COMPARATIVE PRIME-NUMBER THEORY. II

(COMPARISON OF THE PROGRESSIONS $\equiv 1 \bmod k$ AND $\equiv l \bmod k$, $l \not\equiv 1 \bmod k$)

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1. Now we turn, as told in I, to a detailed exposition of the comparison of the progressions

(1. 1)
$$n \equiv 1 \bmod k, \quad n \equiv l \bmod k$$
$$l \not\equiv 1 \bmod k.$$

We shall keep the (usual) notations laid down in I; we denote by $c_1, c_2, \ldots$ positive explicitely calculable numerical constants. As to the results we formulate them as theorems when they mean a progress in one of the problems raised in I no matter how easily it follows from another theorem of this paper; in particular having proved a theorem under HASELGROVE's condition[1] we shall formulate it as a separate theorem for the k's for which this assumption is verified, i. e. for

(1. 2)
$$k = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 19, 24.$$

We begin with the

THEOREM 1. 1. *For the k's in* (1. 2) *we have for $T > c_1$ the inequalities*

$$\max_{T^{\frac{1}{3}} \leqq x \leqq T} \{\psi(x, k, 1) - \psi(x, k, l)\} > \sqrt{T} e_1\left(-41 \frac{\log T \log_3 T}{\log_2 T}\right)$$

and

$$\min_{T^{\frac{1}{3}} \leqq x \leqq T} \{\psi(x, k, 1) - \psi(x, k, l)\} < -\sqrt{T}\, e_1\left(-41 \frac{\log T \log_3 T}{\log_2 T}\right).$$

By this theorem is problem 24 of paper I for the k's in (1. 2) essentially solved in the case $l_1 = 1$ at least. Since as Siegel proved (see SIEGEL [1]) for all $L(s, \chi)$ functions belonging to primitive characters mod k have at least one zero $\varrho^* = \varrho^*(\chi)$ in the domain

(1. 3)
$$\sigma \geqq \frac{1}{2}, \quad |t| \leqq \frac{c_2}{\log_3 (k + e_3(1))}, \qquad (s = \sigma + it).$$

Theorem 1. 1 follows at once from the

[1] We remind the reader that HASELGROVE's assumption for a k means that there is an $1 \geqq A(k) > 0$ such that no $L(s, \chi)$ belonging to mod k vanishes for $0 < \sigma < 1$, $|t| \leqq A(k)$. Further $e_1(x) = e^x$, $e_\nu(x) = e_{\nu-1}(e_1(x))$, $\log_1 x = \log x$, $\log_\nu x = \log_{\nu-1} (\log x)$.

THEOREM 1. 2. *If for the k's in* (1. 2) *and a* $\varrho_0 = \beta_0 + i\gamma_0$ *with*

$$\beta_0 \geqq \frac{1}{2}, \qquad \gamma_0 > 0 \tag{1.4}$$

ϱ_0 *is a zero of an* $L(s, \chi^*)$ *belonging to* mod k *with* $\chi^*(l) \neq 1$ *and*

$$T > \max(c_3, e_2(10|\varrho_0|)),$$

then the inequalities

$$\max_{T^{\frac{1}{3}} \leqq x \leqq T} \{\psi(x, k, 1) - \psi(x, k, l)\} > T^{\beta_0} e_1\left(-41 \frac{\log T \log_3 T}{\log_2 T}\right)$$

and

$$\min_{T^{\frac{1}{3}} \leqq x \leqq T} \{\psi(x, k, 1) - \psi(x, k, l)\} < -T^{\beta_0} e_1\left(-41 \frac{\log T \log_3 T}{\log_2 T}\right)$$

hold.

2. Parallel we state the

THEOREM 2. 1. *For the k's in* (1. 2) *we have for* $T > c_4$ *the inequalities*

$$\max_{T^{\frac{1}{3}} \leqq x \leqq T} \{\Pi(x, k, 1) - \Pi(x, k, l)\} > \sqrt{T} e_1\left(-41 \frac{\log T \log_3 T}{\log_2 T}\right)$$

and

$$\min_{T^{\frac{1}{3}} \leqq x \leqq T} \{\Pi(x, k, 1) - \Pi(x, k, l)\} < -\sqrt{T} e_1\left(-41 \frac{\log T \log_3 T}{\log_2 T}\right).$$

By this theorem is Problem 44 of paper I for the k's in (1. 2) essentially solved in the case $l_1 = 1$ at least. Owing to (1. 3) Theorem 2. 1 follows at once from the

THEOREM 2. 2. *If for the k's in* (1. 2) *and a* $\varrho_0 = \beta_0 + i\gamma_0$ *with*

$$\beta_0 \geqq \frac{1}{2}, \qquad \gamma_0 > 0$$

ϱ_0 *is a zero of an* $L(s, \chi^*)$ *belonging to* mod k *with* $\chi^*(l) \neq 1$ *and*

$$T > \max(c_5, e_2(10|\varrho_0|))$$

then the inequalities

$$\max_{T^{\frac{1}{3}} \leqq x \leqq T} \{\Pi(x, k, 1) - \Pi(x, k, l)\} > T^{\beta_0} e_1\left(-41 \frac{\log T \log_3 T}{\log_2 T}\right)$$

and

$$\min_{T^{\frac{1}{3}} \leqq x \leqq T} \{\Pi(x, k, 1) - \Pi(x, k, l)\} < -T^{\beta_0} e_1\left(-41 \frac{\log T \log_3 T}{\log_2 T}\right)$$

hold.

3. In turn theorems (1. 2) and (2. 2) follow at once from

THEOREM 3. 1. *If for a modulus k Haselgrove's condition holds and for a* ϱ_0 *with* (1. 4) *holds, then for*

$$T > \max\left(c_6,\, e_2(10|\varrho_0|),\, e_2(k),\, e_2\left(\frac{1}{A(k)^3}\right)\right) \tag{3.1}$$

the inequalities

$$\max_{T^{\frac{1}{3}} \leqq x \leqq T} \{\psi(x, k, 1) - \psi(x, k, l)\} > T^{\beta_0} e_1\left(-41 \frac{\log T \log_3 T}{\log_2 T}\right) \tag{3.2}$$

and

$$\min_{T^{\frac{1}{3}} \leqq x \leqq T} \{\psi(x, k, 1) - \psi(x, k, l)\} < -T^{\beta_0} e_1\left(-41 \frac{\log T \log_3 T}{\log_2 T}\right), \tag{3.3}$$

further

$$\max_{T^{\frac{1}{3}} \leqq x \leqq T} \{\Pi(x, k, 1) - \Pi(x, k, l)\} > T^{\beta_0} e_1\left(-41 \frac{\log T \log_3 T}{\log_2 T}\right) \tag{3.4}$$

and

$$\min_{T^{\frac{1}{3}} \leqq x \leqq T} \{\Pi(x, k, 1) - \Pi(x, k, l)\} < -T^{\beta_0} e_1\left(-41 \frac{\log T \log_3 T}{\log_2 T}\right) \tag{3.5}$$

hold.

Since the proofs of (3. 2)—(3. 3) resp. of (3. 4)—(3. 5) run parallel and the last one is a bit more difficult, it will be enough to prove (3. 4)—(3. 5).

4. Before turning to the announcement of essentially different further results, we shall formulate three simple consequences of Theorem 3. 1. First of all taking as ϱ_0 the zero in (1. 3) we get as a mere corollary the

THEOREM 4. 1. *In the interval*

$$0 < x < \max\left(c_7,\, e_2(k),\, e_2\left(\frac{1}{A(k)^3}\right)\right)$$

the functions

$$\psi(x, k, 1) - \psi(x, k, l)$$
$$\Pi(x, k, 1) - \Pi(x, k, l)$$

change certainly their sign, if for k the Haselgrove-condition holds.

Here

$$c_7 = \max\left(c_6,\, e_2(10(1 + c_2))\right) \tag{4.1}$$

can be taken. This theorem constitutes first step towards the solution of the Problems 45 and 25 of the paper I and is probably very far from the best-possible. We conjecture that the „best" interval in Theorem 4. 1 is

(4. 2) $$0 < x < e_1(c_8 k)$$

with a suitable c_8.

A further consequence we can draw from Theorem 3. 1 refers to the functions[2] $U_k(T, 1, l)$, $V_k(T, 1, l)$, resp. This is the

THEOREM 4. 2. *If for a k Haselgrove's condition holds (i. e. for the k's in* (1. 2) *unconditionally) and*[3]

(4. 3) $$T > \exp c_9^2 \left(e_1(k) + e_1\left(\frac{1}{A(k)^3}\right)\right)^2$$

then the inequalities

$$U_k(T, 1, l) > \frac{1}{8 \log 3} \log_2 T$$

and

$$V_k(T, 1, l) > \frac{1}{8 \log 3} \cdot \log_2 T$$

hold.

