COMPARATIVE PRIME-NUMBER THEORY. IV

(PARADIGMA TO THE GENERAL CASE, $k=8$ AND 5)

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1. In papers II and III of this series[1] (see KNAPOWSKI—TURÁN [2]—[3]) we treated systematically the comparison-problem of the residue-classes 1 and $l \bmod k$. Now we turn to the much more difficult general case. In this case our results are very far from being complete. As an introduction to the following analysis as well as to treat a case which our methods can settle to a certain extent completely and without any conjectures, we shall treat in the paper in extenso the case $k=8$. As to the comparison of the residue-classes 1 and $l \bmod 8$ the strongest result is given by the Theorem 5. 1 of the paper II since the Theorem 2. 1 of paper III, which would give much stronger localisation, is not applicable to our cases. In the remaining cases however, when

$$l_1 \text{ and } l_2 \text{ are among } 3, 5, \text{ or } 7 \bmod 8 \tag{1. 1}$$

we shall prove theorems of almost the same strength. More exactly we assert the

THEOREM 1. 1. *For $T > c_1$ and for all pairs l_1, l_2 with $l_1 \neq l_2$ among the numbers* 3, 5, 7 *we have*

$$\max_{T^{1/3} \leq x \leq T} \{\pi(x, 8, l_1) - \pi(x, 8, l_2)\} > \sqrt{T}\, e_1\left(-23\, \frac{\log T \log_3 T}{\log_2 T}\right). \tag{1. 2}$$

Since none of l_1 and l_2 is distinguished to the other, the same inequality holds interchanging l_1 and l_2. This Theorem 1. 1 is obviously in the direction of Problem 3 formulated in the paper [1]. As an easy consequence of Theorem 1. 1 we formulate the[2]

THEOREM 1. 2. *For $T > c_1$ we have the inequality*

$$W_8(T, l_1, l_2) > c_2 \log_2 T \tag{1. 3}$$

if only $l_1 \neq l_2$ and among 3, 5, 7.

According to Theorem 1. 1 of paper [3] we have, again without any conjectures, the inequality

$$W_8(T, 1, l) > c_3 \log_4 T \tag{1. 4}$$

[1] As in previous papers of this series $c_1, c_2, \dots$ are always positive numerical, explicitely calculable constants. l is always supposed to be prime to the modulus $e_\nu(x) = e_{\nu-1}(e^x)$ and $\log_\nu x = \log_{\nu-1}(\log x)$.

[2] We remind the reader that $w_k(T, l_1, l_2)$ stands for the number of sign-changes of $\pi(x, k, l_1) - \pi(x, k, l_2)$ for $0 < x \leq T$.

which is much weaker than (1. 3). Whether this is inherent with the matter (which we think is the case) or not, we cannot decide at present.

2. The fact that the congruence

$$x^2 \equiv l \bmod 8, \qquad l \not\equiv 1 \bmod 8 \tag{2. 1}$$

is not solvable gives at once that Theorem 1. 2 is a simple consequence of the

THEOREM 2. 1. *For $T > c_4$ and for all pairs l_1, l_2 with $l_1 \neq l_2$ among the numbers 3, 5, 7 we have*[3]

$$\max_{T^{1/3} \leqq x} \{\Pi(x, 8, l_1) - \Pi(x, 8, l_2)\} > \sqrt{T}\, e_1\left(-23 \frac{\log T \log_3 T}{\log_2 T}\right). \tag{2. 2}$$

A slight modification of the proof will lead to the

THEOREM 2. 2. *For $T > c_4$ and for all pairs l_1, l_2 with $l_1 \neq l_2$ among the numbers 3, 5, 7 we have*

$$\max_{T^{1/3} \leqq x \leqq T} \{\psi(x, k, l_1) - \psi(x, k, l_2)\} > \sqrt{T}\, e_1\left(-23 \frac{\log T \log_3 T}{\log_2 T}\right). \tag{2. 3}$$

As an immediate corollary we have the following

THEOREM 2. 3. *For $T > c_4$ and for all pairs l_1, l_2 with $l_1 \neq l_2$ among the numbers 3, 5, 7 we have the inequalities*[4]

$$\begin{aligned} U_8(T, l_1, l_2) &> c_5 \log_2 T, \\ V_8(T, l_1, l_2) &> c_5 \log_2 T. \end{aligned} \tag{2. 4}$$

