 T}{\log_2 T}\right).$$

Further from (11. 2) we get, using also (11. 4),

$$(11.7) \qquad J_2 \leqq \max_{x \leqq T}\{\Pi(x,k,l_1) - \Pi(x,k,l_2)\}\cdot\frac{(\log Y_1)^{\nu+\nu_0+1}}{(\nu+\nu_0+1)!}\left(\frac{\nu+\nu_0}{\nu+\nu_0+1}\right)^{\nu+\nu_0}$$

and

$$(11.8) \qquad J_2 \geqq \min_{x \leqq T}\{\Pi(x,k,l_1) - \Pi(x,k,l_2)\}\cdot\frac{(\log Y_1)^{\nu+\nu_0+1}}{(\nu+\nu_0+1)!}\left(\frac{\nu+\nu_0}{\nu+\nu_0+1}\right)^{\nu+\nu_0}.$$

12. Lemma VII and (10. 5) give

$$(12.1) \qquad J(T) = \frac{1}{\varphi(k)}\sum_{\chi(l_1)\neq\chi(l_2)}\!\!{}_{\chi}\,(\bar\chi(l_1)-\bar\chi(l_2)) \sum\nolimits'_{\varrho(\chi)} \frac{(\omega_0 L_1^{\nu_0})^{\varrho}}{\varrho^{\nu_0+1}}\cdot\left(\frac{e^{y_1\varrho}}{\varrho}\right)^{\nu} -$$

$$-\frac{1}{2\pi i}\int_V \left(\frac{e^{y_1 s}}{s}\right)^{\nu}\frac{(\omega_0 L_1^{\nu_0})^s}{s^{\nu_0+1}}\cdot\frac{1}{\varphi(k)}\left\{\sum_\chi (\bar\chi(l_1)-\bar\chi(l_2))\frac{L'}{L}(s,\chi)\right\}ds,$$

where the dash in the sum means that the summation is extended only over the zeros right to V. The last integral is absolutely

$$< 5^{2\nu} e^{\frac{2}{5}\nu y_1} c_{26} k \log^3 k\cdot(\omega_0 L_1^{\nu_0})^{\frac{2}{5}}$$

and hence owing to (1. 4), (10. 2), (10. 4), (8. 1) and (8. 2)

$$(12.2) \qquad < c_{27} T^{0,45}.$$

Further, taking in account that the number of all zeros of all $L(s,\chi)$-functions mod k with imaginary parts between r and $r+1$ is

$$(12.3)\qquad < c_{28}\varphi(k)\log k\,(2+|r|),$$

the contribution of the zeros with $|t_\varrho| > \log^{\frac{1}{10}} T_1$ to the sum on the right of (12. 1) is absolutely (roughly)

$$3c_{28}\sum_{n=[\log^{\frac{1}{10}} T_1]}^{\infty} \log kn \frac{\omega_0 L_1^{\nu_0}}{n^{\nu_0+1}}\frac{e^{\nu y_1}}{n^\nu} <$$

$$< c_{29} T\cdot\left(\frac{1}{2}\log^{\frac{1}{10}} T_1\right)^{-\nu} < c_{29}\frac{T}{T_1}e_1\left(\log^{0,9} T\cdot\log_2 T + 20\frac{\log T}{\log_2 T}\right) < c_{30}T^{0,45}$$

owing to (10. 4), (10. 2), (10. 1) and (1. 4). Hence denoting the remaining sum on the right of (12. 1) by $Z(\nu)$ and collecting the previous estimations we get

$$(12.4)\qquad \frac{(\log Y_1)^{\nu+\nu_0+1}}{(\nu+\nu_0+1)!}\max_{x\le T}\{\Pi(x,k,l_1)-\Pi(x,k,l_2)\} > Z(\nu) - c_{31}T^{0,45}$$

and

$$(12.5)\qquad \frac{(\log Y_1)^{\nu+\nu_0+1}}{(\nu+\nu_0+1)!}\min_{x\le T}\{\Pi(x,k,l_1)-\Pi(x,k,l_2)\} < Z(\nu) + c_{31}T^{0,45}.$$

13. Now we estimate $Z(\nu)$ in (12. 4) and (12. 5) by proper choices of ν from below and above, resp., by aid of lemmata I, II and VI. Let $\varrho_4 = \sigma_4 + it_4$ be one of the zeros of $L(s,\chi)$ $\chi(l_1)\ne\chi(l_2)$ with

$$(13.1)\qquad |t_\varrho| \le \log^{\frac{1}{10}} T_1$$

for which

$$(13.2)\qquad \left|\frac{e^{y_1\varrho}}{\varrho}\right| = \text{maximal.}$$

We write $Z(\nu)$ in the form

$$(13.3)\qquad Z(\nu) = \frac{1}{\varphi(k)}\left(\frac{e^{y_1\sigma_4}}{|\varrho_4|}\right)^\nu\cdot\sum_\chi\Big\{(\bar\chi(l_1)-\bar\chi(l_2))\cdot$$

$$\cdot\sum_{\substack{\varrho(\chi)\\ |t_\varrho|\le\log^{\frac{1}{10}} T_1\\ \varrho \text{ right to } V}} \frac{(\omega_0 L_1^{\nu_0})^\varrho}{\varrho^{\nu_0+1}}\cdot\left(\frac{e^{y_1(\varrho-\sigma_4)}}{\varrho}|\varrho_4|\right)^\nu \overset{\text{def}}{=} \left(\frac{e^{y_1\sigma_4}}{|\varrho_4|}\right)^\nu Z_1(\nu).$$