In contrary to the previous theorems these lower bounds are probably very rough. Still they are as far as we know the first quantitative results in this direction which are at least for the k's in (1. 2) unconditional and thus they mean the first essential steps towards the solution of the Problems 26 and 46 of the paper I.

In order to deduce Theorem 4. 2 from Theorem 3. 1 we take as ϱ_0 a zero from (1. 3) and putting

(4. 4) $$c_9 = \max(c_6, e_2(10(1 + c_2)))$$

Theorem 3. 1 is applicable to each interval

$$\tau^{\frac{1}{3}} \leqq x \leqq \tau$$

whenever

$$\tau > \max\left(c_9, e_2(k), e_2\left(\frac{1}{A(k)^3}\right)\right)$$

i. e. a fortiori when

(4. 5) $$\tau > \exp c_9 \left(e_1(k) + e_1\left(\frac{1}{A(k)^3}\right)\right) \overset{\text{def}}{=} \tau_0 .$$

Hence each of the intervals

$$[\tau_0^{3^{2j}}, \tau_0^{3^{2j+1}}] \overset{\text{def}}{=} I_j$$

contains at least one sign-change of $\psi(x, k, 1) - \psi(x, k, l)$. But then if the integer μ is determined by

(4. 6) $$\tau_0^{3^{2\mu}} \leqq T < \tau_0^{3^{2\mu+2}},$$

[2] We remind the reader that $U_k(T, 1, l)$ resp. $V_k(T, 1, l)$ denote the number of sign-changes of $\psi(x, k, 1) - \psi(x, k, l)$ resp. $\Pi(x, k, 1) - \Pi(x, k, l)$ in the interval $0 < x \leqq T$.

[3] For c_9 see (4.4).

we have

(4. 7) $$U_k(T, 1, l) \geqq \mu, \quad V_k(T, 1, l) \geqq \mu.$$

Since from $c_9 \geqq 1$, (4. 3) and (4. 5) we have

$$\log T > (\log \tau_0)^2$$

i. e. from (4. 6)

$$3^{2\mu+2} > \frac{\log T}{\log \tau_0} > \sqrt{\log T}$$

$$2\mu \geqq \mu + 1 > \frac{1}{4 \log 3} \log_2 T$$

which together with (4. 7) proves Theorem 4. 2 (if Theorem 3. 1 is proved).
An obvious consequence of Theorem 3.1 is the following

THEOREM 4. 3. *Let $L(s, \chi^*)$ an arbitrary L-function* mod *k and (supposing Haselgrove's condition for k)*

$$T > \max\left(c_6, e_2(k), e_2\left(\frac{1}{A(k)^3}\right)\right).$$

If l is such that $\chi^*(l) \neq 1$, then $L(s, \chi)$ does not vanish in the domain

$$\sigma \geqq 41 \frac{\log_3 T}{\log_2 T} + \frac{1}{\log T} \max_{T^{\frac{1}{3}} \leqq x \leqq T} \log \{\psi(x, k, 1) - \psi(x, k, l)\}$$

$$|t| \leqq \frac{1}{10} \log_2 T - 1.$$

5. Turning to $\pi(x, k, 1) - \pi(x, k, l)$ the matter becomes still more difficult. We assert the

THEOREM 5. 1. *If k is one of the moduli* (1. 2) *then for $T > c_{10}$ the inequalities*

(5. 1) $$\max_{e_1(\log_3^{\frac{1}{130}} T) \leqq x \leqq T} \frac{\pi(x, k, 1) - \pi(x, k, l)}{\left(\frac{\sqrt{x}}{\log x}\right)} > \frac{1}{100} \log_5 T$$

and

(5. 2) $$\min_{e_1(\log_3^{\frac{1}{130}} T) \leqq x \leqq T} \frac{\pi(x, k, 1) - \pi(x, k, l)}{\left(\frac{\sqrt{x}}{\log x}\right)} < -\frac{1}{100} \log_5 T$$

hold.

This theorem holds without any conjectures and by it Problem 3 of paper I for the k's in (1. 2) in the case $l_1 = 1$ is essentially solved. This will follow at once from

THEOREM 5. 2. *If Haselgrove's condition holds for a k and*

$$T > \max\left(e_5(c_{11}k),\ e_2\left(\frac{1}{A(k)^3}\right)\right) \tag{5. 3}$$

then the inequalities (5. 1) *and* (5. 2) *hold.*

Again we formulate two simple consequences of Theorem 5. 2. As a mere corollary of it we have the

THEOREM 5. 3. *If Haselgrove's condition holds for a k then the interval*

$$1 \leqq x \leqq \max\left(e_5(c_{11}k),\ e_2\left(\frac{1}{A(k)^3}\right)\right) \tag{5. 4}$$

contains at least one sign-change of $\pi(x, k, 1) - \pi(x, k, l)$.

This theorem constitutes the first step towards the solution of Problem 5 in paper I. Most probably the interval (5. 4) is much too large. An upper bound better than (5. 4) we shall see in paper III.

A further consequence we could draw from Theorem 5. 2 refers to the function[4] $W_k(T, 1, l)$. But the lower bound obtained that way would be very low; in paper III we shall return to the subject and obtain a much better lower bound.

Again this is — at least for the k's in (1. 2) — the first step without conjuctures after Littlewood towards the solution of Problem 6 in paper I.

As we shall see in the course of the proof of Theorem 5. 2 an analogous theorem holds for $\psi(x, k, l)$ resp. $\Pi(x, k, l)$ instead of $\pi(x, k, l)$. However we shall not formulate the corresponding theorems explicitely.

As to the race-problem of SHANKS — RÉNYI we state the following theorem (which is for the k's in (1. 2) unconditional).

THEOREM 5. 4. *If Haselgrove's condition is fulfilled for a k and for T the restriction* (5. 3) *holds then the inequalities*

$$\max_{e_1(\log_3^{\frac{1}{130}} T) \leqq x \leqq T} \frac{\log x}{\sqrt{x}} \left\{\pi(x, k, 1) - \frac{1}{\varphi(k) - 1} \sum_{\substack{(l,k)=1 \\ l \neq 1}} \pi(x, k, l)\right\} > \frac{1}{100} \log_5 T$$

and

$$\min_{e_1(\log_3^{\frac{1}{130}} T) \leqq x \leqq T} \frac{\log x}{\sqrt{x}} \left\{\pi(x, k, 1) - \frac{1}{\varphi(k) - 1} \sum_{\substack{(l,k)=1 \\ l \neq 1}} \pi(x, k, l)\right\} < -\frac{1}{100} \log_5 T$$

hold.[5]

[4] We remind the reader that $W_k(T, 1, l)$ denotes the number of sign-changes of $\pi(x, k, 1) - \pi(x, k, l)$ in the interval $0 < x \leqq T$.

[5] Or in equivalent formulation, under the restriction (5.3) the inequalities

$$\max_{e_1(\log_3^{\frac{1}{130}} T) \leqq x \leqq T} \frac{\log x}{\sqrt{x}} \left\{\pi(x, k, 1) - \frac{1}{\varphi(k)} \pi(x)\right\} > \frac{1}{200} \log_5 T$$

and

$$\min_{e_1(\log_3^{\frac{1}{130}} T) \leqq x \leqq T} \frac{\log x}{\sqrt{x}} \left\{\pi(x, k, 1) - \frac{1}{\varphi(k)} \pi(x)\right\} < -\frac{1}{200} \log_5 T$$

hold.

This theorem again is near to the best-possible. Since its proof needs only slight changes compared to that of Theorem 5. 2 a sketch will be sufficient. Popularly expressed this means that in the race of the quantities $\pi(x, k, l)$ (k fixed) with $(k, l)=1$ the runner $l=1$ is certainly not permanently, from a certain place on, on the last place.

As to a comparison of the above-mentioned theorems with results attainable by older methods we refer to the introductory paper I.