3. In the case $k = 5$ the situation is somewhat different. Again Theorem 5. 1 from the paper [2] settles the cases[5]

$$(1, 2)_5, \quad (1, 3)_5, \tag{3. 1}$$

for the case $(1, 4)_5$ however the Theorems 2. 1 or 2. 2 from the paper [3] furnish a much stronger result. The case $(2, 3)_5$ could be treated with the same results — apart from the occurring constants — as given in the case $k = 8$ by the theorems of this paper; since only slight readjustments are required we shall confine ourselves to indicate them by footnotes. However the remaining cases

$$(2, 4)_5, \quad \text{and} \quad (3, 4)_5 \tag{3. 2}$$

present an unpleasant (or pleasant?) surprise. The complete analoga of Theorems

[3] We remind the reader that $\Pi(x, k, l)$ denotes the sum $\sum_{n \leqq x,\, n \equiv l \bmod k} \frac{\Lambda(n)}{\log n}$ and in the next theorem $\psi(x, k, l)$ the sum $\sum_{n \leqq x,\, n \equiv l \bmod k} \Lambda(n)$.

[4] We remind the reader that $U_k(T, l_1, l_2)$ and $V_k(T, l_1, l_2)$ denote the number of sign-changes of $\psi(x, k, l_1) - \psi(x, k, l_2)$ and $\Pi(x, k, l_1) - \Pi(x, k, l_2)$ for $0 < x \leqq T$, resp.

[5] The case $(l_1, l_2)_k$ means throughout the whole series of these papers the comparison of the progressions $\equiv l_1$, resp. $\equiv l_2 \bmod k$.

2. 1, 2. 2 and 2. 3 of this paper could be proved without essential changes; we shall not do it. But as to

$$\begin{aligned}&\pi(x, 5, 4)-\pi(x, 5, 2)\\&\pi(x, 5, 4)-\pi(x, 5, 3)\end{aligned} \tag{3. 3}$$

our methods do not work in their present shape, even supposing the truth of RIEMANN—PILTZ conjecture;[6] the reasons for it will be explained in the papers V and VI of this series.

4. All in all we have to prove only Theorem 2. 1 in all details. The proof is in principle similar to that of Theorem 3. 2 of paper [3], both proofs reduce the difficulties analogously to one which can be surmounted by a proper use of Lemma II of this paper ($\equiv$theorem 4. 1 of paper [3]). The inconvenient way however, the lower bound in Lemma II depends upon the b_j's, gives some indication where this last difficulty lies and at the same time gives a plausibility to the fact that this difficulty can be surmounted in different situations only on entirely different ways (as it is the case here and later).

Theorem 2. 1 contains three statements; it will suffice to treat one of them, e. g. the case

$$(3, 5)_8 \tag{4. 1}$$

in the notation of (3. 1).

A further moment in the proof is a certain configurational statement concerning the early zeros of the L-functions mod 8 which we could surmount sofar only by some numerical data, kindly furnished to us by C. B. HASELGROVE. They are the following. For the character

$$\chi_1(n) = 1, 1, -1, -1 \bmod 8 \tag{4. 2}$$

$L(s, \chi_1)$ vanishes for $\varrho^{(1)} = \varrho^{(1)}(\chi_1) = \frac{1}{2}+i{\cdot}4{,}89\ldots$, for $\overline{\varrho^{(1)}}$ and apart from these $L(s, \chi_1)$ does not vanish for

$$0<\sigma<1, \quad |t|\leqq 7. \tag{4. 3}$$

For the character

$$\chi_2(n) = 1, -1, 1, -1 \bmod 8 \tag{4. 4}$$

$L(s, \chi_2)$ vanishes for $\varrho^{(2)} = \varrho^{(2)}(\chi_2) = \frac{1}{2}+i{\cdot}6{,}02\ldots$, for $\overline{\varrho^{(2)}}$ and apart from these $L(s, \chi_2)$ does not vanish in the rectangle (4. 3). For the character

$$\chi_3(n) = 1, -1, -1, 1 \bmod 8 \tag{4. 5}$$

$L(s, \chi_3)$ vanishes for $\varrho^{(3)} = \varrho^{(3)}(\chi_3) = \frac{1}{2}+i{\cdot}3{,}57\ldots$, for $\overline{\varrho^{(3)}}$ and apart from these $L(s, \chi_3)$ does not vanish in the rectangle (4. 3). Further data of HASELGROVE and Mr. D. DAVIES show a similar situation for $k|24$ and $k=5$, respectively. The mentioned configurational statement is in the case (4. 1) that there is a positive (numerical) α_1 such that the rectangle

$$0<\sigma<1, \quad |t|\leqq\alpha_1 \qquad (s=\sigma+it) \tag{4. 6}$$

[6] We remind the reader that this conjecture asserts that no $L(s, \chi)$ functions vanish in the half-plane $\sigma>\frac{1}{2}$ $(s=\sigma+it)$.

