COMPARATIVE PRIME-NUMBER THEORY. III

(CONTINUATION OF THE STUDY OF COMPARISON OF THE PROGRESSIONS $\equiv 1 \bmod k$ AND $\equiv l \bmod k$)

By

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1. In the previous paper of this series[1] we established a part of our results concerning the comparison of the residue classes 1 and $l \bmod k$. As mentioned in II we could deduce a (very low) lower bound for

$$W_k(T, 1, l), \tag{1.1}$$

the number of sign-changes of

$$\pi(x, k, 1) - \pi(x, k, l)$$

for $0 < x \leqq T$. Now we are going to show first how by an appropriate modification of the *proof* of the Theorem 5. 1, resp. 5. 2 we can obtain a much better lower bound for $W(T, 1, l)$ (though these theorems themselves remain untouched being essentially best-possible ones). Most probably this will be also rather rough, but as to the moduli

$$k = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 19, 24 \tag{1.2}$$

as far as we know, our result is the first theorem in this direction proved without any conjectures. Keeping the convention of I and II we denote by $c_1, c_2, \ldots$ explicitly calculable positive numerical constants;[2] then we assert the

THEOREM 1. 1. *For $T > c_1$ we have for the moduli k in* (1. 2) *the inequality*

$$W_k(T, 1, l) > c_2 \log_4 T.$$

More generally we have

THEOREM 1. 2. *If for a k the Haselgrove-condition[3] holds then for*

$$T > \max\left(e_4(k^{c_3}),\ e_2\left(\frac{2}{A(k)^3}\right)\right) \tag{1.3}$$

the inequality

$$W_k(T, 1, l) > k^{-c_3} \log_4 T$$

holds.

[1] We shall quote the previous papers of this series for the sake of brevity by I and II, respectively.

[2] And also $e_1(x) = e^x$ and $e_\nu(x) = e_1(e_{\nu-1}(x))$, further $\log_1 x = \log x$ and $\log_\nu x = \log(\log_{\nu-1} x)$.

[3] We remind the reader that by the Haselgrove-condition for a k we mean the existence of an $A = A(k)$ with $0 < A(k) \leqq 1$ such that no $L(s, \chi)$ belonging to mod k vanishes for $0 < \sigma < 1$ and $|t| \leqq A(k)$ $(s = \sigma + it)$.

Obviously it suffices to prove Theorem 1. 2. As another immediate consequence we assert the

THEOREM 1. 3. *If for a k the Haselgrove-condition holds then the interval*

$$0 < x \leqq \max\left(e_4(k^{c_3}), e_2\left(\frac{2}{A(k)^3}\right)\right)$$

contains at least one sign-change of

$$\pi(x, k, 1) - \pi(x, k, l)$$

for all $l \not\equiv 1 \bmod k$.

We can prove the analoga of Theorems 1. 1 and 1. 2 for

$$\pi(x, k, 1) - \frac{1}{\varphi(k)-1} \sum_{\substack{(l,k)=1 \\ l \neq 1}} \pi(x, k, l). \tag{1.4}$$

Denoting the number of sign-changes of this function for $0 < x \leqq T$ by $S_k(T)$ we have the

THEOREM 1. 4. *If for a k the Haselgrove-condition holds, then for*

$$T > \max\left(e_4(k^{c_3}), e_2\left(\frac{2}{A(k)^3}\right)\right)$$

the inequality

$$S_k(T) > k^{-c_3} \log_4 T$$

holds.[4]

Since the proof of this theorem follows closely that of Theorem 1. 2 we shall omit its details.

By these theorems steps are made towards the solution of Problems 6 and 5 of paper I.

2. The previous results of this paper and all results of paper II refer to *all* l's with $(l, k) = 1$, $l \not\equiv 1 \bmod k$. For certain l's however these theorems are capable to an essential refinement. The l's in question are those for which the congruence

$$x^2 \equiv l \bmod k \tag{2.1}$$

has exactly as many (incongruent) solutions as the congruence

$$x^2 \equiv 1 \bmod k. \tag{2.2}$$

Such l's occur „rather often", in particular when k is a prime. Then we assert the

[4] The same holds for the number of sign-changes of $\pi(x, k, 1) - \frac{1}{\varphi(k)} \pi(x)$ and mutatis mutandis for $\pi(x, k, 1) - \frac{1}{\varphi(k)} \int_2^x \frac{dv}{\log v}$.

THEOREM 2. 1. *For the k's in* (1. 2) *and l's satisfying the condition* (2. 1)—(2. 2) *for* $T > c_4$ *the inequalities*

$$\max_{T^{1/3} \leqq x \leqq T} \{\pi(x, k, 1) - \pi(x, k, l)\} > \sqrt{T} e_1\left(-42 \frac{\log T \log_3 T}{\log_2 T}\right) \tag{2. 3}$$

and

$$\min_{T^{1/3} \leqq x \leqq T} \{\pi(x, k, 1) - \pi(x, k, l)\} < -\sqrt{T} e_1\left(-42 \frac{\log T \log_3 T}{\log_2 T}\right) \tag{2. 4}$$

hold.

This is an elegant special case of the

THEOREM 2. 2. *For the moduli k in* (1. 2) *and l satisfying* (2. 1)—(2. 2), *if* $\varrho_0 = \beta_0 + i\gamma_0$ *with* $\beta_0 \geqq \frac{1}{2}$ *is such that* $L(\varrho_0, \chi_1) = 0$ *with* $\chi_1(l) \neq 1$, *then we have for*

$$T > \max(c_5, e_2(10|\varrho_0|))$$

the inequalities

$$\max_{T^{1/3} \leqq x \leqq T} \{\pi(x, k, 1) - \pi(x, k, l)\} > T^{\beta_0} e_1\left(-42 \frac{\log T \log_3 T}{\log T}\right) \tag{2. 5}$$

and

$$\min_{T^{1/3} \leqq x \leqq T} \{\pi(x, k, 1) - \pi(x, k, l)\} < -T^{\beta_0} e_1\left(-42 \frac{\log T \log_3 T}{\log_2 T}\right). \tag{2. 6}$$

As remarked in paper II it is again Siegel's theorem[5] (see SIEGEL [1]) which makes Theorem 2. 1 a special case of Theorem 2. 2.

3. Theorems 2. 1 and 2. 2 are unconditional. Passing, however, to general k's, we have as in paper II to use Haselgrove-condition. Then we have

THEOREM 3. 1. *If for a k the Haselgrove-condition holds and l satisfies the condition* (2. 1)—(2. 2) *then for*

$$T > \max\left(c_7, e_2(k), e_2\left(\frac{1}{A(k)^3}\right)\right)$$

the inequalities (2. 3) *and* (2. 4) *hold.*

All the theorems formulated in 2 and 3 are owing to Siegel's above-mentioned theorem consequences of the

THEOREM 3. 2. *If for a k the Haselgrove-condition holds and l satisfies* (2. 1)—(2. 2) *and if further* $\varrho_0 = \beta_0 + i\gamma_0$, $\beta_0 \geqq \frac{1}{2}$ *is a zero of an* $L(s, \chi_1)$ *with* $\chi_1(l) \neq 1$, *then for*

$$T > \max\left(c_7, e_2(k), e_2\left(\frac{1}{A(k)^3}\right), e_2(10|\varrho_0|)\right) \tag{3. 1}$$

the inequalities (2. 5)—(2. 6) *hold.*

[5] This theorem asserts that *all* $L(s, \chi)$ functions belonging to primitive χ' s mod k have at least one zero in the domain

$$\sigma \geqq \frac{1}{2}, \qquad |t| \leqq \frac{c_6}{\log_3 (k + e_3(1))}.$$

As easy corollary we have for the l's in (2. 1)—(2. 2) the following improvement of our Theorems 1. 1 and 1. 2.