6. Now we turn to the proof of Theorem 3. 1 and to that of (3. 4)–(3. 5), resp. The proof will be based on the following theorem (see TURÁN [1]) which we formulate as

LEMMA I. *Let* $n \leqq N$ *and with a* $0 < \varkappa \leqq \frac{\pi}{2}$

(6. 1) $$|z_1| \geqq |z_2| \geqq \ldots \geqq |z_n|$$

(6. 2) $$\varkappa \leqq |\text{arc } z_j| \leqq \pi \qquad (j=1, 2, \ldots, n)$$

and the b_j*-numbers are restricted by*

(6. 3) $$B \overset{\text{def}}{=} \min_{\mu} \text{Re}\,(b_1 + \ldots + b_\mu) > 0.$$

Then to each non-negative m *there are integers* ν_1 *and* ν_2 *with*

(6. 4) $$m \leqq \nu_1, \quad \nu_2 \leqq m + N\left(3 + \frac{\pi}{\varkappa}\right)$$

such that the inequalities

(6. 5) $$\text{Re} \sum_{j=1}^{n} b_j z_j^{\nu_1} \geqq B \frac{|z_1|^{\nu_1}}{2N+1} \left\{ \frac{N}{24e\left(m + N\left(3 + \frac{\pi}{\varkappa}\right)\right)} \right\}^{2N}$$

and

(6. 6) $$\text{Re} \sum_{j=1}^{n} b_j z_j^{\nu_2} \leqq -B \frac{|z_1|^{\nu_1}}{2N+1} \left\{ \frac{N}{24e\left(m + N\left(3 + \frac{\pi}{\varkappa}\right)\right)} \right\}^{2N}$$

hold.

For the proof of this lemma we refer to the original paper. However a modified form of it will be used in paper III of this series, whose proof follows quite closely that of Lemma I and there we shall sketch the proof indicating the (slight) changes.

7. We shall further need a lemma, which is adopted for our present purposes from a paper of one of us (see KNAPOWSKI [1]). Let

(7. 1) $$T > \max\left(c_{12}, e_2(k), e_2\left(\frac{1}{A(k)^3}\right)\right)$$

and $\varrho = \sigma_\varrho + it_\varrho$ should run over all non-trivial zeros of all $L(s, \chi)$-functions belonging to mod k. Then we assert the

LEMMA II. *Under Haselgrove's condition for the T's in* (7. 1) *there is a* y_1 *with*

$$\frac{1}{20}\log_2 T \leqq y_1 \leqq \frac{1}{10}\log_2 T \tag{7.2}$$

and that for all ϱ's the inequalities

$$\pi \geqq \left| \operatorname{arc} \frac{e^{it_\varrho y_1}}{\varrho} \right| \geqq c_{12} \frac{A(k)^3}{k(1+|t_\varrho|)^6 \log^3 k(2+|t_\varrho|)} \tag{7.3}$$

and

$$\pi \geqq \left| \operatorname{arc} \frac{e^{\frac{1}{2}it_\varrho y_1}}{\varrho} \right| \geqq c_{12} \frac{1}{k(1+|t_\varrho|) \log^3 k(2+|t_\varrho|)} \tag{7.4}$$

hold.

Actually we shall use in paper II only (7. 3); (7. 4) will be used only in later papers of this series. We shall give here the full proof in order to avoid repetitions (as we shall see, however, the proof of (7. 3) alone, would amount to the same).

8. Let us call the ϱ's and $\frac{\varrho}{2}$'s together as λ's (which are not necessarily different) and put

$$\lambda = \sigma_\lambda + it_\lambda . \tag{8.1}$$

Let for a fixed λ and $\varepsilon > 0$ $E(\varepsilon, \lambda)$ denote the x-set in

$$I \overset{\text{def}}{=} \left[\frac{1}{20}\log_2 T, \frac{1}{10}\log_2 T\right] \tag{8.2}$$

with the property

$$\left| \operatorname{tg}(t_\lambda x) - \frac{t_\lambda}{\sigma_\lambda} \right| < \frac{\varepsilon}{1+t_\lambda^2} . \tag{8.3}$$

Let us estimate its $|E(\varepsilon, \lambda)|$-measure. We define for our λ and fixed integer q the number $x_q(\lambda)$ by

$$\begin{gathered} q\pi \leqq t_\lambda x_q(\lambda) < (q+1)\pi \\ \operatorname{tg}(t_\lambda x_q(\lambda)) = \frac{t_\lambda}{\sigma_\lambda} . \end{gathered} \tag{8.4}$$

However q is restricted by the requirement $x_q(\lambda)$ being in I; we denote the set of q's by Q and their number by $|Q|$. Let first be $t_\lambda > 0$. Since from (8. 2)

$$\frac{t_\lambda \log_2 T}{20} \leqq t_\lambda x_q(\lambda) \leqq \frac{t_\lambda \log_2 T}{10},$$

we have roughly

$$|Q| < 2 + \frac{t_\lambda \log_2 T}{20} < (1+t_\lambda)\log_2 T. \tag{8.5}$$

Denoting by $E_q(\varepsilon, \lambda)$ with a fixed q the subset of $E(\varepsilon, \lambda)$ with

(8. 6) $$q\pi \leqq t_\lambda x < (q+1)\pi$$

we have in $E_q(\varepsilon, \lambda)$ owing to (8. 3)

$$|\mathrm{tg}\,(t_\lambda x) - \mathrm{tg}\,(t_\lambda x_q(\lambda))| < \frac{\varepsilon}{1+t_\lambda^2}$$

and thus

$$E_q(\varepsilon, \lambda) \subset E_q'(\varepsilon, \lambda)$$

where $E_q'(\varepsilon, \lambda)$ is defined by

(8. 7) $$|x - x_q(\lambda)| \leqq \frac{\varepsilon}{1+t_\lambda^2} \cdot \frac{1}{t_\lambda}$$
$$q\pi \leqq t_\lambda x < (q+1)\pi.$$

Hence

$$|E(\varepsilon, \lambda)| = |\bigcup_{q\in Q} E_q(\varepsilon, \lambda)| \leqq |\bigcup_{q\in Q} E_q'(\varepsilon, \lambda)| \leqq (1+|t_\lambda|)\log_2 T \frac{\varepsilon}{(1+t_\lambda^2)|t_\lambda|}$$

using (8. 5) and the same holds obviously for $t_\lambda < 0$. Hence for the measure $|H|$ of the subset H of I, for which

(8. 8) $$\left|\mathrm{tg}\,(t_\lambda x) - \frac{t_\lambda}{\sigma_\lambda}\right| > \frac{\varepsilon}{1+t_\lambda^2}$$

holds for *all* of our λ's, we obtain the lower bound

(8. 9) $$|I| - \sum_\lambda \frac{\varepsilon}{1+t_\lambda^2} \frac{1+|t_\lambda|}{|t_\lambda|} \log_2 T \geqq \log_2 T \left\{\frac{1}{20} - \varepsilon \sum_\lambda \frac{1+|t_\lambda|}{|t_\lambda|} \frac{1}{1+t_\lambda^2}\right\}.$$

Writing the last sum in the form

$$\sum_{|t_\lambda|\leqq 1} + \sum_{|t_\lambda|>1}$$

the first sum is, owing to the definition of $A(k)$ and the inequality (χ fixed!)

(8. 10) $$\sum_{\substack{\varrho(\chi) \\ r \leqq t_\varrho < r+1}} 1 < c_{13} \log k(1+|r|),$$

for all real r's at most

$$c_{14} \frac{k \log k}{A(k)},$$

the second

$$c_{15} k \sum_{r=1}^{\infty} \frac{\log k(1+r)}{r^2} < c_{16} k \log k.$$

Hence (8. 9) gives

(8. 11) $$|H| > \log_2 T \left\{\frac{1}{20} - c_{17} \varepsilon \frac{k \log k}{A(k)}\right\}.$$

Thus choosing

$$\varepsilon = \frac{1}{40c_{17}} \frac{A(k)}{k \log k} \tag{8.12}$$

we get

$$|H| > \frac{1}{40} \log_2 T > 0$$

i. e. there exists an y_1 in I such that

$$\left| \operatorname{tg}(t_\lambda y_1) - \frac{t_\lambda}{\sigma_\lambda} \right| > \frac{1}{40c_{17}} \frac{A(k)}{k \log k} \cdot \frac{1}{1+t_\lambda^2} \tag{8.13}$$

for *all* of our λ's.

9. Now we assert that this y_1 has the property required by the lemma. For the proof we remark first as well-known the inequality

$$\sigma_\lambda > \frac{c_{18}}{\log k(1+|t_\lambda|)} \tag{9.1}$$

(using also Haselgrove's condition and the functional equation). We distinguish two cases.