contains a unique couple of (complex-conjugate) simple zeros[7]

(4. 7) $$\varrho', \overline{\varrho'}$$

of $L(s, \chi_1)\, L(s, \chi_2)$ and that for all other non-trivial zeros of $L(s, \chi_1)\, L(s, \chi_2)$ the inequality

(4. 8) $$|\varrho| > |\varrho'| + \alpha_2$$

is fulfilled, again with a positive (numerical) α_2. In order to make our proof for the case (4. 1) easier adaptable to the other cases as well as to the case $k=5$, say, we shall not use the Haselgrove-values, only α_1 and α_2. (However in the paper VIII we shall need his data to a much larger extent.)

Mutatis mutandis we may assert the following Theorems.

THEOREM 4. 1. *For $T > c_4$ and $l = 3, 5, 7$ we have*

$$\max_{T^{1/3} \leqq x \leqq T} \left\{ \pi(x, 8, l) - \frac{1}{4}\pi(x) \right\} > \sqrt{T} \cdot e_1\left(-23 \frac{\log T \log_3 T}{\log_2 T}\right)$$

and[8]

$$\min_{T^{1/3} \leqq x \leqq T} \left\{ \pi(x, 8, l) - \frac{1}{4}\pi(x) \right\} < -\sqrt{T} \cdot e_1\left(-23 \frac{\log T \log_3 T}{\log_2 T}\right).$$

Further the

THEOREM 4. 2. *For $T > c_4$ and $l = 3, 5, 7$ we have for the number $Z_l(T)$ of sign-changes of the function*

$$\pi(x, 8, l) - \frac{1}{4}\pi(x)$$

in $0 < x \leqq T$ the lower bound

$$c_5 \log_2 T.$$

For a comparison of our above-stated results with the ones, attainable by older methods, we refer to our introductory paper [1].

5. In order to make, as told in paper I, this paper as self-consistent as possible, we shall quote explicitely our two main tools here, proved in papers [2] and [3] as Lemma I and II, respectively.

LEMMA I. *For $T > c_6$ there is a y_1 with*

(5. 1) $$\frac{1}{20} \log_2 T \leqq y_1 \leqq \frac{1}{10} \log_2 T$$

[7] Necessarily on the line $\sigma = \frac{1}{2}$.

[8] The same holds replacing $\pi(x)$ by $\int\limits_2^x \frac{dv}{\log v}$.

such that for all non-trivial zeros $\varrho = \sigma_\varrho + it_\varrho$ *of all* $L(s, \chi)$*-functions belonging to* mod 8 *we have*

(5. 2) $$\pi \geqq \left| \operatorname{arc} \frac{e^{it_\varrho y_1}}{\varrho} \right| \geqq \frac{c_7}{t_\varrho^6 \log^3(2+|t_\varrho|)}.$$

As to Lemma II let be

(5. 3) $$1 = |z_1| \geqq |z_2| \geqq \ldots \geqq |z_n|$$

and with a $0 < \varkappa \leqq \frac{\pi}{2}$

(5. 4) $$\varkappa \leqq |\operatorname{arc} z_j| \leqq \pi \qquad (j = 1, 2, \ldots, n).$$

Let us be given a positive integer m and the index h so that

(5. 5) $$|z_h| > \frac{4n}{m + n\left(3 + \frac{\pi}{\varkappa}\right)}$$

and fixed. Further we define B and the index h_1 by

(5. 6) $$B = \min_{h \leqq \xi < h_1} \operatorname{Re} \sum_{j=1}^{\xi} b_j$$

if there is an index h_1 with

(5. 7) $$|z_{h_1}| < |z_h| - \frac{2n}{m + n\left(3 + \frac{\pi}{\varkappa}\right)}$$

and

(5. 8) $$B = \min_{h \leqq \xi \leqq n} \operatorname{Re} \sum_{j=1}^{\xi} b_j$$

otherwise.[9] Then we have the

LEMMA II. *If* $B > 0$ *then there are integers* ν_1 *and* ν_2 *with*

(5. 9) $$m + 1 \leqq \nu_1, \qquad \nu_2 \leqq m + n\left(3 + \frac{\pi}{\varkappa}\right)$$

such that

(5. 10) $$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_1} \geqq \frac{B}{2n+1} \left\{ \frac{n}{24\left(m + n\left(3 + \frac{\pi}{\varkappa}\right)\right)} \right\}^{2n} \cdot \left(\frac{|z_h|}{2}\right)^{m + n\left(3 + \frac{\pi}{\varkappa}\right)}$$