THEOREM 3. 3. *For $T>c_1$ we have for the moduli k in* (1. 2) *and l's satisfying* (2. 1)—(2. 2) *the inequality*

$$W_k(T, 1, l) > c_8 \log_2 T.$$

THEOREM 3. 4. *If for a k the Haselgrove-condition holds and*

$$T > \max\left(c_9, e_2(2k), e_2\left(\frac{2}{A(k)^3}\right)\right)$$

then for the l's satisfying (2. 1)—(2. 2) *the inequality*

$$W_k(T, 1, l) > c_8 \log_2 T$$

holds.

As to a comparison of the above stated theorems to those attainable by older methods we refer to our paper I.

4. The basis of the proofs in paper II was the Lemma I. As to its proof we referred to the paper of one of us (see TURÁN [1]) since we mentioned we shall need for our later purposes a slightly more general form of this lemma. For an exposition of this, let m be an arbitrary non-negative integer and

$$1 = |z_1| \geqq |z_2| \geqq \ldots \geqq |z_n| \tag{4. 1}$$

further with a $0 < \varkappa \leqq \frac{\pi}{2}$ for $j = 1, 2, \ldots, n$

$$\varkappa \leqq |\operatorname{arc} z_j| \leqq \pi. \tag{4. 2}$$

Let the index h be such that

$$|z_h| > \frac{4n}{m + n\left(3 + \frac{\pi}{\varkappa}\right)} \tag{4. 3}$$

and fixed. Further we define A and the index h_1 by

$$A = \min_{h \leqq \xi < h_1} \operatorname{Re} \sum_{j=1}^{\xi} b_j \tag{4. 4}$$

if there is a $h_1 \leqq n$ with

$$|z_{h_1}| < |z_h| - \frac{2n}{m + n\left(3 + \frac{\pi}{\varkappa}\right)} \tag{4. 5}$$

and

$$A = \min_{h \leqq \xi \leqq n} \operatorname{Re} \sum_{j=1}^{\xi} b_j \tag{4.6}$$

otherwise. Then we assert the[6]

THEOREM 4. 1. *If $A > 0$ then there are integers ν_1 and ν_2 with*

$$m+1 \leqq \nu_1, \quad \nu_2 \leqq m+n\left(3+\frac{\pi}{\varkappa}\right) \tag{4.7}$$

such that

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_1} \geqq \frac{A}{2n+1} \left\{ \frac{n}{24\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)} \right\}^{2n} \left(\frac{|z_h|}{2}\right)^{m+n\left(3+\frac{\pi}{\varkappa}\right)} \tag{4.8}$$

and

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_2} \leqq -\frac{A}{2n+1} \left\{ \frac{n}{24\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)} \right\}^{2n} \left(\frac{|z_h|}{2}\right)^{m+n\left(3+\frac{\pi}{\varkappa}\right)}. \tag{4.9}$$

5. Since the proof of this theorem differs very little from that of the above mentioned Lemma I of the paper II a sketch will suffice. We can take without any change from TURÁN [1] the following

LEMMA I. *The polynomial*

$$F(z) = z^d + a_1 z^{d-1} + \ldots + a_d$$

with real coefficients and with all zeros in the domain

$$|z| \leqq 1, \quad \varkappa \leqq |\operatorname{arc} z| \leqq \pi$$

can be multiplied by a polynomial $\varphi(z)$ with real coefficients so that writing

$$F(z)\,\varphi(z) = \sum_\nu e_\nu z^\nu$$

we have

a) $$e_\nu \geqq 0 \qquad (\nu = 0, 1, \ldots)$$

b) *the degree of $F(z)\varphi(z)$ cannot exceed* $\dfrac{d}{2}\left(1+\dfrac{\pi}{\varkappa}\right)$

c) $$\sum_\nu e_\nu \leqq 2^d$$

d) *the coefficient of the highest power of z in $F(z)\varphi(z)$ is at least 3^{-d}.*

Having this lemma we introduce as in TURÁN [1] to our z_j-numbers (which can be assumed to be all different) the numbers ξ_ν by

$$\xi_{2j-1} = z_j, \quad \xi_{2j} = \bar{z}_j \qquad (j = 1, 2, \ldots, n) \tag{5.1}$$

[6] A similar modification of the second main theorem (see TURÁN [2], SÓS-TURÁN [1]), which increases essentially the applicability of it, had been found by one of us (see KNAPOWSKI [1]).

and let $\eta_1, \eta_2, \ldots, \eta_l$ with

$$1 = |\eta_1| \geqq |\eta_2| \geqq \ldots \geqq |\eta_l| \tag{5.2}$$

be the maximal number of different ones among the ξ's. Certainly we have beside (5. 2) also

$$\varkappa \leqq |\operatorname{arc} \eta_j| \leqq \pi \qquad (j = 1, \ldots, l) \tag{5.3}$$

and

$$n \leqq l \leqq 2n. \tag{5.4}$$

Since with η_j also $\bar{\eta}_j$ is among our η_j's, the polynomial

$$\Phi(z) \overset{\text{def}}{=} \prod_{j=1}^{l} (z - \eta_j) \tag{5.5}$$

has real coefficients. We apply the Lemma I to $\Phi(z)$ as $F(z)$ (i. e. in case $d = l$) and denoting by $\varphi^*(z)$ the polynomial corresponding to $\varphi(z)$ we have

$$\Phi(z)\,\varphi^*(z) \overset{\text{def}}{=} \sum_{\nu \leqq \frac{l}{2}\left(1 + \frac{\pi}{\varkappa}\right)} e_\nu' z^\nu \tag{5.6}$$

with non-negative e_ν'-coefficients, with

$$\sum e_\nu' \leqq 2^l \tag{5.7}$$

and with the leading coefficient

$$\geqq 3^{-l}. \tag{5.8}$$

6. Let (somewhat differently from TURÁN [1]) be

$$\delta = |z_h| - \frac{2n}{m + n\left(3 + \frac{\pi}{\varkappa}\right)} \left(\geqq \frac{|z_h|}{2}\right) \tag{6.1}$$

and

$$G(z) = \left(\frac{24}{|z_h| - \delta}\right)^{2n} \frac{1 + z + z^2 + \ldots + z^{l-1}}{\delta^{m + n\left(3 + \frac{\pi}{\varkappa}\right)}}, \tag{6.2}$$

$$\Psi(z) \overset{\text{def}}{=} G(z)\,\Phi(z)\,\varphi^*(z) \overset{\text{def}}{=} \sum_\nu e_\nu'' z^\nu.$$

$\Psi(z)$ is obviously a polynomial of degree

$$\leqq \frac{l}{2}\left(1 + \frac{\pi}{\varkappa}\right) + (l - 1) \leqq n\left(3 + \frac{\pi}{\varkappa}\right) - 1 \tag{6.3}$$

with non-negative coefficients and if the exact degree of $\Phi(z)\varphi^*(z)$ is N_0, then owing to (5. 8) we have for $\nu = N_0, N_0 + 1, \ldots, N_0 + l - 1$ the inequality

$$\text{coeffs. } z^\nu \text{ in } \Psi(z) \geqq \left(\frac{24}{|z_h| - \delta}\right)^{2n} 3^{-l} \delta^{-m - n\left(3 + \frac{\pi}{\varkappa}\right)} \geqq \left(\frac{8}{|z_h| - \delta}\right)^{2n} \delta^{-m - n\left(3 + \frac{\pi}{\varkappa}\right)}. \tag{6.4}$$