Case a). λ is such that

$$|\cos(t_\lambda y_1)| \leqq \frac{\sigma_\lambda}{4(1+|t_\lambda|)} \left(< \frac{1}{4} \right). \tag{9.2}$$

Then we have

$$|\sin(t_\lambda y_1)| > \frac{1}{2}$$

and hence from this, (9. 2) and (9.1) we get

$$\left| \operatorname{Im} \frac{e^{it_\lambda y_1}}{\lambda} \right| = \left| \frac{\sigma_\lambda \sin(t_\lambda y_1) - t_\lambda \cos(t_\lambda y_1)}{\sigma_\lambda^2 + t_\lambda^2} \right| > \tag{9.3}$$

$$> \frac{\frac{\sigma_\lambda}{2} - \frac{\sigma_\lambda}{4}}{1+t_\lambda^2} > \frac{c_{18}}{4} \frac{1}{(1+t_\lambda^2) \log k(1+|t_\lambda|)}.$$

Case b). λ is such that

$$|\cos(t_\lambda y_1)| > \frac{\sigma_\lambda}{4(1+|t_\lambda|)}. \tag{9.4}$$

Then we have from this, (8. 13) and (9. 1)

$$\left| \operatorname{Im} \frac{e^{it_\lambda y_1}}{\lambda} \right| = \sigma_\lambda |\cos(t_\lambda y_1)| \frac{\left| \operatorname{tg}(t_\lambda y_1) - \frac{t_\lambda}{\sigma_\lambda} \right|}{\sigma_\lambda^2 + t_\lambda^2} \geqq$$

$$\geqq \frac{1}{160c_{17}} \frac{A(k)\sigma_\lambda^2}{(1+|t_\lambda|)^5 k \log k} > c_{19} \frac{A(k)}{(1+|t_\lambda|)^5 k \log^3 k(1+|t_\lambda|)}.$$

This and (9. 3) gives for all of our λ's

$$\left|\operatorname{Im}\frac{e^{it_\lambda y_1}}{\lambda}\right| > c_{20}\frac{A(k)}{(1+|t_\lambda|)^5 k\log^3 k(1+|t_\lambda|)}. \tag{9.5}$$

But then, since roughly

$$\left|\operatorname{Re}\frac{e^{it_\lambda y_1}}{\lambda}\right| = \frac{|\sigma_\lambda\cos(t_\lambda y_1)+t_\lambda\sin(t_\lambda y_1)|}{\sigma_\lambda^2+t_\lambda^2} \leqq \frac{1+|t_\lambda|}{t_\lambda^2} \leqq 4\frac{1+|t_\lambda|}{A(k)^2},$$

we get from this and (9. 5) for all of our λ's

$$\left|\operatorname{tg}\left(\operatorname{arc}\frac{e^{it_\lambda y_1}}{\lambda}\right)\right| \geqq c_{21}\frac{A(k)^3}{(1+|t_\lambda|)^6 k\log^3 k(2+|t_\lambda|)}. \tag{9.6}$$

We now fix arc z with

$$-\pi < \operatorname{arc} z \leqq \pi.$$

If for a λ we have

$$\pi \geqq \left|\operatorname{arc}\frac{e^{it_\lambda y_1}}{\lambda}\right| \geqq \frac{\pi}{4},$$

we have nothing to prove. If

$$\frac{\pi}{4} > \left|\operatorname{arc}\frac{e^{it_\lambda y_1}}{\lambda}\right| > 0$$

then owing to the inequality

$$|\operatorname{tg}\varphi| \leqq \frac{4}{\pi}|\varphi| \qquad |\varphi| \leqq \frac{\pi}{4}$$

the Lemma II follows from (9. 6) at once.

10. We shall further need the

LEMMA III. *If $s=\sigma+it$, then there is a connected path V in the vertical strip $\frac{1}{5}\leqq\sigma\leqq\frac{2}{5}$ say, symmetrical to the real axis, consisting alternately of horizontal and vertical segments and increasing monotonically from $-\infty$ to $+\infty$ on which for all $L(s,\chi)$-functions mod k the inequality*

$$\left|\frac{L'}{L}(s,\chi)\right| < c_{22}k\log^3 k(2+|t|)$$

holds.

The (simple) proof is mutatis mutandis the same as that of the Appendix III in the book of one of us (see TURÁN [3]) and can be omitted. The symmetry was not important there.

11. Now we can turn to the proof of (3. 4)–(3. 5). Let for $\sigma>1$

$$F(s) \overset{\text{def}}{=} \frac{1}{\varphi(k)}\sum_\chi\left(1-\bar\chi(l)\right)\frac{L'}{L}(s,\chi) = \sum_{n\equiv l \bmod k}\frac{\Lambda(n)}{n^s} - \sum_{n\equiv 1 \bmod k}\frac{\Lambda(n)}{n^s}. \tag{11.1}$$

Then with y_1 of Lemma II and with an integer $\nu \geqq 2$, to be determined later, putting

$$J(\nu) \overset{\text{def}}{=} \frac{1}{2\pi i} \int\limits_{(2)} \left(\frac{e^{y_1 s}}{s} \right)^{\nu} F(s)\, ds, \tag{11.2}$$

we obtain, owing to the well-known integral-formula

$$\frac{1}{2\pi i} \int\limits_{(2)} \frac{x^s}{s^{\nu}}\, ds = \begin{cases} \dfrac{1}{(\nu-1)!} \log^{\nu-1} x & \text{for} \quad x \geqq 1 \\ 0 & \phantom{\text{for}} \quad 0 < x < 1 \end{cases}$$

the formula

$$J(\nu) = \frac{1}{(\nu-1)!} \Bigg\{ \sum_{\substack{n \leqq e^{\nu y_1} \\ n \equiv l \bmod k}} \Lambda(n) \log^{\nu-1} \frac{e^{\nu y_1}}{n} - \sum_{\substack{n \leqq e^{\nu y_1} \\ n \equiv 1 \bmod k}} \Lambda(n) \log^{\nu-1} \frac{e^{\nu y_1}}{n} \Bigg\}.$$

Or introducing

$$\sum_{\substack{n \leqq x \\ n \equiv l \bmod k}} \frac{\Lambda(n)}{\log n} - \sum_{\substack{n \leqq x \\ n \equiv 1 \bmod k}} \frac{\Lambda(n)}{\log n} \overset{\text{def}}{=} g(x, k, l) \tag{11.3}$$

we get

$$J(\nu) = \frac{1}{(\nu-1)!} \int\limits_{1}^{e^{\nu y_1}} \log x \cdot \log^{\nu-1} \frac{e^{\nu y_1}}{x}\, d_x g(x, k, l) = \tag{11.4}$$

$$= -\frac{1}{(\nu-1)!} \int\limits_{1}^{e^{\nu y_1}} g(x, k, l) \left\{ \log x \cdot \log^{\nu-1} \frac{e^{\nu y_1}}{x} \right\}' dx.$$

The function

$$\log x \cdot \log^{\nu-1} \frac{e^{\nu y_1}}{x}$$

increases for $1 \leqq x \leqq e^{y_1}$ and decreases for $x \geqq e^{y_1}$ and hence from (11.4)

$$J(\nu) = -\frac{1}{(\nu-1)!} \int\limits_{1}^{e^{y_1}} g(x, k, l) \left\{ \log x \cdot \log^{\nu-1} \frac{e^{\nu y_1}}{x} \right\}' dx + \tag{11.5}$$

$$+ \frac{1}{(\nu-1)!} \int\limits_{e^{y_1}}^{e^{\nu y_1}} g(x, k, l) \left| \left\{ \log x \cdot \log^{\nu-1} \frac{e^{\nu y_1}}{x} \right\}' \right| dx \overset{\text{def}}{=} J_1(\nu) + J_2(\nu).$$

For the (trivial) upper estimation of $|J_1(\nu)|$ we have from (7.2)

$$|J_1(\nu)| \leqq \max_{x \leqq \log^{\frac{1}{10}} T} |g(x,k,l)| \cdot \frac{1}{(\nu-1)!} \int_1^{e^{y_1}} \left| \left(\log x \cdot \log^{\nu-1} \frac{e^{\nu y_1}}{x} \right)' \right| dx =$$

$$= \max_{x \leqq \log^{\frac{1}{10}} T} |g(x,k,l)| \frac{1}{(\nu-1)!} \int_1^{e^{y_1}} \left(\log x \cdot \log^{\nu-1} \frac{e^{\nu y_1}}{x} \right)' dx =$$

$$= \frac{1}{(\nu-1)!} y_1((\nu-1)y_1)^{\nu-1} \max_{x \leqq \log^{\frac{1}{10}} T} |g(x,k,l)|.$$

Since trivially

$$\max_{x \leqq \log^{\frac{1}{10}} T} |g(x,k,l| < \sum_{\substack{p,m \\ p^m \leqq \log^{\frac{1}{10}} T}} 1 < \log T$$

we get from (7.2)

(11.6) $$|J_1(\nu)| \leqq \frac{y_1((\nu-1)y_1)^{\nu-1}}{(\nu-1)!} \log T.$$

As to $J_2(\nu)$ we have

$$J_2(\nu) \leqq \frac{y_1((\nu-1)y_1)^{\nu-1}}{(\nu-1)!} \max_{x \leqq e^{\nu y_1}} g(x,k,l)$$

and hence

(11.7) $$\max_{x \leqq e^{\nu y_1}} g(x,k,l) > \frac{(\nu-1)!}{y_1((\nu-1)y_1)^{\nu-1}} J(\nu) - \log T$$

and analogously

(11.8) $$\min_{x \leqq e^{\nu y_1}} g(x,k,l) < \frac{(\nu-1)!}{y_1((\nu-1)y_1)^{\nu-1}} J(\nu) + \log T.$$

We now restrict ν by

(11.9) $$\frac{\log T}{y_1} - \log^{0,9} T \leqq \nu \leqq \frac{\log T}{y_1}.$$

If we want to prove (3.2)—(3.3) instead of (3.4)—(3.5), we have in (11.4) $\left(\log^{\nu-1} \frac{e^{\nu y_1}}{x}\right)'$ instead of $\left(\log x \log^{\nu-1} \frac{e^{\nu y_1}}{x}\right)'$ i. e. the term corresponding to $J_1(\nu)$ does not appear at all.