[9] Of course we *can* choose (5. 8) as the definition of h_1 always, but in *some* cases; like in paper [3], it was more advantageous to choose $h_1 < n$.

and

$$\text{(5.11)}\qquad \operatorname{Re}\sum_{j=1}^{n} b_j z_j^{\nu_2} < -\frac{B}{2n+1}\left\{\frac{n}{24\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)}\right\}^{2n}\left(\frac{|z_h|}{2}\right)^{m+n\left(3+\frac{\pi}{\varkappa}\right)}.$$

Finally we shall need again the *Lemma* III.

LEMMA III. *There is a connected path V in the vertical strip $\frac{1}{5}\leqq\sigma\leqq\frac{2}{5}$, symmetrical to the real axis, consisting alternately of horizontal and vertical segments and increasing from $-\infty$ to $+\infty$ monotonically, such that for all L-functions* mod 8 *the inequality*

$$\left|\frac{L'}{L}(s,\chi)\right| < c_8\log^3(2+|t|)$$

holds.

Since the proof is mutatis mutandis the same as in the book of one of us (see TURÁN [1], Appendix III), we shall omit it.

6. Next we turn to the proof of Theorem 2.1. We start with the notation (4.2)—(4.4) from the integral

$$\text{(6.1)}\qquad J(T)=\frac{1}{4\pi i}\int_{(2)}\left(\frac{e^{y_1 s}}{s}\right)^{\nu}\frac{1}{s^{\alpha+1}}\left\{\frac{L'}{L}(s,\chi_1)-\frac{L'}{L}(s,\chi_2)\right\}ds,$$

where y_1 is defined by Lemma I the integer ν is restricted at present only by the requirement

$$\text{(6.2)}\qquad \frac{\log T}{y_1}-\log^{0,9}T\leqq\nu\leqq\frac{\log T}{y_1}$$

and the (numerical constant) integer $\alpha\geqq 1$ will be chosen appropriately only later (in dependence *only* upon the constants α_1 and α_2 in (4.6) resp. (4.8)). Performing the integration in (6.1) we get

$$\text{(6.3)}\qquad J(T)=\sum_{\substack{n\leqq e^{\nu y_1}\\ n\equiv 5 \bmod 8}}\Lambda(n)\frac{\log^{\nu+\alpha}\frac{e^{\nu y_1}}{n}}{(\nu+\alpha)!}-\sum_{\substack{n\leqq e^{\nu y_1}\\ n\equiv 3 \bmod 8}}\Lambda(n)\frac{\log^{\nu+\alpha}\frac{e^{\nu y_1}}{n}}{(\nu+\alpha)!}=$$

$$=\int_1^{e^{\nu y_1}}\frac{\log^{\nu+\alpha}\frac{e^{\nu y_1}}{x}}{(\nu+\alpha)!}\,d_x\{\Pi(x,8,5)-\Pi(x,8,3)\}=\int_1^{e^{\nu y_1}}\{\Pi(x,8,3)-\Pi(x,8,5)\}\cdot$$

$$\cdot\left(\log x\,\frac{\log^{\nu+\alpha}\frac{e^{\nu y_1}}{x}}{(\nu+\alpha)!}\right)' dx.$$

Writing the integral as

$$(6.4)\qquad \int_{1}^{e_1\left(\frac{\nu}{\nu+\alpha+1}y_1\right)} + \int_{e_1\left(\frac{\nu}{\nu+\alpha+1}y_1\right)}^{e_1(\nu y_1)} \overset{\text{def}}{=} J' + J'',$$

the second factor in the integral in (6. 3) is of constant sign in each of our intervals. Hence if c_4 is sufficiently large, we have

$$(6.5)\qquad |J'| < e^{y_1} \int_{1}^{e_1\left(\frac{\nu y_1}{\nu+\alpha+1}\right)} \left(\log x \frac{\log^{\nu+\alpha}\frac{e^{\nu y_1}}{x}}{(\nu+\alpha)!}\right)' dx <$$

$$< \frac{y_1 e^{y_1}}{(\nu+\alpha)!}(\nu y_1)^{\nu+\alpha} < e_1\left(21\frac{\log T \log_3 T}{\log_2 T}\right)$$

using (5. 1) and (6. 2). Further from (6. 2)