Analogously as in TURÁN [1] (and as in SÓS—TURÁN [1]) we get the existence of an R with

$$\delta \leqq |z| \leqq |z_h| \tag{6. 5}$$

such that on the whole periphery of the circle $|z| = R$ the inequality

$$|\Phi(z)| \geqq 2\left(\frac{|z_h| - \delta}{4}\right)^l$$

holds and the same for all partial products

$$(z-\eta_{i_1})(z-\eta_{i_2})\ldots(z-\eta_{i_j}) \qquad (1 \leqq i_1 < i_2 < \ldots < i_j \leqq l). \tag{6. 6}$$

Let the index ω be defined uniquely by

$$1 = |\eta_1| \geqq |\eta_2| \geqq \ldots \geqq |\eta_\omega| > R > |\eta_{\omega+1}| \geqq \ldots \geqq |\eta_l|, \tag{6. 7}$$

(possibly also $\omega = l$) and let[7] as in TURÁN [1]

$$H_1(z) \overset{\text{def}}{=} \prod_{j=\omega+1}^{l} (z-\eta_j) \overset{\text{def}}{=} \sum_{j=0}^{l-\omega} c_j^{(1)} z^j. \tag{6. 8}$$

The $c_j^{(1)}$-coefficients are obviously *real* and the estimation

$$|c_j^{(1)}| \leqq \binom{l-\omega}{j} \qquad (j = 0, 1, \ldots, (l-\omega)) \tag{6. 9}$$

holds $\left(\binom{0}{0} \text{ means } 1\right)$.

7. Next we consider as in TURÁN [1] the auxiliary polynomial $H_2(z)$ of degree $\leqq \omega - 1$, defined by the requirements $(N_0$ in (6. 4))

$$H_2(\eta_j) = \frac{1}{\eta_j^{m+1+N_0} H_1(\eta_j)} \qquad (j = 1, 2, \ldots, \omega). \tag{7. 1}$$

Writing $H_2(z)$ in the form

$$\begin{aligned} H_2(z) = c_0^{(2)} + c_1^{(2)}(z-\eta_1) + c_2^{(2)}(z-\eta_1)(z-\eta_2) + \ldots \\ \ldots + c_{\omega-1}^{(2)}(z-\eta_1)(z-\eta_2)\ldots(z-\eta_{\omega-1}) \end{aligned} \tag{7. 2}$$

the Nörlund integral-representation

$$c_j^{(2)} = \frac{1}{2\pi i} \int\limits_{|w|=R} \frac{dw}{w^{m+1+N_0} H_1(w)(w-\eta_1)\ldots(w-\eta_{j+1})}$$

gives owing to (6. 6) the estimation

$$|c_j^{(2)}| < \frac{1}{2R^{m+N_0}} \left(\frac{4}{|z_h| - \delta}\right)^l \qquad (j = 0, 1, \ldots, \omega - 1). \tag{7. 3}$$

[7] If $\omega = l$ the product means 1.

Writing $H_2(z)$ in the form

$$H_2(z) \overset{\text{def}}{=} \sum_{\gamma=0}^{\omega-1} c_\gamma^{(3)} z^\gamma \tag{7.4}$$

the coefficients are owing to (7. 1) *real* and analogously as in TURÁN [1] we get

$$|c_\gamma^{(3)}| \leqq \frac{1}{2R^{m+N_0}} \left(\frac{4}{|z_h|-\delta}\right)^l \binom{\omega}{\gamma+1} \qquad (\gamma=0, 1, \dots, \omega-1). \tag{7.5}$$

We need further the polynomial

$$H_3(z) \overset{\text{def}}{=} H_1(z) H_2(z) \overset{\text{def}}{=} \sum_{\nu=0}^{l-1} c_\nu^{(4)} z^\nu. \tag{7.6}$$

The coefficients $c_\nu^{(4)}$ are obviously *real* and we have

$$\sum_{\nu=0}^{l-1} |c_\nu^{(4)}| \leqq \left(\sum_{\nu=0}^{l-\omega} |c_\nu^{(1)}|\right)\left(\sum_{\nu=0}^{\omega-1} |c_\nu^{(3)}|\right) \leqq \frac{1}{2} R^{-m-N_0} \left(\frac{8}{|z_h|-\delta}\right). \tag{7.7}$$

8. Finally we consider the auxiliary polynomials ($\Psi(z)$ from (6. 2))

$$\Psi(z) + z^{N_0} H_3(z) \overset{\text{def}}{=} \sum_\nu c_\nu^{(5)} z^\nu \tag{8.1}$$

$$\Psi(z) - z^{N_0} H_3(z) \overset{\text{def}}{=} \sum_\nu c_\nu^{(6)} z^\nu. \tag{8.2}$$

Their coefficients are obviously real; but as in TURÁN [1] one can see, they are *non-negative*. Further we get from the definition of the $\Psi(z)$ and $\Phi(z)$ polynomials for $j=1, 2, \dots, l$,

$$\sum_\nu c_\nu^{(5)} \eta_j^\nu = \eta_j^{N_0} H_3(\eta_j),$$

$$\sum_\nu c_\nu^{(6)} \eta_j^\nu = -\eta_j^{N_0} H_3(\eta_j),$$

resp., and hence owing to the definition of the $H_3(z)$ polynomials as in TURÁN [1]

$$\sum_\nu c_\nu^{(5)} \eta_j^{m+1+\nu} = \begin{cases} 1 & \text{for } j=1, 2, \dots, \omega \\ 0 & \text{for } j=\omega+1, \dots, l, \end{cases} \tag{8.3}$$

$$\sum_\nu c_\nu^{(6)} \eta^{m+1+\nu} = \begin{cases} -1 & \text{for } j=1, 2, \dots, \omega \\ 0 & \text{for } j=\omega+1, \dots, l \end{cases} \tag{8.4}$$

resp.[8] Owing to the definition of the η_j's, ξ_j's and z_j's (8. 3) and (8. 4) hold with ξ's instead of the η's, the two categories being

$$|z_j| > R \qquad |z_j| < R \tag{8.5}$$

resp.; hence defining the index μ by

$$1 = |z_1| \geqq |z_2| \geqq \dots \geqq |z_\mu| > R > |z_{\mu+1}| \geqq \dots \tag{8.6}$$

[8] In (8.3) and (8.4) the second category can be empty (and also later).

we have

$$\sum_\nu c_\nu^{(5)} z_j^{m+1+\nu} = \begin{cases} 1 & \text{for } j=1,2,\dots,\mu \\ 0 & \text{for } j=\mu+1,\dots,n, \end{cases} \tag{8.7}$$

$$\sum_\nu c_\nu^{(6)} z_j^{m+1+\nu} = \begin{cases} -1 & \text{for } j=1,2,\dots,\mu \\ 0 & \text{for } j=\mu+1,\dots,n \end{cases} \tag{8.8}$$

resp. Now we may observe first that from (6. 5) $|z_h| \geqq |z_\mu|$, i. e.

$$h \leqq \mu;$$

further in the case (4. 5) from (6. 5) and (6. 1)

$$|z_\mu| > R \geqq \delta = |z_h| - \frac{2n}{m+n\left(3+\frac{\pi}{\varkappa}\right)} > |z_{h_1}|,$$

i. e.

$$\mu < h_1$$

and $\mu \leqq n$ otherwise. Hence in the case (4. 5)

$$h \leqq \mu < h_1 \tag{8.9}$$

and

$$h \leqq \mu \leqq n \tag{8.10}$$

otherwise.