12. Using Lemma III we obtain that on V the inequality

(12.1) $$|F(s)| < c_{22} k \log^3 (2 + |t|)$$

holds. Cauchy's integral-theorem gives at once

$$(12.2)\qquad \left| J(\nu) - \frac{1}{\varphi(k)} \sum_{\chi} (1-\bar{\chi}(l)) \sum{}' \left(\frac{e^{\varrho y_1}}{\varrho}\right)^{\nu} \right| = \left| \frac{1}{2\pi i} \int_{(V)} \left(\frac{e^{y_1 s}}{s}\right)^{\nu} F(s)\, ds \right| < c_{23} k \log^3 k \left(5e^{\frac{2}{5} y_1}\right)^{\nu},$$

where the dash means that the summation is extended at fixed χ only to those $\varrho = \varrho(\chi)$ zeros of $L(s, \chi)$ which are right to V. Owing to (11. 9) and (7. 2) we get from (12. 2)

$$(12.3)\qquad \left| J(\nu) - \frac{1}{\varphi(k)} \sum_{\chi} (1-\bar{\chi}(l)) \sum{}' \left(\frac{e^{\varrho y_1}}{\varrho}\right)^{\nu} \right| < c_{24} T^{\frac{2}{5}} k \log^3 k e_1\left(40 \frac{\log T}{\log_2 T}\right).$$

Further owing to (8. 10) and (11. 9) we get for the absolute value of the sum containing the zeros with

$$|\mathrm{Im}\, \varrho| \geqq \log^{\frac{1}{10}} T$$

the upper bound

$$c_{25} \log^2 kT \cdot T \cdot e_1\left(-\frac{\nu}{10} \log_2 T\right),$$

and since from (11. 9), (7. 2) and (3. 1) we have

$$\frac{\nu}{10} \log_2 T > \frac{\log_2 T}{10} \left\{\frac{\log T}{y_1} - \log^{0,9} T\right\} > \log T - \log^{0,95} T$$

we get for this sum using (3. 1) the upper bound

$$c_{25} \log^2 kT \cdot e_1(\log^{0,95} T) < e_1\left(\frac{\log T}{\log_2 T}\right).$$

This and (12. 3) give owing to (3. 1)

$$\left| J(\nu) - \frac{1}{\varphi(k)} \sum_{\chi} (1-\bar{\chi}(l)) \sum_{|t_\varrho| \leqq \log^{\frac{1}{10}} T}{}' \left(\frac{e^{\varrho y_1}}{\varrho}\right)^{\nu} \right| < T^{\frac{2}{5}} e_1\left(\frac{\log T \log_3 T}{\log_2 T}\right).$$

We write it in the form (owing to the symmetry of V)

$$\left| J(\nu) - \left(\frac{e^{\beta_0 y_1}}{|\varrho_0|}\right)^{\nu} \frac{1}{\varphi(k)} \mathrm{Re} \sum_{\chi} (1-\bar{\chi}(l)) \cdot \sum_{|t_\varrho| \leqq \log^{\frac{1}{10}} T}{}' \left(\frac{e^{(\varrho-\beta_0) y_1} |\varrho_0|}{\varrho}\right)^{\nu} \right| < T^{\frac{2}{5}} e_1\left(\frac{\log T \log_3 T}{\log_2 T}\right).$$

Hence (11. 7) and (11. 8) give

$$(12.4)\qquad \max_{x \leqq T} g(x, k, l) \geqq \frac{(\nu-1)!}{y_1((\nu-1)y_1)^{\nu-1}} \left\{\left(\frac{e^{\beta_0 y_1}}{|\varrho_0|}\right)^{\nu} \frac{1}{\varphi(k)} \cdot \right.$$

$$\left. \cdot \mathrm{Re} \sum_{\chi(l) \neq 1} (1-\bar{\chi}(l)) \sum_{|t_\varrho| \leqq \log^{\frac{1}{10}} T}{}' \left(\frac{e^{(\varrho-\beta_0) y_1} |\varrho_0|}{\varrho}\right)^{\nu} - T^{0,45} \right\} - \log T$$

and analogously

$$\min_{x \leqq T} g(x,k,l) \leqq \frac{(\nu-1)!}{y_1((\nu-1)y_1)^{\nu-1}} \cdot \left\{ \left(\frac{e^{\beta_0 y_1}}{|\varrho_0|}\right)^{\nu} \frac{1}{\varphi(k)} \operatorname{Re} \sum_{\chi(l) \neq 1} \chi (1-\bar{\chi}(l)) \cdot \right.$$

(12.5)

$$\left. \cdot \sum_{|t_\varrho| \leqq \log^{\frac{1}{10}} T}' \left(\frac{e^{(\varrho-\beta_0)y_1}|\varrho_0|}{\varrho}\right)^{\nu} + T^{0,45} \right\} + \log T.$$

13. The integer ν was restricted so far only by (11.9). In order to prove (3.5) we shall use the inequality (6.5) in Lemma I. For this sake we observe first that the number of terms in the inner sum of (12.4) is owing to (8.10) at most

$$3c_{13}k \log(k \log^{\frac{1}{10}} T) \cdot \log^{\frac{1}{10}} T$$

which in turn is owing to (3.1)

$$(13.1) \qquad < c_{26} \log^{\frac{1}{10}} T (\log_2 T)^2 < \log^{0,11} T \overset{\text{def}}{=} N.$$

Let further

$$(13.2) \qquad m = \frac{\log T}{y_1} - \log^{0,9} T.$$

The role of the z_j's in Lemma I is played by the numbers

$$(13.3) \qquad \frac{e^{(\varrho-\beta_0)y_1}|\varrho_0|}{\varrho}$$

and that of the b_j's by numbers $(1-\bar{\chi}(l))$. But then we have for $\mu = 1, 2, \ldots$

$$(13.4) \qquad \operatorname{Re}(b_1 + b_2 + \ldots + b_\mu) \geqq 1 - \cos\frac{2\pi}{\varphi(k)} > \frac{8}{k^2}$$

i. e. (6.3) is satisfied. Further Lemma II, (3.1) and (13.3) give that

$$(\pi \geqq) |\operatorname{arc} z_j| = \left|\operatorname{arc} \frac{e^{it_\varrho y_1}}{\varrho}\right| \geqq c_{12} \frac{A(k)^3}{k(1+|t_\varrho|)^6 \log^3 k(2+|t_\varrho|)} \geqq$$

(13.5)

$$\geqq \frac{c_{12}}{2} \frac{A(k)^3}{k \log^{\frac{3}{5}} T \cdot \log^3 k(2+\log^{\frac{1}{10}} T)} >$$

$$> c_{27} \log^{-\frac{3}{5}} T \cdot (\log_2 T)^{-5} > \log^{-\frac{3}{4}} T \overset{\text{def}}{=} \varkappa.$$

Since with the choices (13.1)—(13.2)—(13.5) the interval $\left[m, m+N\left(3+\frac{\pi}{\varkappa}\right)\right]$ is contained in the interval (11.9), we may choose as ν the value ν_1 in (6.5). Further we have owing to (3.1)

$$|\gamma_0| < \frac{1}{10} \log_2 T < \log^{\frac{1}{10}} T$$

and hence $\varrho=\varrho_0$ occurs among our ϱ's; thus

$$\max_j |z_j| \geqq 1. \tag{13.6}$$

Then (6. 5) of Lemma I gives from (12. 4)

$$\max_{x\leqq T} g(x,k,l) \geqq \frac{(\nu_1-1)!}{y_1((\nu_1-1)y_1)^{\nu_1-1}}\left\{\frac{8}{k^2}\cdot\frac{1}{3}\log^{-0,11}T\cdot\right.$$
$$\left.\cdot\left(\frac{e^{\beta_0 y_1}}{|\varrho_0|}\right)^{\nu_1}\left(\frac{\log^{0,11}T}{24e\log T}\right)^{2\log^{0,11}T} - T^{0,45}\right\} - \log T. \tag{13.7}$$