The role of z_j's are played by the numbers

$$\frac{e^{y_1(\varrho-\sigma_4)}}{\varrho}|\varrho_4|$$

(and hence owing to the definition of ϱ_4 the condition $\max_j |z_j| = 1$ in Lemma II is satisfied) and that of b_j's by the numbers

$$(\bar{\chi}(l_1)-\bar{\chi}(l_2))\frac{(\omega_0 L_1^{\nu_0})^{\varrho}}{\varphi(k)\varrho^{\nu_0+1}}.$$

Then

$$\operatorname{arc} z_j = \operatorname{arc}\frac{e^{it_\varrho y_1}}{\varrho}$$

and hence from (10. 3) a lower bound for $|\operatorname{arc} z_j|$ is

$$c_7\frac{A(k)^3}{k(1+\log^{\frac{1}{10}}T_1)^6\log^3(2k\log^{\frac{1}{10}}T_1)}$$

which owing to (1. 4)

$$(13.4)\qquad >\log^{-\frac{2}{3}}T_1 \overset{\text{def}}{=} \varkappa.$$

Let further

$$(13.5)\qquad m=\left[\frac{\log T_1}{y_1}-\log^{0,9}T_1\right].$$

As to the index h let

$$(13.6)\qquad z_h=\frac{e^{y_1(\varrho_5-\sigma_4)}}{\varrho_5}|\varrho_4|,$$

where $\varrho_5=\frac{1}{2}+it_5$ is any zero of any $L(s,\chi)$ mod k with

$$(13.7)\qquad L_1-1\leqq t_5<L_1.$$

The number of z_j's is owing to (12. 3) and (1. 4)

$$\leqq c_{32}k\log^{\frac{1}{10}}T_1\cdot\log k(2+\log^{\frac{1}{10}}T_1)<\log^{\frac{1}{10}}T_1\cdot(\log_2 T_1)^3$$

if c_2 is sufficiently large in dependence of c_1 and c_{10}. Hence

$$(13.8)\qquad n<\log^{\frac{1}{10}}T_1\cdot(\log_2 T_1)^3.$$

We have to verify (5. 5). Since

$$(13.9)\qquad \frac{4n}{m+n\left(3+\frac{\pi}{\varkappa}\right)}<\frac{4\log^{\frac{1}{10}}T_1(\log_2 T_1)^3}{\frac{1}{2}10\frac{\log T_1}{\log_2 T_1}}$$

and

$$(13.10)\qquad |z_h|=\frac{e^{y_1\left(\frac{1}{2}-\sigma_4\right)}}{|\varrho_5|}|\varrho_4|>\frac{e^{-\frac{1}{2}y_1}}{1/2+L_1}\cdot\frac{1}{2}>\frac{1}{3c_1k^{10}}\log^{-\frac{1}{20}}T_1>\log^{-\frac{1}{10}}T_1$$

if only c_2 is sufficiently large in dependence upon c_1 and c_{10}. This and (13. 9) prove (5. 5) indeed.

14. In order to apply Lemma II we have to estimate B of this lemma from below. This will be done by Lemma VI or rather by its corollary (9. 13). This gives, the estimation

$$B \geqq e_1\left(\frac{1}{30} L_1^2\right) - \frac{1}{\varphi(k)} \sum_\chi 2 \sum_{L_1 - 1 \leqq |t_\varrho|}^{\varrho(\chi)} \frac{(\omega_0 L_1^{\nu_0})}{|\varrho|^{\nu_0+1}} > \tag{14.1}$$

$$> e_1\left(\frac{1}{30} L_1^2\right) - c_{33} \log(kL_1) \cdot \frac{\omega_0}{L_1} 2^{\nu_0} > e_1\left(\frac{1}{30} L_1^2\right) - e_1\left(\frac{2L_1^2}{\log L_1}\right) > e_1\left(\frac{1}{40} L_1^2\right) > 1$$

if c_1 is sufficiently large in dependence upon c_{10}. Finally we have to verify that the interval, given in (5.7) for ν_1 and ν_2 is contained in (10.4). The first part of this assertion follows from (13.5) at once; further from (13.5), (13.9) and (13.4) we have

$$m + n\left(3 + \frac{\pi}{\varkappa}\right) \leqq \frac{\log T_1}{y_1} - \log^{0,9} T_1 + \tag{14.2}$$

$$+ \log^{\frac{1}{10}} T_1 (\log_2 T_1)^3 (3 + \pi \log^{\frac{2}{3}} T_1) < \frac{\log T_1}{y_1}$$

indeed if and only if c_2 is sufficiently large in dependence of c_1 and c_{10}. Hence choosing $\nu = \nu_1$ we get from (13.9) and (14.2)