9. Now we can complete the sketch of the proof of Theorem 4. 1. Multiplying in (8. 7) by b_j, summing for $j=1,\dots,n$ and taking real parts we get

$$\sum_\nu c_\nu^{(5)} \left\{ \operatorname{Re} \sum_{j=1}^{n} b_j z_j^{m+\nu+1} \right\} = \operatorname{Re} \sum_{j=1}^{\mu} b_j \geqq A \tag{9.1}$$

from (4. 4) and (4. 6). Since the degree of the polynomial in (8. 1) (and in (8. 2)) is owing to (6. 3), (5. 6) and (7. 6) at most

$$n\left(3+\frac{\pi}{\varkappa}\right) - 1$$

we get from (9. 1) owing to the non-negativity of $c_\nu^{(5)}$'s

$$\max_{\substack{m+1 \leqq \nu \leqq m+n\left(3+\frac{\pi}{\varkappa}\right) \\ \nu \text{ integer}}} \operatorname{Re} \sum_{j=1}^{n} b_j z_j^\nu \geqq \frac{A}{\sum_\nu c_\nu^{(5)}}. \tag{9.2}$$

But from (8. 1), (6. 2), (5. 7) and (7. 7) we get

$$\begin{aligned} \sum_\nu c_\nu^{(5)} &\leqq \sum_\nu e_\nu'' + \sum_\nu |c_\nu^{(4)}| = \left(\frac{24}{|z_h|-\delta}\right)^{2n} 2n\delta^{-m-n\left(3+\frac{\pi}{\varkappa}\right)} 4^n + \\ &+ \frac{1}{2}\delta^{-m-n\left(3+\frac{\pi}{\varkappa}\right)} \left(\frac{8}{|z_h|-\delta}\right)^{2n} < (2n+1)\left(\frac{48}{|z_h|-\delta}\right)^{2n} \delta^{-m-n\left(3+\frac{\pi}{\varkappa}\right)}. \end{aligned} \tag{9.3}$$

Since from (6. 1) we get

$$\frac{48}{|z_h|-\delta}=\frac{24\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)}{n} \tag{9. 4}$$

and from (4. 3) and (6. 1)

$$\delta \geqq \frac{1}{2}|z_h|, \tag{9. 5}$$

the proof of (4. 8) follows from (9. 2), (9. 3), (9. 4) and (9. 5). Similarly for (4. 9) starting from (8. 8).

10. In order to prove all the theorems stated previously it will be enough to prove the Theorems 1. 2 and 3. 2. We shall start with the first; the proof will have much common with that of Theorem 5. 2 in paper II (these parts will be only sketched) and again some ideas of Littlewood, Ingham and Skewes (see LITTLEWOOD [1], INGHAM [1], SKEWES [1]) will be used.

We shall use from paper II the following theorem (see (3. 4) and (3. 5) there).

If for a modulus k Haselgrove-condition holds and for a χ' with $\chi'(l)\neq 1$ the function $L(s,\chi')$ vanishes at $\varrho_0=\sigma_0+it_0$, then for

$$T_1>\max\left(c_{10},\, e_2(10|\varrho_0|),\, e_2(k),\, e_2\left(\frac{1}{A(k)^3}\right)\right) \tag{10. 1}$$

the inequalities

$$\max_{T_1^{1/3}\leqq x\leqq T_1}\{\Pi(x,k,1)-\Pi(x,k,l)\}>T_1^{\beta_0}e_1\left(-41\frac{\log T_1\log_3 T_1}{\log_2 T_1}\right) \tag{10. 2}$$

and

$$\min_{T_1^{1/3}\leqq x\leqq T_1}\{\Pi(x,k,1)-\Pi(x,k,l)\}<-T_1^{\beta_0}e_1\left(-41\frac{\log T_1\log_3 T_1}{\log_2 T_1}\right) \tag{10. 3}$$

hold.

For our present aims we need this theorem in a slightly modified form. Since obviously we have for $x\geqq 2$ the inequality

$$|\{\pi(x,k,1)-\pi(x,k,l)\}-\{\Pi(x,k,1)-\Pi(x,k,l)\}|<c_{11}x^{\frac{1}{2}},$$

we get, if

$$\beta_0\geqq\frac{1}{2}+42\frac{\log_3 T_1}{\log_2 T_1}, \tag{10. 4}$$

for the T_1's in (10. 1) the inequalities[9]

$$\max_{T_1^{1/3}\leqq x\leqq T_1}\{\pi(x,k,1)-\pi(x,k,l)\}>T_1^{\beta_0}e_1\left(-42\frac{\log T_1\log_3 T_1}{\log_2 T_1}\right)$$

[9] Replacing if necessary c_{10} in (10.1) by a larger constant c.

and

$$\min_{T_1^{1/3} \leqq x \leqq T_1} \{\pi(x, k, 1) - \pi(x, k, l)\} < -T_1^{\beta_0} e_1\left(-42 \frac{\log T_1 \log_3 T_1}{\log_2 T_1}\right).$$

Hence if (10. 1) and (10. 4) are satisfied, the function $\pi(x, k, 1) - \pi(x, k, l)$ has at least one sign-change in the interval

$$T_1^{1/3} \leqq x \leqq T_1. \tag{10.5}$$

Now let T be in the range (1. 3) and we consider the

Case I. There is a character χ' with $\chi'(l) \neq 1$ such that $L(s, \chi')$ has a zero $\varrho_0' = \beta_0 + i\gamma_0$ with

$$\beta_0 \geqq \frac{1}{2} + 43 \frac{\log_3 T}{\log_2 T}, \quad |\gamma_0| \leqq \frac{1}{40} \log_2 T. \tag{10.6}$$

Then one can see at once that replacing in (10. 1) and (10. 4) T_1 by $e_1\left(\frac{\log T}{\log_3 T}\right)$ these inequalities are amply satisfied (choosing in (1. 3) c_3 sufficiently large) and a fortiori replacing T_1 by the values

$$T^{3^{-\nu}}$$

with

$$\nu = 0, 1, \ldots, \left[\frac{1}{2} \log_4 T\right].$$

Hence by (10. 5) the function $\pi(x, k, 1) - \pi(x, k, l)$ has in the case I for $0 < x \leqq T$ at least

$$\frac{1}{4} \log_4 T$$

sign-changes.

11. *Case II. None of the $L(s, \chi)$ functions with $\chi(l) \neq 1$ vanishes for*

$$\sigma \geqq \frac{1}{2} + 43 \frac{\log_3 T}{\log_2 T}, \quad |t| \leqq \frac{1}{40} \log_2 T. \tag{11.1}$$

This part of the argument will have much resemblance to that of Theorem 5. 2 of paper II but this time the parameter τ will be chosen differently, as

$$\tau = k^{c_{12}}, \tag{11.2}$$

c_{12} sufficiently large and

$$q = \log^2 \tau. \tag{11.3}$$

Putting

$$g(r) \overset{\text{def}}{=} e^{-\frac{r}{2}} \Big\{ \sum_{\substack{n \leqq e^r \\ n \equiv 1 \bmod k}} \Lambda(n) - \sum_{\substack{n \leqq e^r \\ n \equiv l \bmod k}} \Lambda(n) \Big\} \tag{11.4}$$

we start as in paper II (formula (15. 3)) from the inequality

$$(11.5) \qquad \left| g(r) + \frac{1}{\varphi(h)} \sum_{\chi} (1-\bar{\chi}(l)) \sum_{1<|t_\varrho| \leqq \frac{1}{40}\log_2 T} {}_{\varrho(\chi)} \frac{e^{it_\varrho r}}{it_\varrho} \right| < c_{13} \log k,$$

if only r is restricted by

$$(11.6) \qquad 0 < r \leqq \log_3 T$$

(see (15. 2) of paper II). Denoting again

$$(11.7) \qquad G_\omega(r) \stackrel{\text{def}}{=} \tau \left(\frac{\sin \frac{\tau(r-\omega)}{2}}{\frac{\tau(r-\omega)}{2}} \right)^2$$

we shall not work with a single ω-value as in paper II but with several ones; more exactly if c_{14} is sufficiently large then for

$$(11.8) \qquad \nu = 0, 1, 2, \ldots, \left(\left[\frac{\log_4 T}{6k^{c_{14}} \log_2 (k^{c_{12}})} \right] - 1 \right)$$

we restrict ω_ν at present only by

$$(11.9) \qquad q^{(3\nu+1)k^{c_{14}}} \leqq \omega_\nu \leqq q^{(3\nu+2)k^{c_{14}}}.$$