$$J'' \leqq \max_{x \leqq T}\{\Pi(x, 8, 5) - \Pi(x, 8, 3)\} \cdot \int_{e_1\left(\frac{\nu y_1}{\nu+\alpha+1}\right)}^{e_1(\nu y_1)} \left(-\log x \frac{\log^{\nu+\alpha}\frac{e^{\nu y_1}}{x}}{(\nu+\alpha)!}\right)' dx =$$

$$= \max_{x \leqq T}\{\Pi(x, 8, 5) - \Pi(x, 8, 3)\} \frac{1}{(\nu+\alpha)(\nu+\alpha)!}\left(\nu y_1 \frac{\nu+\alpha}{\nu+\alpha+1}\right)^{\nu+\alpha+1};$$

hence

$$(6.6)\qquad \max_{x \leqq T}\{\Pi(x, 8, 5) - \Pi(x, 8, 3)\} \geqq (\nu+\alpha)(\nu+\alpha)!\left(\frac{\nu+\alpha+1}{\nu y_1(\nu+\alpha)}\right)^{\nu+\alpha+1} \cdot$$

$$\cdot\left\{J(T) - e_1\left(21\frac{\log T \log_3 T}{\log_2 T}\right)\right\}$$

and analogously

$$(6.7)\qquad \min_{x \leqq T}\{\Pi(x, 8, 5) - \Pi(x, 8, 3)\} \leqq (\nu+\alpha)(\nu+\alpha)!\left(\frac{\nu+\alpha+1}{\nu y_1(\nu+\alpha)}\right)^{\nu+\alpha+1} \cdot$$

$$\cdot\left\{J(T) + e_1\left(21\frac{\log T \log_3 T}{\log_2 T}\right)\right\}.$$

Replacing Π by ψ the necessary (slight) change is the same as at the end of 10 in paper [2] and can be omitted.

7. Using Lemma III Cauchy's integral-theorem gives at once

$$(7.1)\qquad \left|J(T) - \frac{1}{2}\sum\nolimits'_{\varrho(\chi_1)}\frac{1}{\varrho^{\alpha+1}}\left(\frac{e^{y_1\varrho}}{\varrho}\right)^{\nu} + \frac{1}{2}\sum\nolimits'_{\varrho(\chi_2)}\frac{1}{\varrho^{\alpha+1}}\left(\frac{e^{y_1\varrho}}{\varrho}\right)^{\nu}\right| =$$

$$= \frac{1}{4\pi}\left|\int_{V}\left(\frac{e^{y_1 s}}{s}\right)^{\nu}\frac{1}{s^{\alpha+1}}\left\{\frac{L'}{L}(s, \chi_1) - \frac{L'}{L}(s, \chi_2)\right\}ds\right| < c_9\left(5e^{\frac{2}{5}y_1}\right)^{\nu},$$

where the dash indicates that the summation is to be performed only upon zeros lying right to V. Owing to (6. 2) this is for sufficiently large c_4

$$< c_{10} T^{0,45}. \tag{7. 2}$$

Since further as well-known the number of non-trivial zeros of $L(s, \chi)$ with

$$r \leqq t_\varrho < r+1$$

is

$$< c_{11} \log(2+|r|),$$

the contribution of the ϱ's with

$$|\mathrm{Im}\, \varrho| > \log^{\frac{1}{10}} T$$

to the sum (7. 1) is owing to (5. 1) absolutely

$$< c_{12} \left(\frac{e^{y_1}}{\log^{\frac{1}{10}} T} \right)^{\nu} \leqq c_{12}.$$

Collecting this, (7. 1) and (7. 2) and (6. 6) we got

$$\max_{x \leqq T} \{\Pi(x, 8, 5) - \Pi(x, 8, 3)\} > \frac{(\nu+\alpha)(\nu+\alpha)!}{2} \left(\frac{\nu+\alpha+1}{\nu y_1(\nu+\alpha)} \right)^{\nu+\alpha+1} \cdot \tag{7. 3}$$

$$\cdot \left\{ \sum_{\substack{\varrho(\chi_1) \\ |t_\varrho| \leqq \log^{\frac{1}{10}} T}}' \varrho^{-\alpha-1} \left(\frac{e^{y_1 \varrho}}{\varrho} \right)^{\nu} - \sum_{\substack{\varrho(\chi_2) \\ |t_\varrho| \leqq \log^{\frac{1}{10}} T}}' \varrho^{-\alpha-1} \left(\frac{e^{y_1 \varrho}}{\varrho} \right)^{\nu} - c_{13} T^{0,45} \right\}.$$