From (11. 9) and (7. 2) we have

$$(e^{\beta_0 y_1})^{\nu_1} = (e^{y_1\nu_1})^{\beta_0} \geqq T^{\beta_0} e_1(-y_1\log^{0,9}T) \geqq T^{\beta_0} e_1\left(-\frac{1}{10}\log^{0,9}T\cdot\log_2 T\right), \tag{13.8}$$

further, using also (3. 1),

$$|\varrho_0|^{\nu_1} \leqq \left(\frac{1}{10}\log_2 T\right)^{20\frac{\log T}{\log_2 T}} < e_1\left(20\frac{\log T\log_3 T}{\log_2 T}\right) \tag{13.9}$$

and finally

$$\frac{(\nu_1-1)!}{y_1((\nu_1-1)y_1)^{\nu_1-1}} \geqq \left(\frac{10}{e\log_2 T}\right)^{20\frac{\log T}{\log_2 T}} > e_1\left(-20\frac{\log T\log_3 T}{\log_2 T}\right).$$

From this, (13. 7), (13. 8) and (13. 9) gives

$$\max_{x\leqq T} g(x,k,l) \geqq 2T^{\beta_0} e_1\left(-41\frac{\log T\log_3 T}{\log_2 T}\right). \tag{13.10}$$

Since for $x \leqq T^{1/3}$ evidently

$$|g(x,k,l)| < T^{\frac{1}{3}}$$

holds, (3. 4) follows from (13. 10) at once. The proof of (3. 5) runs analogously only we have to apply (6. 6) to (12. 5).

14. Next we turn to the proof of Theorem 5. 2 in which we also use ideas of Littlewood, Ingham and Skewes (see LITTLEWOOD [1], INGHAM [1], SKEWES [1]). We shall distinguish two cases (l fixed!).

Case I. There is an $L(s,\chi')$ mod k with $\chi'(l)\neq 1$ which has a zero $\varrho'=\sigma'+it'$ with

$$\sigma' \geqq \frac{1}{2} + 42\frac{\log_3 T}{\log_2 T}, \quad |t'| \leqq \frac{1}{40}\log_2 T. \tag{14.1}$$

In this case (3. 4) – (3. 5) is clearly applicable i. e. there are τ_1 and τ_2 with

$$T^{\frac{1}{3}} \leqq \tau_1, \quad \tau_2 \leqq T \tag{14.2}$$

so that

$$\Pi(\tau_1, k, 1) - \Pi(\tau_1, k, l) > T^{\frac{1}{2}} e_1\left(\frac{\log T \log_3 T}{\log_2 T}\right) \tag{14.3}$$

and the same lower bound for $\Pi(\tau_2, k, l) - \Pi(\tau_2, k, 1)$. Since evidently for $x \geqq 2$

$$|\{\Pi(x, k, 1) - \Pi(x, k, l)\} - \{\pi(x, k, 1) - \pi(x, k, l)\}| < 2\sqrt{x},$$

choosing c_{11} in (5.3) sufficiently large we have

$$\pi(\tau_1, k, 1) - \pi(\tau_1, k, l) > \frac{1}{2}\sqrt{T} e_1\left(\frac{\log T \log_3 T}{\log_2 T}\right) \tag{14.4}$$

and the same lower bound for $\pi(\tau_2, k, l) - \pi(\tau_2, k, 1)$, which settles amply the Theorem 5.2 for Case I.

Case II. No $L(s, \chi)$ mod k with $\chi(l) \neq 1$ vanishes in the domain

$$\sigma \geqq \frac{1}{2} + 42\frac{\log_3 T}{\log_2 T} \tag{14.5}$$

$$|t| \leqq \frac{1}{40}\log_2 T.$$

Let

$$\tau \overset{\text{def}}{=} \frac{1}{20\varphi(k)} \frac{\log_4 T}{\log_5 T \cdot \log_6 T} \; (< \log_4 T) \tag{14.6}$$

and with a sufficiently large c_{28}

$$q \overset{\text{def}}{=} 1 + [c_{28} \log_5 T]. \tag{14.7}$$

15. Let χ be an arbitrary character mod k and χ the corresponding primitive character mod k^* with $k^*|k$. Then we have[6] for $x \geqq 2$

$$\left| \sum_{n \leqq x} \Lambda(n)\chi^*(n) - \left\{E_0 x - \sum_{|t_\varrho| \leqq \frac{1}{40}\log_2 T} {}_{\varrho} \frac{x^\varrho}{\varrho} - v_0(\chi^*)\log x - d_0(\chi^*)\right\}\right| < \tag{15.1}$$
$$< c_{29}\left\{\frac{x}{\log_2 T}(\log^2 x + \log^2(k \log_2 T)) + \log x\right\}$$

where

$$E_0 = \begin{cases} 1 & \text{for} \quad \chi = \chi_0 \\ 0 & \text{elsewhere,} \end{cases}$$

$\varrho = \varrho(\chi)$ runs over the non-trivial zeros of $L(s, \chi)$ (identical with those of $L(s, \chi^*)$) and

$$\frac{L'}{L}(s, \chi^*) = \frac{v_0(\chi^*)}{s} + d_0(\chi^*) + \dots. \tag{15.2}$$

[6] See PRACHAR [1], pp. 228–229.

We have $v_0(\chi^*)=0$ or 1 and since[7]

$$\left| d_0(\chi^*) + \sum_{|t_\varrho| \leqq 1} \frac{1}{\varrho} \right| < c_{30} \log k,$$

taking in account also the functional-equation and (14. 5) we get

$$|d_0(\chi^*)| < c_{31} \log k. \tag{15. 3}$$

Since further we have[8]

$$\left| \sum_{n \leqq x} \Lambda(n)\chi(n) - \sum_{n \leqq x} \Lambda(n)\chi^*(n) \right| < c_{32} \log k \log x,$$

we get for all characters mod k

$$\left| \sum_{n \leqq x} \Lambda(n)\chi(n) - E_0 x + \sum_{|t_\varrho| \leqq \frac{1}{40} \log_2 T} \frac{x^\varrho}{\varrho} \right| <$$

$$< c_{33} \left\{ \log k \log x + \frac{x}{\log_2 T} (\log^2 x + \log^2 (k \log_2 T)) \right\}.$$

Dividing by $\sqrt{x}$ and replacing x by e^r (i. e. $r \geqq \log 2$) and putting

$$g(r) \overset{\text{def}}{=} e^{-\frac{r}{2}} \{ \sum_{\substack{n \leqq e^r \\ n \equiv 1 \bmod k}} \Lambda(n) - \sum_{\substack{n \leqq e^r \\ n \equiv l \bmod k}} \Lambda(n) \} \tag{15. 4}$$

we obtain the inequality

$$\left| g(r) + \frac{1}{\varphi(k)} \sum_{\chi} (1 - \bar{\chi}(l)) \sum_{|t_\varrho| \leqq \frac{1}{40} \log_2 T}^{\varrho(\chi)} \frac{e^{\left(\varrho - \frac{1}{2}\right) r}}{\varrho} \right| <$$

(15. 5)

$$< c_{34} \left\{ r e^{-\frac{r}{2}} \log k + \frac{e^{\frac{r}{2}}}{\log_2 T} (r^2 + \log^2 (k \log_2 T)) \right\}.$$

Since owing to the functional-equation and (14. 5) we have

$$\frac{1}{\varphi(k)} \left| \sum_{\chi} (1 - \bar{\chi}(l)) \sum_{|t_\varrho| \leqq 1}^{\varrho} \frac{1}{\varrho} \right| < c_{35} \log k,$$

(15. 5) gives

$$\left| g(r) + \frac{1}{\varphi(k)} \sum_{\chi} (1 - \bar{\chi}(l)) \sum_{1 < |t_\varrho| \leqq \frac{1}{40} \log_2 T}^{\varrho(\chi)} \frac{e^{\left(\varrho - \frac{1}{2}\right) r}}{\varrho} \right| <$$

$$< c_{36} \left\{ \log k + \frac{e^{\frac{r}{2}}}{\log_2 T} (r^2 + \log^2 (k \log_2 T)) \right\}.$$

[7] See PRACHAR [1], p. 233.
[8] See PRACHAR [1], p. 234.