(11. 8) and (11. 9) give evidently that for our ν's

$$(11.10) \qquad \frac{5}{4} \omega_\nu < \log_3 T$$

i. e. we can multiply in (11. 5) by any of our $G_{\omega_\nu}(r)$-functions and integrate over $\left(\frac{3}{4}\omega_\nu, \frac{5}{4}\omega_\nu\right)$ respectively. This gives, owing to

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \tau \left(\frac{\sin \frac{\tau(r-\omega)}{2}}{\frac{\tau(r-\omega)}{2}} \right)^2 dr = 1,$$

the inequality (for all of our ν's)

$$\left| \frac{1}{2\pi} \int_{\frac{3}{4}\omega_\nu}^{\frac{5}{4}\omega_\nu} g(r) G_{\omega_\nu}(r) dr + \frac{1}{\varphi(k)} \sum_{\chi} (1-\bar{\chi}(l)) \cdot \right.$$

$$\left. \cdot \sum_{1<|t_\varrho| \leqq \frac{1}{40}\log_2 T} {}_{\varrho(\chi)} \frac{1}{2\pi i t_\varrho} \int_{\frac{3}{4}\omega_\nu}^{\frac{5}{4}\omega_\nu} e^{it_\varrho r} G_{\omega_\nu}(r) dr \right| < c_{15} \log k$$

or proceeding as in 16 of paper II for each of our ν's

$$(11.11)\quad \left|\frac{1}{2\pi}\int_{\frac{3}{4}\omega_\nu}^{\frac{5}{4}\omega_\nu} g(r)G_{\omega_\nu}(r)dr + \frac{1}{\varphi(k)}\sum_\chi (1-\bar{\chi}(l)) \sum_{1<|t_\varrho|\leq\tau}{}^{\varrho(\chi)} \left(1-\frac{|t_\varrho|}{\tau}\right)\frac{e^{it_\varrho\omega_\nu}}{it_\varrho}\right| < c_{16}\log k$$

where c_{16} does *not* depend upon the choice of c_{12}!

12. The reasoning of 17 of paper II gives again — instead of the reference to (5. 3) of paper II to (1. 3) of the present paper with sufficiently large c_3 — that

$$(12.1)\quad \frac{1}{\varphi(k)}\sum_\chi (1-\bar{\chi}(l)) \sum_{1<|t_\varrho|\leq\tau}{}^{\varrho(\chi)} \left(1-\frac{|t_\varrho|}{\tau}\right)\frac{e^{i\frac{t_\varrho\pi}{2\tau}}}{it_\varrho} > \frac{1}{32\pi}\log\tau$$

and

$$(12.2)\quad \frac{1}{\varphi(k)}\sum_\chi (1-\chi(l)) \sum_{1<|t_\varrho|\leq\tau}{}^{\varrho(\chi)} \left(1-\frac{|t_\varrho|}{\tau}\right)\frac{e^{-i\frac{t_\varrho\pi}{2\tau}}}{it_\varrho} < -\frac{1}{32\pi}\log\tau.$$

Next we determine for ν's in (11. 8) the numbers γ_ν so that

$$(12.3)\quad \left\|\frac{t_\varrho}{2\pi}\gamma_\nu\right\| < \frac{1}{q}$$

with the q in (11. 3) should hold for all $0 < t_\varrho \leq \tau$ of *all* $L(s, \chi)$-functions belonging to mod k; here again $\|x\|$ stands for the distance of x from the next integer. As before we get for the number of these t_ϱ's — choosing c_{12} in (11. 2) sufficiently large — the upper bound

$$2\frac{\varphi(k)}{\pi}\tau\log\tau$$

and hence Dirichlet's theorem assures the existence of an γ_ν with (12. 3) and

$$(12.4)\quad q^{\left(3\nu+\frac{3}{2}\right)k^{c_{14}}} \leq \gamma_\nu \leq q^{\left(3\nu+\frac{3}{2}\right)k^{c_{14}}+\frac{2\varphi(k)}{\pi}\tau\log\tau}.$$

Then we have from (12. 1), (11. 2) and (11. 3)[10]

$$(12.5)\quad \frac{1}{\varphi(k)}\sum_\chi (1-\chi(l)) \sum_{1<|t_\varrho|\leq\tau}{}^{\varrho(\chi)} \left(1-\frac{|t_\varrho|}{\tau}\right)\frac{e^{it_\varrho\left(\gamma_\nu+\frac{\pi}{2\tau}\right)}}{it_\varrho} >$$

$$> \frac{1}{32\pi}\log\tau - \frac{1}{\varphi(k)}\left|\sum_\chi (1-\bar{\chi}(l)) \sum_{1<|t_\varrho|\leq\tau}{}^{\varrho(\chi)} \left(1-\frac{|t_\varrho|}{\tau}\right)\cdot\right.$$

$$\left.\cdot\frac{e^{i\frac{t_\varrho\pi}{2\tau}}}{t_\varrho}\left(e^{2\pi i\frac{t_\varrho\gamma_\nu}{2\pi}}-1\right)\right| > \frac{1}{32\pi}\log\tau - \frac{2}{\varphi(k)}\sum_\chi \sum_{1\leq|t_\varrho|\leq\tau}{}^{\varrho(\chi)} \frac{2\pi}{q}\cdot$$

$$\cdot\frac{1}{|t_\varrho|} > \frac{1}{32\pi}\log\tau - \frac{8}{q}\log\tau\log k\tau > \frac{1}{40\pi}\log\tau$$

[10] Also the well-known inequality $\sum_{\lambda\leq t_\varrho\leq\lambda+1}^{\varrho(\chi)} 1 < c_{17}\log k(2+|\lambda|)$ is used for all χ's mod k.

if c_{12} in (11. 2) is sufficiently large. Analogously we get

$$\frac{1}{\varphi(k)}\sum_{\chi}(1-\bar{\chi}(l))\sum_{\substack{\varrho(\chi)\\ 1<|t_\varrho|\leqq\tau}}\left(1-\frac{|t_\varrho|}{\tau}\right)e^{it_\varrho\left(\gamma_\nu-\frac{\pi}{2\tau}\right)}<-\frac{1}{40\pi}\log\tau. \tag{12.6}$$

In order to use the quantities $\gamma_\nu\pm\frac{\pi}{2\tau}$ as ω_ν's in (11. 11), we have to verify (11. 9) for all ν's in (11. 8). So far we made no restrictions upon c_{14} in (11. 8). Let now c_{12} beyond the previous requirements be so large that for the c_{16} in (11. 11) the inequality

$$c_{16}<\frac{c_{12}}{80\pi} \tag{12.7}$$

holds and then we fix it; then we choose c_{14} in (11. 9) with $c_{14}>1$ so large that

$$2\frac{\varphi(k)}{\pi}\tau\log\tau<\frac{1}{4}k^{c_{14}} \tag{12.8}$$

and then fix it. Since owing to $c_{12}>2$ we have

$$\{\log^2(k^{c_{12}})\}^{\frac{1}{4}k^{c_{14}}}>(\log 9)^{\frac{k}{2}}>2,$$

this means owing to (12. 8) and (12. 4)

$$\gamma_\nu+\frac{\pi}{2\tau}<q^{(3\nu+1,75)k^{c_{14}}}+1<q^{(3\nu+2)k^{c_{14}}}. \tag{12.9}$$