If again $\varrho_0 = \beta_0 + i\gamma_0$ is that zero of $L(s, \chi_1)\, L(s, \chi_2)$ in

$$\sigma \geqq \frac{1}{2}, \quad |t| \leqq \log^{\frac{1}{10}} T$$

for which $\left| \frac{e^{y_1 \varrho}}{\varrho} \right|$ is maximal, (7. 3) gives at once

$$\max_{x \leqq T} \{\Pi(x, 8, 5) - \Pi(x, 8, 3)\} > \frac{(\nu+\alpha)(\nu+\alpha)!}{2} \left(\frac{\nu+\alpha+1}{\nu y_1(\nu+\alpha)} \right)^{\nu+\alpha+1} \cdot$$

$$\cdot \left\{ \left(\frac{e^{\beta_0 y_1}}{|\varrho_0|} \right)^{\nu} \left(\sum_{\substack{\varrho(\chi_1) \\ |t_\varrho| \leqq \log^{\frac{1}{10}} T}}' \varrho^{-\alpha-1} \left(\frac{e^{y_1(\varrho-\beta_0)}}{\varrho} |\varrho_0| \right)^{\nu} - \right. \right.$$

$$\left. \left. - \sum_{\substack{\varrho(\chi_2) \\ |t_\varrho| \leqq \log^{\frac{1}{10}} T}}' \varrho^{-\alpha-1} \left(\frac{e^{y_1(\varrho-\beta_0)}}{\varrho} |\varrho_0| \right)^{\nu} \right) - c_{13} T^{0,45} \right\}$$

and taking in account that $L(s, \chi_1)\, L(s, \chi_2)$ is real on the real axis and also the symmetry of V, we obtain also the inequality

$$(7.4)\quad \max_{x\leq T}\{\Pi(x,8,5)-\Pi(x,8,3)\} > \frac{(\nu+\alpha)(\nu+\alpha)!}{2}\left(\frac{\nu+\alpha+1}{\nu y_1(\nu+\alpha)}\right)^{\nu+\alpha+1}\cdot$$

$$\cdot\left\{\left(\frac{e^{\beta_0 y_1}}{|\varrho_0|}\right)^{\nu}\left(\operatorname{Re}\sum_{\substack{\varrho(\chi_1)\\ |t_\varrho|\leq \log^{\frac{1}{10}} T}}{}' \varrho^{-\alpha-1}\left(\frac{e^{y_1(\varrho-\beta_0)}}{\varrho}|\varrho_0|\right)^{\nu}-\right.\right.$$

$$\left.\left.-\operatorname{Re}\sum_{\substack{\varrho(\chi_2)\\ |t_\varrho|\leq \log^{\frac{1}{10}} T}}{}' \varrho^{-\alpha-1}\left(\frac{e^{y_1(\varrho-\beta_0)}}{\varrho}|\varrho_0|\right)^{\nu}\right)-c_{13}T^{0,45}\right\}.$$

Analogous *upper* bound can be given[10] for

$$(7.5)\quad \min_{x\leq T}\{\Pi(x,8,5)-\Pi(x,8,3)\}$$

with a changed sign of inequality on the right.

8. Now we shall determine ν beyond (6. 2) by using appropriately Lemma II. The z_j-numbers will be again of course the numbers

$$\frac{e^{y_1(\varrho-\beta_0)}}{\varrho}|\varrho_0|;$$

the definition of ϱ_0 gives

$$\max_j |z_j|=1$$

immediately. As b_j's the ($\pm\varrho^{-\alpha-1}$)-numbers will serve this time, with proper signs; let

$$(8.1)\quad m=\left[\frac{\log T}{y_1}-\log^{0,9} T\right].$$

As to $\varkappa$ of Lemma II we have owing to (5. 2) and $|t_\varrho|\leq\log^{\frac{1}{10}}T$ for sufficiently large c_4 again

$$(8.2)\quad \varkappa=\log^{-\frac{2}{3}} T,$$

further for n the upper bound

$$(8.3)\quad \log^{\frac{1}{10}} T\cdot(\log_2 T)^3.$$

However in the choice of the indices h and h_1 we have now a situation, different from that in paper [3]. We choose now simply

$$(8.4)\quad h_1=n$$

[10] Sofar e. g. the case $(2, 3)_5$ runs quite parallel, only in (6. 1) the factor $\frac{L'}{L}(s,\chi_1)-\frac{L'}{L}(s,\chi_2)$ has to be replaced by $\frac{i}{2}\left\{\frac{L'}{L}(s;\chi^*)-\frac{L'}{L}(s,\overline{\chi^*})\right\}$, $\chi^*(n)=1, i, -i, -1 \bmod 5$.