Making the restriction

$$0 < r < \log_3 T \tag{15.6}$$

this and (5.3) give

$$\left| g(r) + \frac{1}{\varphi(k)} \sum_{\chi} (1 - \bar{\chi}(l)) \sum_{\substack{\varrho(\chi) \\ 1 < |t_\varrho| \leq \frac{1}{40}\log_2 T}} \frac{e^{\left(\varrho - \frac{1}{2}\right) r}}{\varrho} \right| < c_{37} \log k. \tag{15.7}$$

16. Next we consider the sum

$$S_\chi(r, T) \stackrel{\text{def}}{=} \sum_{\substack{\varrho(\chi) \\ 1 < |t_\varrho| \leq \frac{1}{40}\log_2 T}} \frac{e^{\left(\varrho - \frac{1}{2}\right) r}}{\varrho}. \tag{16.1}$$

Then we have

$$\begin{aligned} \left| S_\chi(r, T) - \sum_{\substack{\varrho(\chi) \\ 1 < |t_\varrho| \leq \frac{1}{40}\log_2 T}} \frac{e^{it_\varrho r}}{it_\varrho} \right| &\leq \sum_{1 < |t_\varrho| \leq \frac{1}{40}\log_2 T} \left| \frac{e^{\left(\sigma_\varrho - \frac{1}{2}\right) r}}{\sigma_\varrho + it_\varrho} - \frac{1}{it_\varrho} \right| \leq \\ &\leq \sum_{1 < |t_\varrho| \leq \frac{1}{40}\log_2 T} \frac{\left| e^{\left(\sigma_\varrho - \frac{1}{2}\right) r} - 1 \right|}{|\varrho|} + \sum_{1 < |t_\varrho| \leq \frac{1}{40}\log_2 T} \frac{1}{|t_\varrho|\,|\varrho|}. \end{aligned} \tag{16.2}$$

Since from (15.6) and (14.5) we have (if c_{11} in (5.3) is large enough)

$$\left| e^{\left(\sigma_\varrho - \frac{1}{2}\right) r} - 1 \right| < 84 \frac{(\log_3 T)^2}{\log_2 T},$$

i.e. the right-side of (16.2), using also (5.3),

$$\leq 1 + c_{38} \sum_{1 < |t_\varrho| \leq \frac{1}{40}\log_2 T} \frac{1}{|\varrho|\,|t_\varrho|},$$

this is again

$$< c_{39} \log k$$

and hence from (15.7) we get

$$\left| g(r) + \frac{1}{\varphi(k)} \sum_{\chi} (1 - \bar{\chi}(l)) \sum_{1 < |t_\varrho| \leq \frac{1}{40}\log_2 T} \frac{e^{it_\varrho r}}{it_\varrho} \right| < c_{40} \log k. \tag{16.3}$$

With q and τ in 14 we restrict a new parameter ω by

$$q^{\frac{\varphi(k)}{6\pi}\tau \log \tau} \leq \omega \leq q^{\frac{3}{\pi}\varphi(k)\tau \log \tau}. \tag{16.4}$$

Defining $G_\omega(r)$ by

$$G_\omega(r) = \tau\left(\frac{\sin\frac{\tau(r-\omega)}{2}}{\frac{\tau(r-\omega)}{2}}\right)^2 \tag{16.5}$$

we multiply in (16.3) by $\frac{1}{2\pi}G_\omega(r)$ and integrate with respect to r between $\frac{3}{4}\omega$ and $\frac{5}{4}\omega$; the requirement (15.6) is not violated since from (16.4), (14.6) and (14.7) we have

$$\frac{5}{4}\omega \leqq \frac{5}{4}q^{\frac{3}{\pi}\varphi(k)\tau\log\tau} \leqq \frac{5}{4}e_1\left\{\frac{4}{\pi}(\log_6 T + \log c_{28})\cdot\frac{1}{20}\frac{\log_4 T}{\log_5 T\log_6 T}\cdot\log_5 T\right\} < \log_3 T$$

again if c_{11} in (5.3) is sufficiently large. Using also the relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tau\cdot\left(\frac{\sin\frac{\tau(r-\omega)}{2}}{\frac{\tau(r-\omega)}{2}}\right)^2 dr = 1,$$

we get from (16.3) the inequality

$$\left|\frac{1}{2\pi}\int_{\frac{3}{4}\omega}^{\frac{5}{4}\omega} g(r)G_\omega(r)\,dr + \frac{1}{\varphi(k)}\sum_\chi(1-\bar\chi(l))\sum_{1<|t_\varrho|\leqq\frac{1}{40}\log_2 T}\frac{1}{2\pi i t_\varrho}\cdot\right. \tag{16.6}$$
$$\left.\cdot\int_{\frac{3}{4}\omega}^{\frac{5}{4}\omega} e^{it_\varrho r}G_\omega(r)\,dr\right| < c_{40}\log k.$$

17. We complete the inner integrals to $\int_{-\infty}^{\infty}$. For this sake we consider first the sum

$$Z_1 \stackrel{\text{def}}{=} \left|\sum_{1<|t_\varrho|\leqq\tau}{}^{\varrho(\chi)}\frac{1}{2\pi i t_\varrho}\int_{\frac{5}{4}\omega}^{\infty} e^{it_\varrho r}G_\omega(r)\,dr\right|.$$

This is obviously (roughly)

$$< \frac{c_{41}}{\tau\omega}\sum_{1<|t_\varrho|\leqq\tau}\frac{1}{|t_\varrho|} < c_{42}\log k \tag{17.1}$$

and the same holds for the sum

$$(17.2)\qquad \left| \sum_{1<|t_\varrho|\leqq\tau} \frac{1}{2\pi i t_\varrho} \int_{-\infty}^{\frac{3}{4}\omega} e^{it_\varrho r} G_\omega(r)\,dr \right|.$$

Let further be

$$Z_2 \overset{\text{def}}{=} \left| \sum_{\tau<|t_\varrho|\leqq\frac{1}{40}\log_2 T}^{\varrho(x)} \frac{1}{2\pi i t_\varrho} \int_{\frac{5}{4}\omega}^{\infty} e^{it_\varrho r} G_\omega(r)\,dr \right| =$$

$$(17.3)\qquad = \left| \sum_{\tau\leqq|t_\varrho|\leqq\frac{1}{40}\log_2 T}^{\varrho(x)} \frac{e^{it_\varrho\omega}}{2\pi i t_\varrho} \int_{\frac{\tau\omega}{4}}^{\infty} e^{i\frac{t_\varrho}{\tau}y} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}} \right)^2 dy \right|.$$

Then

$$(17.4)\qquad Z_2 \leqq \frac{1}{2\pi} \sum_{\tau\leqq|t_\varrho|\leqq\frac{1}{40}\log_2 T} \frac{1}{|t_\varrho|} \left| \int_{\frac{\tau\omega}{4}}^{\infty} e^{i\frac{t_\varrho}{\tau}y} \cdot \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}} \right)^2 dy \right|.$$

Partial integration gives that the last integral is absolutely

$$< \frac{\tau}{|t_\varrho|} \frac{64}{\tau^2\omega^2} + \frac{\tau}{|t_\varrho|} \cdot \frac{c_{43}}{\tau\omega} < \frac{c_{44}}{|t_\varrho|\omega}$$

and hence from (17. 4), (8. 10), (16. 4) and (14. 6)

$$(17.5)\qquad Z_2 < \frac{c_{45}}{\omega} \sum_{|t_\varrho|\geqq\tau} \frac{1}{t_\varrho^2} < \frac{c_{46}}{\omega} \left(\frac{\log k}{\tau} + \frac{\log \tau}{\tau} \right) < c_{47}$$

(since from (5. 3) we have $\tau \geqq 3$) and the same for the sum

$$(17.6)\qquad \left| \sum_{\tau\leqq|t_\varrho|\leqq\frac{1}{40}\log_2 T} \frac{1}{2\pi i t_\varrho} \int_{-\infty}^{\frac{3\omega}{4}} \right|.$$

Taking in account that — as well-known —

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it_\varrho r} G_\omega(r)\,dr = \frac{e^{it_\varrho\omega}}{2\pi} \int_{-\infty}^{\infty} e^{i\frac{t_\varrho}{\tau}y} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}} \right)^2 dy =$$

$$= \begin{cases} \left(1 - \frac{|t_\varrho|}{\tau}\right) e^{it_\varrho\omega} & \text{if} \quad |t_\varrho| \leqq \tau \\ 0 & \phantom{\text{if}} \quad |t_\varrho| \geqq \tau, \end{cases}$$

we get from (16. 6), (17. 1), (17. 2), (17. 5) and (17. 6) the important inequality

$$(17.7)\quad \left| \frac{1}{2\pi} \int_{\frac{3}{4}\omega}^{\frac{5}{4}\omega} g(r) G_\omega(r)\, dr + \frac{1}{\varphi(k)} \sum_\chi (1-\bar\chi(l)) \cdot \sum_{\substack{\varrho(\chi) \\ 1<|t_\varrho|\le\tau}} \left(1-\frac{|t_\varrho|}{\tau}\right) \frac{e^{it_\varrho\omega}}{it_\varrho} \right| < c_{48} \log k.$$