Since further, as before, we have

$$\gamma_\nu-\frac{\pi}{2\tau}>q^{\left(3\nu+\frac{3}{2}\right)k^{c_{14}}}-1>q^{(3\nu+1)k^{c_{14}}} \tag{12.10}$$

the requirement (11. 9) is verified indeed and hence from (11. 11), (12. 5), (12. 6) and (12. 7) we get

$$\frac{1}{2\pi}\int\limits_{\frac{3}{4}\left(\gamma_\nu-\frac{\pi}{2\tau}\right)}^{\frac{5}{4}\left(\gamma_\nu+\frac{\pi}{2\tau}\right)}g(r)G_{\gamma_\nu+\frac{\pi}{2\tau}}(r)\,dr<-\frac{1}{40\pi}\log\tau+c_{16}\log k<-\frac{1}{80\pi}\log\tau$$

and analogously

$$\frac{1}{2\pi}\int\limits_{\frac{3}{4}\left(\gamma_\nu-\frac{\pi}{2\tau}\right)}^{\frac{5}{4}\left(\gamma_\nu-\frac{\pi}{2\tau}\right)}g(r)G_{\gamma_\nu-\frac{\pi}{2\tau}}(r)\,dr>\frac{1}{80\pi}\log\tau.$$

These last two inequalities show at once that $g(r)$ changes sign necessarily in the intervals

$$(q^{3\nu k^{c_{14}}}<)\ \frac{3}{4}\left(\gamma_\nu-\frac{\pi}{2\tau}\right)\leqq r\leqq\frac{5}{4}\left(\gamma_\nu+\frac{\pi}{2\tau}\right)(<q^{(3\nu+3)k^{c_{14}}})$$

for all of our ν's; since these ranges are distinct, taking in account also (11. 8), the proof of Theorem 1. 2 is finished.

13. Now we turn to the proof of Theorem 3. 2. Since (7. 1) of paper II is by the assumptions of our theorem simply satisfied, Lemma II of the paper II is applicable and hence we have a y_1 such that

$$\frac{1}{20}\log_2 T\leqq y_1\leqq\frac{1}{10}\log_2 T \tag{13. 1}$$

and for all ϱ's of all $L(s,\chi)$ functions mod k (writing $\varrho=\sigma_\varrho+it_\varrho$) the inequalities

$$\pi\geqq\left|\operatorname{arc}\frac{e^{it_\varrho y_1}}{\varrho}\right|\geqq c_{18}\frac{A(k)^3}{k(1+|t_\varrho|)^6\log^3 k(2+|t_\varrho|)} \tag{13. 2}$$

and

$$\pi\geqq\left|\operatorname{arc}\frac{e^{i\frac{t_\varrho}{2}y_1}}{\varrho}\right|\geqq c_{18}\frac{A(k)^3}{k(1+|t_\varrho|)^6\log^3 k(2+|t_\varrho|)} \tag{13. 3}$$

hold; we fix this y_1. The integer ν is restricted at present only by the requirement

$$\frac{\log T}{y_1}-\log^{0,9}T\leqq\nu\leqq\frac{\log T}{y_1}. \tag{13. 4}$$

Let the residue-classes

$$\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_Q\\ \alpha_1',\alpha_2',\ldots,\alpha_Q'\end{matrix} \tag{13. 5}$$

be the incongruent solutions of the congruences

$$x^2\equiv l \bmod k,\qquad x^2\equiv 1 \bmod k \tag{13. 6}$$

respectively. Then we start from the integral

$$J_\nu(T,k)\stackrel{\text{def}}{=}\frac{1}{2\pi i}\int\limits_{(2)}\left\{\left(\frac{e^{y_1 s}}{s}\right)^\nu\frac{1}{\varphi(k)}\sum_\chi\left(1-\bar\chi(l)\right)\frac{L'}{L}(s,\chi)-\right. \tag{13. 7}$$

$$\left.-2^{\nu-1}\sum_{j=1}^{Q}\left(\frac{e^{\frac{1}{2}y_1 s}}{s}\right)^\nu\frac{1}{\varphi(k)}\sum_\chi\left(\bar\chi(\alpha_j')-\bar\chi(\alpha_j)\right)\frac{L'}{L}(s,\chi)\right\}ds.$$

Inserting the known Dirichlet-series for $\frac{L'}{L}(s, \chi)$ we get

(13.8) $$J_\nu(T, k) = \sum_{\substack{n \leq e^{y_1 \nu} \\ n \equiv l \bmod k}} \Lambda(n) \frac{\log^{\nu-1} \frac{e^{\nu y_1}}{n}}{(\nu-1)!} - \sum_{\substack{n \leq e^{y_1 \nu} \\ n \equiv 1 \bmod k}} \Lambda(n) \frac{\log^{\nu-1} \frac{e^{\nu y_1}}{n}}{(\nu-1)!} +$$

$$+ \sum_{j=1}^{Q} \sum_{\substack{n \leq e^{\frac{1}{2} y_1 \nu} \\ n \equiv \alpha_j' \bmod k}} \Lambda(n) \frac{\log^{\nu-1} \frac{e^{\nu y_1}}{n^2}}{(\nu-1)!} - \sum_{j=1}^{Q} \sum_{\substack{n \leq e^{\frac{1}{2} y_1 \nu} \\ n \equiv \alpha_j \bmod k}} \Lambda(n) \frac{\log^{\nu-1} \frac{e^{\nu y_1}}{n^2}}{(\nu-1)!}.$$

The contribution of the primes to the first two sums is obviously

(13.9) $$\sum_{\substack{p \leq e^{\nu y_1} \\ p \equiv l \bmod k}} \log p \cdot \frac{\log^{\nu-1} \frac{e^{\nu y_1}}{p}}{(\nu-1)!} - \sum_{\substack{p \leq e^{\nu y_1} \\ p \equiv 1 \bmod k}} \log p \cdot \frac{\log^{\nu-1} \frac{e^{\nu y_1}}{p}}{(\nu-1)!}.$$

However the contribution of the p^2's to the first two sums and that of the p's to the latter ones in (13. 8) cancels, owing to (13. 5). Hence we have on the one hand

(13. 10) $$\left| J_\nu(T, k) - \sum_{\substack{p \leq e^{y_1 \nu} \\ p \equiv l \bmod k}} \log p \cdot \frac{\log^{\nu-1} \frac{e^{\nu y_1}}{p}}{(\nu-1)!} + \sum_{\substack{p \leq e^{y_1 \nu} \\ p \equiv 1 \bmod k}} \log p \cdot \frac{\log^{\nu-1} \frac{e^{\nu y_1}}{p}}{(\nu-1)!} \right| <$$

$$< c_{19} \left\{ \frac{(\nu y_1)^{\nu-1}}{(\nu-1)!} (e^{\frac{1}{3} y_1 \nu}) y_1 \nu + k \frac{(\nu y_1)^{\nu-1}}{(\nu-1)!} (e^{\frac{1}{4} y_1 \nu}) y_1 \nu \right\} < c_{20} T^{\frac{1}{3}} e_1\left(21 \frac{\log T \log_3 T}{\log_2 T}\right),$$

using (13. 1), (13. 4) and (3. 1). Now the two sums on the left can be written in the form

(13. 11) $$\int_1^{e^{\nu y_1}} \frac{\log^{\nu-1} \frac{e^{\nu y_1}}{x}}{(\nu-1)!} \log x \cdot d\{\pi(x, k, l) - \pi(x, k, 1)\} =$$

$$= \int_1^{e^{\nu y_1}} (\pi(x, k, l) - \pi(x, k, 1)) \left\{ \frac{\log x}{(\nu-1)!} \log^{\nu-1} \frac{e^{\nu y_1}}{x} \right\}' dx = \int_1^{e^{y_1}} + \int_{e^{y_1}}^{e^{\nu y_1}}.$$