and if ϱ' is defined in (4. 7), let h be such that

$$(8.5) \qquad \frac{e^{y_1(\varrho'-\beta_0)}}{\varrho'}|\varrho_0| \overset{\text{def}}{=} z_h, \quad \frac{e^{y_1(\overline{\varrho'}-\beta_0)}}{\overline{\varrho'}}|\varrho_0| \overset{\text{def}}{=} z_{h-1}.$$

Then we have from (4. 6), (4. 7) and (8. 2) for sufficiently large c_4

$$(8.6) \qquad |z_h| > e_1\left(-\frac{y_1}{2}\right) \geqq \log^{-\frac{1}{20}} T > \frac{4}{3+\frac{\pi}{\varkappa}} > \frac{4n}{m+n\left(3+\frac{\pi}{\varkappa}\right)}$$

i. e. (5. 5) is by this choice satisfied. Hence Lemma II is applicable; the resulting v_1 and v_2 satisfy the restriction (6. 2) owing to (8. 1) and the inequality

$$n\left(3+\frac{\pi}{\varkappa}\right) < \frac{1}{2}\log^{0,9} T$$

follows quite roughly (see (8. 3) and (8. 2)). Before applying it, we have to orientate about B in (5. 8); this will be done by the proper choice of the integer α in (6. 1) and by use of the configurational inequality (4. 8).

We distinguish two cases.

Case I.[11] ϱ' is a zero of $L(s, \chi_2)$ (and hence *not* of $L(s, \chi_1)$). Since χ_2 is real, we have without loss of generality

$$(8.7) \qquad 0 < \operatorname{arc} \varrho' \leqq \operatorname{arc\,tg} 2\alpha_1 \left(<\frac{\pi}{2}\right).$$

Evidently each of our remaining $\sum_{j=1}^{\xi} b_j$-sums contains the terms

$$(8.8) \qquad -\left(\frac{1}{(\varrho')^{\alpha+1}}+\frac{1}{(\overline{\varrho'})^{\alpha+1}}\right) = \frac{2}{|\varrho'|^{\alpha+1}}\cos\left((\alpha+1)\operatorname{arc}\varrho'+\pi\right).$$

(8. 7) gives at once that for an infinity of integer α's

$$(8.9) \qquad |(\alpha+1)\operatorname{arc}\varrho'+\pi| \leqq \frac{\pi}{4} \bmod (-\pi, \pi].$$

For these α's we have certainly

$$B \geqq 2\frac{\sqrt{2}}{2}|\varrho'|^{-\alpha-1} - \sum\nolimits_{\varrho} |\varrho|^{-\alpha-1},$$

where the last sum contains owing to (4. 8) all zeros of $L(s, \chi_1)\, L(s, \chi_2)$ with

$$|\varrho| \geqq |\varrho'| + \alpha_2;$$

hence we can find a sufficiently large α from those satisfying (8. 9) so that

$$(8.10) \qquad B > |\varrho'|^{-\alpha-1}.$$

[11] Which is *actually* owing to (4. 2) and (4. 4) impossible but, as told, we want to use only the configurational assertion (4. 7)—(4. 8).

Case II. ϱ' is a zero of $L(s, \chi_1)$ (and not of $L(s, \chi_2)$). The only change in this case is that instead of (8. 8) we have

$$\frac{1}{(\varrho')^{\alpha+1}}+\frac{1}{(\overline{\varrho'})^{\alpha+1}}=\frac{2}{|\varrho'|^{\alpha+1}}\cos((\alpha+1)\operatorname{arc}\varrho')$$

and hence instead of (8. 9) the first requirement for α is

$$|(\alpha+1)\operatorname{arc}\varrho'|\leqq\frac{\pi}{4}\operatorname{mod}(-\pi,\pi].$$

Hence (8. 10) can be attained in all cases;[12] we fix α that way. Hence the application of Lemma II gives an integer ν_1 so that

$$\left(\frac{e^{\beta_0 y_1}}{|\varrho_0|}\right)^{\nu_1}\frac{1}{2}\operatorname{Re}\Bigg\{\sum_{\substack{\varrho(\chi_1)\\|t_\varrho|\leqq\log^{\frac{1}{10}}T}}'\varrho^{-\alpha-1}\left(\frac{e^{y_1(\varrho-\beta_0)}}{\varrho}|\varrho_0|\right)^{\nu_1}-$$