$$Z_1(\nu_1) > \frac{1}{3} \log^{-\frac{1}{10}} T_1 (\log_2 T_1)^{-3} \left(\frac{y_1}{26 \log T_1}\right)^{2 \log^{\frac{1}{10}} T_1 (\log_2 T_1)^3} \cdot$$

$$\cdot \left(\frac{|z_h|}{2}\right)^{m+n\left(3+\frac{\pi}{\varkappa}\right)} > e_1\{-2 \log^{\frac{1}{10}} T_1 \cdot (\log_2 T_1)^5\} \cdot \left(\frac{e^{y_1\left(\frac{1}{2} - \sigma_4\right)}}{2|\varrho_5|} |\varrho_4|\right)^{\nu_1} \cdot \left(\frac{|z_h|}{2}\right)^{n\left(3+\frac{\pi}{\varkappa}\right)}$$

or by (13.11), (13.9), (13.4), (13.7), (13.6), (10.4) and (1.4), choosing c_2 sufficiently large,

$$Z_1(\nu_1) > e_1\left(-21 \frac{\log T_1 \cdot \log_3 T_1}{\log_2 T_1}\right) \cdot e^{y_1\left(\frac{1}{2} - \sigma_4\right)\nu_1} |\varrho_4|^{\nu_1}. \tag{14.3}$$

Hence from (13.3) and (10.4) we get

$$Z(\nu_1) > e^{\frac{\nu_1 y_1}{2}} \cdot e_1\left(-21 \frac{\log T_1 \log_3 T_1}{\log_2 T_1}\right) > \sqrt{T_1}\, e_1\left(-22 \frac{\log T_1 \log_3 T_1}{\log_2 T_1}\right).$$

We get from (14.3) and (11.4)

$$Z(\nu_1) > \sqrt{T}\, e_1\left(-23 \frac{\log T \log_3 T}{\log_2 T}\right)$$

and analogously

$$Z(\nu_2) < -\sqrt{T}\, e_1\left(-23 \frac{\log T \log_3 T}{\log_2 T}\right).$$

Going back to (12.4) and (12.5) this gives

$$\max_{x \leqq T} \{\Pi(x, k, l_1) - \Pi(x, k, l_2)\} > \tag{14.4}$$

$$> \sqrt{T} \cdot e_1\left(-23 \frac{\log T \log_3 T}{\log_2 T}\right) (\nu_1 + \nu_0 + 1)! \log^{-\nu_1 - \nu_0 - 1} Y_1$$

and

(14. 5)
$$\min_{x \leqq T} \{\Pi(x,k,l_1)-\Pi(x,k,l_2)\} \leqq$$
$$\leqq -\sqrt{T}\, e_1\left(-23\frac{\log T \log_3 T}{\log_2 T}\right)\cdot(\nu_2+\nu_0+1)!\log^{-\nu_2-\nu_0-1} Y_1.$$

Since from (10. 6)

(14. 6)
$$(\nu_1+\nu_0+1)!\log^{-\nu_1-\nu_0-1} Y_1 > \left(\frac{\nu_1+\nu_0+1}{e\log Y_1}\right)^{\nu_1+\nu_0+1} =$$
$$=\left(\frac{\nu_1+\nu_0+1}{e(\log\omega_0+\nu_0\log L_1+\nu_1 y_1)}\right)^{\nu_1+\nu_0+1} >$$
$$> e\left(\frac{1}{\nu_1}\log\omega_0+\frac{\nu_0}{\nu_1}\log L_1+\frac{1}{10}\log_2 T\right)^{-\nu_1-\nu_0-1},$$

we get using (7. 1), (5. 7) and (13. 5)
$$\frac{1}{\nu_1}\log\omega_0 \leqq \frac{\log(c_{10}k^3)}{9\dfrac{\log T_1}{\log_2 T_1}} < 1$$

and using (8. 2), (5. 7) and (13. 5)
$$\frac{\nu_0}{\nu_1}\log L_1 < \frac{2L_1^2}{\left(9\dfrac{\log T_1}{\log_2 T_1}\right)} < 1$$

if only c_2 is sufficiently large in dependence upon c_1 and c_{10}. Hence from (14. 6) and (1. 4)
$$(\nu_1+\nu_0+1)!\log^{-\nu_1-\nu_0-1} Y_1 > (\log_2 T)^{-\nu_1-\nu_0-1} >$$
$$> e_1\left\{-\log_3 T\left(1+2\frac{L_1^2}{\log L_1}+20\frac{\log T}{\log_2 T}\right)\right\} > e_1\left(-21\frac{\log T\log_3 T}{\log_2 T}\right),$$

choosing c_2 sufficiently large in dependence upon c_1 and c_{10}. Similarly
$$(\nu_2+\nu_0+1)!\log^{-\nu_2-\nu_0-1} Y_1 > e_1\left(-21\frac{\log T\log_3 T}{\log_2 T}\right).$$

Putting these into (14. 4) and (14. 5) Theorem 1. 1 follows at once.

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(Received 10 January 1962)

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