For the absolute value of the first integral we get simply (as in 10 of paper II) the upper bound

$$e_1\left(21 \frac{\log T \log_3 T}{\log_2 T}\right).$$

As to the second one, since the factor

$$-\log^{\nu-1}\frac{e^{\nu y_1}}{x}\cdot\log x$$

is increasing for $e^{y_1}\leqq x\leqq e^{\nu y_1}$ we get respectively the upper and lower bound

$$\max_{x\leqq e^{\nu y_1}}\{\pi(x,k,l)-\pi(x,k,1)\}\cdot\frac{y_1}{(\nu-1)!}((\nu-1y_1)^{\nu-1},$$

$$\min_{x\leqq e^{\nu y_1}}\{\pi(x,k,l)-\pi(x,k,1)\}\cdot\frac{y_1}{(\nu-1)!}((\nu-1)y_1)^{\nu-1}.$$

Taking in account also (13.4), (13.10) gives

$$\max_{x\leqq T}\{\pi(x,k,l)-\pi(x,k,1)\}>\{J_\nu(T,k)-T^{0,4}\}\frac{(\nu-1)!}{y_1}((\nu-1)y_1)^{-\nu+1}\tag{13.12}$$

and analogously for

$$\min_{x\leqq T}\{\pi(x,k,l)-\pi(x,k,1)\}.\tag{13.13}$$

14. We define the broken line V symmetrically to the real axis in the vertical strip $\frac{1}{5}\leqq\sigma\leqq\frac{2}{5}$ exactly as we did in **10** of the paper II and we shift the path of integration in (13.7) to V. For the absolute value of this last integral we get as in 11 of paper II the upper bound

$$c_{21}\{5^\nu\cdot e^{\frac{2}{5}\nu y_1}k\log k+2^\nu\cdot k\cdot 5^\nu\cdot e^{\frac{1}{5}\nu y_1}k\log^3 k\}.$$

Taking in account (13.1), (13.4) and (3.1) this is

$$<T^{0,45}$$

choosing in (3.1) c_7 sufficiently large. With the contribution of the residua this and (13.12) give

$$\max_{x\leqq T}\{\pi(x,k,l)-\pi(x,k,1)\}>\frac{(\nu-1)!}{y_1}((\nu-1)y_1)^{-\nu+1}\cdot\tag{14.1}$$

$$\cdot\left\{\frac{1}{\varphi(k)}\sum_\chi((1-\bar\chi(l))\sum\nolimits'_{\varrho(\chi)}\left(\frac{e^{y_1\varrho}}{\varrho}\right)^\nu-\right.$$

$$\left.-2^{\nu-1}\sum_{j=1}^{Q}\frac{1}{\varphi(k)}\sum_\chi(\bar\chi(\alpha_j')-\bar\chi(\alpha_j))\sum\nolimits'_{\varrho(\chi)}\left(\frac{e^{\frac{y_1}{2}\varrho}}{\varrho}\right)^\nu-2T^{0,45}\right\}$$

and analogously for

$$\min_{x\leqq T}\{\pi(x,k,l)-\pi(x,k,1)\};$$

here and later the dash means that the summation has to be extended only over the ϱ's lying to the right of V.

Next we estimate roughly the contribution of the ϱ's right to V and with

$$|t_\varrho| > \log^{\frac{1}{10}} T.$$

Owing to the already used inequality

$$\sum_{\substack{\varrho(x) \\ r \leq t_\varrho < r+1}} 1 < c_{22} \log k(1+|r|) \tag{14.2}$$

this contribution cannot absolutely exceed

$$c_{23}\left(\frac{e^{y_1}}{\log^{\frac{1}{10}} T}\right)^{v} \cdot 2^{v} < c_{24} e_1\left(20 \frac{\log T}{\log_2 T}\right).$$

Hence from (14. 1) we get

$$\max_{x \leq T}\{\pi(x,k,l) - \pi(x,k,1)\} > \frac{(v-1)!}{y_1}((v-1)y_1)^{-v+1} \cdot$$

$$\cdot\left\{\frac{1}{\varphi(k)} \sum_{\chi} (1-\bar{\chi}(l)) \sum_{\substack{\varrho(x) \\ |t_\varrho| \leq \log^{\frac{1}{10}} T}}' \left(\frac{e^{y_1\varrho}}{\varrho}\right)^{v} - \right.$$

$$\left. -2^{v-1} \sum_{j=1}^{Q} \frac{1}{\varphi(k)} \sum_{\chi} (\bar{\chi}(\alpha_j') - \bar{\chi}(\alpha_j)) \sum_{\substack{\varrho(x) \\ |t_\varrho| \leq \log^{\frac{1}{10}} T}}' \left(\frac{e^{\frac{1}{2}y_1\varrho}}{\varrho}\right)^{v} - 3T^{0,45}\right\}.$$

For our later aims we write it in a different form. Let $\varrho_1 = \sigma_1 + it_1$ be such a zero in our domain of an $L(s,x) \bmod k$ with $\chi(l) \neq 1$ for which

$$\left|\frac{e^{y_1\varrho}}{\varrho}\right| = \text{maximal}. \tag{14.3}$$

Owing to the functional-equation we have

$$\sigma_1 \geq \frac{1}{2}. \tag{14.4}$$

Then we have, since our sums are obviously real, the inequality

$$\max_{x \leq T}\{\pi(x,k,l) - \pi(x,k,1)\} > \frac{(v-1)!}{y_1}((v-1)y_1)^{-v+1} \cdot \tag{14.5}$$

$$\cdot\left\{-3T^{0,45} + \left(\frac{e^{y_1\sigma_1}}{|\varrho_1|}\right)^{v} \operatorname{Re}\left(\frac{1}{\varphi(k)} \sum_{\chi} (1-\bar{\chi}(l)) \sum_{\substack{\varrho(x) \\ |t_\varrho| \leq \log^{\frac{1}{10}} T}}' \left(\frac{e^{y_1(\varrho-\sigma_1)}}{\varrho}|\varrho_1|\right)^{v} - \right.\right.$$

$$\left.\left. -\frac{1}{2\varphi(k)} \sum_{j=1}^{Q} \sum_{\chi} (\bar{\chi}(\alpha_j') - \chi(\alpha_j)) \cdot \sum_{\substack{\varrho(x) \\ |t_\varrho| \leq \log^{\frac{1}{10}} T}}' \left(2\frac{e^{y_1\left(\frac{\varrho}{2}-\sigma_1\right)}}{\varrho}|\varrho_1|\right)^{v}\right)\right\}$$

and analogously for

$$\min_{x \leqq T} \{\pi(x, k, l) - \pi(x, k, 1)\}. \tag{14.6}$$

15. The assertions of Theorem 3. 2 will be proved by two appropriate choices of the integer ν which was so far restricted only by (13. 4); the possibility of these choices will be given by Theorem 4. 1. The z_j-numbers will be of course the numbers

$$\frac{e^{y_1(\varrho - \sigma_1)}}{\varrho} |\varrho_1|, \quad \frac{2e^{y_1\left(\frac{\varrho}{2} - \sigma_1\right)}}{\varrho} |\varrho_1|; \tag{15.1}$$

in what follows we shall call them z_j's of the first resp. of second category. That the maximal absolute value of the z_j's of first category is 1, is trivial from (14. 3); since from (13. 1), owing to the definition of V and ϱ_1

$$2\frac{e^{y_1\left(\sigma_\varrho \frac{1}{2} - \sigma_1\right)}}{|\varrho|} |\varrho_1| = 2e^{-\frac{y_1 \sigma_\varrho}{2}} \left| \frac{e^{y_1(\varrho - \sigma_1)}}{\varrho} \varrho_1 \right| \leqq 2e^{-\frac{y_1 \sigma_\varrho}{2}} \leqq 2e^{-\frac{y_1}{10}} \leqq 2e^{-\frac{1}{200}\log_2 T} < 1,$$

we have

$$\max_j |z_j| = 1. \tag{15.2}$$

As m we chose

$$m = \left[\frac{\log T}{y_1} - \log^{0,9} T\right]. \tag{15.3}$$

As to the arguments of our z_j's owing to (13. 2) and (13. 3) the restriction (4. 2) is satisfied with

$$c_{18} \frac{A(k)^3}{k(1 + \log^{\frac{1}{10}} T)^6 \log^3 k(2 + \log^{\frac{1}{10}} T)}.$$