$$-\sum_{\substack{\varrho(\chi_2)\\|t_\varrho|\leqq\log^{\frac{1}{10}}T}}'\varrho^{-\alpha-1}\left(\frac{e^{y_1(\varrho-\beta_0)}}{\varrho}|\varrho_0|\right)^{\nu_1}\Bigg\}>\left(\frac{e^{\beta_0 y_1}}{|\varrho_0|}\right)^{\nu_1}\frac{c_{14}}{2n+1}\cdot$$

$$\cdot\left\{\frac{n}{24\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)}\right\}^{2n}\left(\frac{e^{y_1\left(\frac{1}{2}-\beta_0\right)}}{2|\varrho'|}|\varrho_0|\right)^{m+n\left(3+\frac{\pi}{\varkappa}\right)}\geqq\frac{c_{14}}{2n+1}|2\varrho'|^{-\frac{\log T}{y_1}}\cdot$$

$$\cdot\left(\frac{n}{24\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)}\right)^{2n}e_1\left\{\left(\frac{y_1}{2}\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)\right\}\cdot\left(\frac{e^{-\beta_0 y_1}}{|\varrho_0|}\right)^{-n\left(3+\frac{\pi}{\varkappa}\right)}.\text{[13]}$$

Taking in account (6. 2), (8. 3) and (8. 1) and $\beta_0\leqq 1$ this is for sufficiently large c_4

$$>c_{15}\sqrt{T}\,e_1\left(-20\log|2\varrho'|\cdot\frac{\log T}{\log_2 T}\right)$$

and analogously

$$\left(\frac{e^{\beta_0 y_1}}{|\varrho_0|}\right)^{\nu_2}\frac{1}{2}\operatorname{Re}\Bigg\{\sum_{\substack{\varrho(\chi_1)\\|t_\varrho|\leqq\log^{\frac{1}{10}}T}}'\varrho^{-\alpha-1}\left(\frac{e^{y_1(\varrho-\beta_0)}}{\varrho}|\varrho_0|\right)^{\nu_2}-$$

$$-\sum_{\substack{\varrho(\chi_2)\\|t_\varrho|\leqq\log^{\frac{1}{10}}T}}'\varrho^{-\alpha-1}\left(\frac{e^{y_1(\varrho-\beta_0)}}{\varrho}|\varrho_0|\right)^{\nu_2}\Bigg\}<-c_{15}\sqrt{T}\cdot e_1\left(-20\log|2\varrho'|\cdot\frac{\log T}{\log_2 T}\right).$$

[12] In the case $(2, 3)_5$ the role of the „dominating part" $\pm((\varrho')^{-\alpha-1}+(\overline{\varrho'})^{-\alpha-1})$ will be played owing to the appearence of complex characters by $\pm 2i((\varrho')^{-\alpha-1}-(\overline{\varrho'})^{-\alpha-1})=|\varrho'|^{-\alpha-1}\cdot\sin\left((\alpha+1)\operatorname{arc}\varrho'\pm\frac{\pi}{2}\right)$ but the reasoning runs quite parallel.

[13] Since $|\varrho_0|>1$ and $m+n\left(3+\frac{\pi}{\varkappa}\right)-\nu_1<n\left(3+\frac{\pi}{\varkappa}\right)$.

Hence choosing c_4 sufficiently large this and (7. 4) give

$$\max_{x \leq T} \{\Pi(x, 8, 5) - \Pi(x, 8, 3)\} > \tag{8. 9}$$

$$> \frac{\nu_1+\alpha}{2}(\nu_1+\alpha)!\left(\frac{\nu_1+\alpha+1}{\nu_1 y_1(\nu_1+\alpha)}\right)^{\nu_1+\alpha+1} c_{15}\sqrt{T}\cdot e_1\left(-c_{16}\frac{\log T}{\log_2 T}\right).$$

Since from (6. 2) for sufficiently large c_4

$$\frac{\nu_1+\alpha}{2}(\nu_1+\alpha)!\left(\frac{\nu_1+\alpha+1}{\nu_1 y_1(\nu_1+\alpha)}\right)^{\nu_1+\alpha+1} > e_1\left(-21\frac{\log T \log_3 T}{\log_2 T}\right)$$

(8. 9) give

$$\max_{x \leq T} \{\Pi(x, 8, 5) - \Pi(x, 8, 3)\} > \sqrt{T}\cdot e_1\left(-22\frac{\log T \log_3 T}{\log_2 T}\right)$$

and analogously

$$\min_{x \leq T} \{\Pi(x, 8, 5) - \Pi(x, 8, 3)\} < -\sqrt{T}\cdot e_1\left(-22\frac{\log T \log_3 T}{\log_2 T}\right).$$

From these Theorem 2. 1 follows at once.

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