Owing to (3. 1) this is

$$> \frac{c_{27}}{\log^{\frac{3}{5}} T \cdot (\log_2 T)^5}$$

choosing c_7 in (3. 1) sufficiently large. Hence we may choose

$$\varkappa = \log^{-\frac{2}{3}} T. \tag{15.4}$$

As to n we have owing to (14. 2) the upper bound

$$c_{28} k \log^{\frac{1}{10}} T \cdot \log_2 T < \log^{\frac{1}{10}} T \cdot (\log_2 T)^3. \tag{15.5}$$

As to the b_j's, evidently they have the form

$$\frac{1}{\varphi(k)}(1 - \bar{\chi}(l)) \quad \text{and} \quad \frac{1}{2\varphi(k)}(\bar{\chi}(\alpha'_\mu) - \bar{\chi}(\alpha_\mu)); \tag{15.6}$$

they can be called correspondingly to b_j of the first resp. of the second category. Finally, we have to determine the indices h and h_1. Let simply be

$$h = 1 \tag{15.7}$$

then (4.3) is evidently satisfied owing to (15.4). As to h_1 our aim is to rule out the „inconvenient" b_j's of second category and therefore owing to (4.4) let h_1 be the minimal index of the z_j's of the second category. Then we have certainly

$$z_{h_1} = 2\frac{e^{y_1\left(\frac{\varrho_2}{2}-\sigma_1\right)}}{\varrho_2}|\varrho_1|. \tag{15.8}$$

with a certain $\varrho_2 = \sigma_2 + it_2$ and we have to verify (4.5). Owing to the maximality expressed in (14.3) we have

$$|z_{h_1}| = 2e^{-\frac{y_1\sigma_\varrho}{2}}\left|\frac{e^{y_1(\varrho_2-\sigma_1)}}{\varrho_2}\varrho_1\right| \leqq 2e^{-\frac{y_1\sigma_\varrho}{2}} \leqq 2e^{-\frac{y_1}{10}} \leqq 2\log^{-\frac{1}{200}} T$$

i. e.

$$|z_h| - |z_{h_1}| \geqq 1 - 2\log^{-\frac{1}{200}} T > \frac{1}{2}. \tag{15.9}$$

Since from (15.4) for sufficiently large c_7 in (3.1) we have

$$\frac{2n}{m+n\left(3+\frac{\pi}{\varkappa}\right)} < \frac{2}{3+\pi\log^{\frac{2}{3}} T} < \frac{1}{2}$$

(4.5) follows from this and (15.9). Since we have from (3.1)

$$|\gamma_0| < \log^{\frac{1}{10}} T$$

there is at least one b of the first category existing and since

$$\frac{1}{\varphi(k)}\operatorname{Re}(1-\bar{\chi}(l)) > \frac{8}{k^3}$$

we have owing to the definition of h_1

$$A = \min_{1\leqq\xi<h_1} \operatorname{Re}\sum_{j=1}^{\xi} b_j > \frac{8}{k^3}. \tag{15.10}$$

Hence applying Theorem 4.1 we obtain the existence of integer v_1 and v_2 numbers such that

$$\frac{\log T}{y_1} - \log^{0,9} T \leqq v_1; \quad v_2 \leqq \frac{\log T}{y_1} - \log^{0,9} T + \tag{15.11}$$

$$+ (3+\pi\log^{\frac{2}{3}} T)\log^{\frac{1}{10}} T(\log_2 T)^3 < \frac{\log T}{y_1}$$

(choosing c_7 in (3. 1) sufficiently large and taking in account (15. 5)) i. e. (13. 4) is by these choices fulfilled. Thus

$$\mathrm{Re}\left\{\frac{1}{\varphi(k)}\sum_{\chi}(1-\bar{\chi}(l))\sum'_{\substack{\varrho(\chi)\\ |t_\varrho|\le\log^{\frac{1}{10}}T}}\left(\frac{e^{y_1(\varrho-\sigma_r)}}{\varrho}|\varrho_1|\right)^{\nu_1}-\right.$$

$$(15.12)\qquad \left.-\frac{1}{2\varphi(k)}\sum_{j=1}^{Q}\sum_{\chi}(\bar{\chi}(\alpha'_j)-\bar{\chi}(\alpha_j))\sum'_{\substack{\varrho(\chi)\\ |t_\varrho|\le\log^{\frac{1}{10}}T}}\left(2\frac{e^{y_1\left(\frac{\varrho}{2}-\sigma_1\right)}}{\varrho}|\varrho_1|\right)^{\nu_1}\right\}>$$

$$>\frac{8}{k^3}\frac{1}{3\log^{\frac{1}{10}}T(\log_2 T)^3}\cdot\left(\frac{y_1}{24\log T}\right)^{2\log^{\frac{1}{10}}T(\log_2 T)^3}>e_1(-\log^{\frac{1}{5}}T)$$

with (3. 1) with sufficiently large c_7 and analogous inequality in the opposite direction with ν_2 instead of ν_1 (with $-e_1(-\log^{\frac{1}{5}}T)$ on the right).

16. To conclude the proof we need a lower bound for

$$\left(\frac{e^{y_1\sigma_1}}{|\varrho_1|}\right)^{\nu_1}.$$

Owing to the maximum-definition of ϱ_1 in (14. 3) we have

$$\left(\frac{e^{y_1\sigma_1}}{|\varrho_1|}\right)^{\nu_1}\ge\left(\frac{e^{y_1\beta_0}}{|\varrho_0|}\right)^{\nu_1}=\frac{(e^{\nu_1 y_1})^{\beta_0}}{|\varrho_0|^{\nu_1}}$$

and hence from (15. 13) and (3. 1)

$$(16.1)\quad \ge T^{\beta_0}e_1(-\log^{0,9}T\cdot\log_2 T)\cdot(\log_2 T)^{-20\frac{\log T}{\log_2 T}}>T^{\beta_0}e_1\left(-21\frac{\log T\log_3 T}{\log_2 T}\right).$$

Collecting (15. 14), (16. 1) and (14. 5) we got

$$\max_{x\le T}\{\pi(x,k,l)-\pi(x,k,1)\}>\frac{(\nu_1-1)!}{y_1}((\nu_1-1)y_1)^{-\nu_1+1}\cdot$$

$$\cdot\left\{T^{\beta_0}e_1\left(-21\frac{\log T\log_3 T}{\log_2 T}\right)-3T^{0,45}\right\}>T^{\beta_0}e_1\left(-42\frac{\log T\log_3 T}{\log_2 T}\right)$$

using (15. 13). Since $\beta_0\ge\frac{1}{2}$ and

$$\max_{x\le T^{\frac{1}{3}}}\{\pi(x,k,l)-\pi(x,k,1)\}<T^{\frac{1}{3}}$$

we get

$$\max_{T^{\frac{1}{3}} \leqq x \leqq T} \{\pi(x, k, l) - \pi(e, k, 1)\} > T^{\beta_0} e_1\left(-43 \frac{\log T \log_3 T}{\log_2 T}\right)$$

which proves the one-half of Theorem 3. 2; the second half of the proof runs after (14. 6) and (15. 14) on the same lines.

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