18. Next — adapting properly the idea of LITTLEWOOD — we consider the sum

$$(18.1)\quad Z_3 = \frac{1}{\varphi(k)} \sum_\chi (1-\bar\chi(l)) \sum_{\substack{\varrho(\chi) \\ 1<|t_\varrho|\le\tau}} \left(1-\frac{|t_\varrho|}{\tau}\right) \cdot \frac{e^{it_\varrho \frac{\pi}{2\tau}}}{it_\varrho}.$$

This is obviously real and hence

$$(18.2)\quad Z_3 = \frac{1}{\varphi(k)} \sum_\chi \left\{ (1-\mathrm{Re}\,\bar\chi(l)) \sum_{\substack{\varrho(\chi) \\ 1<|t_\varrho|\le\tau}} \left(1-\frac{|t_\varrho|}{\tau}\right) \cdot \frac{\sin \frac{\pi t_\varrho}{2\tau}}{t_\varrho} - \right.$$

$$\left. -\mathrm{Im}\,\bar\chi(l) \sum_{\substack{\varrho(\chi) \\ 1<|t_\varrho|\le\tau}} \left(1-\frac{|t_\varrho|}{\tau}\right) \frac{\cos \frac{\pi t_\varrho}{2\tau}}{t_\varrho} \right\} =$$

$$= \frac{1}{\varphi(k)} \sum_\chi \left\{ (1-\mathrm{Re}\,\bar\chi(l)) \sum_{\substack{\varrho(\chi) \\ 1<t_\varrho\le\tau}} \left(1-\frac{t_\varrho}{\tau}\right) \frac{\sin \frac{\pi t_\varrho}{2\tau}}{t_\varrho} - \right.$$

$$\left. -\mathrm{Im}\,\bar\chi(l) \sum_{\substack{\varrho(\chi) \\ 1<t_\varrho\le\tau}} \left(1-\frac{t_\varrho}{\tau}\right) \frac{\cos \frac{\pi t_\varrho}{2\tau}}{t_\varrho} \right\} + \frac{1}{\varphi(k)} \sum_\chi \left\{ (1-\mathrm{Re}\,\chi(l)) \cdot \right.$$

$$\left. \cdot \sum_{\substack{\varrho(\bar\chi) \\ 1<t_\varrho\le\tau}} \left(1-\frac{t_\varrho}{\tau}\right) \frac{\sin \frac{\pi t_\varrho}{2\tau}}{t_\varrho} - \mathrm{Im}\,\chi(l) \sum_{\substack{\varrho(\bar\chi) \\ 1\le t_\varrho\le\tau}} \left(1-\frac{t_\varrho}{\tau}\right) \frac{\cos \frac{\pi t_\varrho}{2\tau}}{t_\varrho} \right\} =$$

$$= \frac{2}{\varphi(k)} \sum_\chi \left\{ (1-\mathrm{Re}\,\bar\chi(l)) \sum_{\substack{\varrho(\chi) \\ 1<t_\varrho\le\tau}} \left(1-\frac{t_\varrho}{\tau}\right) \frac{\sin \frac{\pi t_\varrho}{2\tau}}{t_\varrho} - \right.$$

$$\left. -\mathrm{Im}\,\bar\chi(l) \sum_{\substack{\varrho(\chi) \\ 1<t_\varrho\le\tau}} \left(1-\frac{t_\varrho}{\tau}\right) \frac{\cos \frac{\pi t_\varrho}{2\tau}}{t_\varrho} \right\} \overset{\mathrm{def}}{=} Z_3' + Z_3''$$

where

$$(18.3)\qquad Z_3' = \frac{2}{\varphi(k)} \sum_{\chi} (1 - \operatorname{Re}\bar{\chi}(l)) \sum_{\substack{\varrho(\chi)\\ 1<t_\varrho \leqq \tau}} \left(1 - \frac{t_\varrho}{\tau}\right) \frac{\sin \frac{\pi t_\varrho}{2\tau}}{t_\varrho}$$

$$(18.4)\qquad Z_3'' = \frac{2}{\varphi(k)} \sum_{\chi} \operatorname{Im}\bar{\chi}(l) \sum_{\substack{\varrho(\chi)\\ 1<t_\varrho \leqq \tau}} \left(1 - \frac{t_\varrho}{\tau}\right) \frac{\cos \frac{\pi t_\varrho}{2\tau}}{t_\varrho}.$$

As to the last sum we remark that the contribution of the real characters is obviously 0; we are going to investigate the contribution $U(\chi)$ of a single complex χ taking in account that for the $N(y, \chi)$-number of $\varrho(\chi)$'s with

$$0<\sigma<1, \quad 0<t_\varrho \leqq y, \quad y \geqq 1$$

the formula

$$(18.5)\qquad \left| N(y,\chi) - \frac{y}{2\pi}\log\frac{k^* y}{2\pi e}\right| < c_{49} \log ky$$

holds (k^* at the beginning of **15**). We have

$$U(\chi) \overset{\text{def}}{=} \sum_{\substack{\varrho(\chi)\\ 1<t_\varrho \leqq \tau}} \left(1 - \frac{t_\varrho}{\tau}\right) \frac{\cos \frac{\pi t_\varrho}{2\tau}}{t_\varrho} = \int_1^\tau \left(1 - \frac{y}{\tau}\right) \frac{\cos\frac{\pi y}{2\tau}}{y} dN(y,\chi) =$$

$$= - \int_1^\tau N(y,\chi) \left\{ \frac{d}{dy} \left(1 - \frac{y}{\tau}\right) \frac{\cos\frac{\pi y}{2\tau}}{y} \right\} dy$$

and hence, since k^* belonging to χ and to $\bar{\chi}$ coincide, and $\operatorname{Im}\bar{\chi}(l) = -\operatorname{Im}\chi(l)$, we have also

$$(\operatorname{Im}\bar{\chi}(l))\cdot U(\chi) + (\operatorname{Im}\chi(l))U(\bar{\chi}) = - \int_1^\tau \left\{ \operatorname{Im}\bar{\chi}(l)\left(N(y,\chi) - \frac{y}{2\pi}\log\frac{k^*y}{2\pi e}\right) + \right.$$

$$\left. + \operatorname{Im}\chi(l)\left(N(y,\bar{\chi}) - \frac{y}{2\pi}\log\frac{k^*y}{2\pi e}\right)\right\} \frac{d}{dy} \left\{ \left(1 - \frac{y}{\tau}\right) \frac{\cos\frac{\pi y}{2\tau}}{y} \right\} dy.$$

Thus

$$(18.6)\qquad |U(\chi)\operatorname{Im}\bar{\chi}(l) + U(\bar{\chi})\operatorname{Im}\chi(l)| < c_{50} \int_1^\tau \log ky \left| \frac{d}{dy} \left\{ \left(1 - \frac{y}{\tau}\right) \frac{\cos\frac{\pi y}{2\tau}}{y} \right\} \right| dy.$$

Since

$$\left|\frac{d}{dy}\left\{\left(1-\frac{y}{\tau}\right)\frac{\cos\frac{\pi y}{2\tau}}{y}\right\}\right|<\frac{\pi}{\tau y}+\frac{1}{y^2},$$

the last integral is

$$<c_{51}\log k$$

and hence from this, (18. 6) and (18. 4) we get

$$Z_3''<c_{52}\log k. \tag{18.7}$$

As to Z_3 we get from (18. 3) at once

$$Z_3'>\frac{2}{\varphi(k)\tau}\sum_{\chi}(1-\mathrm{Re}\,\bar{\chi}(l))\sum_{1<t_\varrho\leq\tau}{}_{\varrho(\chi)}\left(1-\frac{t_\varrho}{\tau}\right)>\frac{1}{\varphi(k)\tau}\sum_{\chi}(1-\mathrm{Re}\,\bar{\chi}(l))\sum_{\frac{\tau}{4}\leq t_\varrho\leq\frac{\tau}{2}}{}_{\varrho(\chi)}1. \tag{18.8}$$

From (18. 5) we get for the inner sum the lower bound (independently of χ)

$$\frac{\tau\log\tau}{8\pi}-c_{52}\tau\log k.$$

But choosing in (5. 3) c_{11} sufficiently large and using (14. 6) this is

$$>\frac{\tau\log\tau}{16\pi}$$

and hence from (18. 8)

$$Z_3'>\frac{\log\tau}{16\pi}.$$

This and (18. 7) give from (5. 3)

$$Z_3>\frac{1}{32\pi}\log\tau \tag{18.9}$$

and analogously

$$Z_4\overset{\text{def}}{=}\frac{1}{\varphi(k)}\sum_{\chi}(1-\bar{\chi}(l))\sum_{1<|t_\varrho|\leq\tau}{}_{\varrho(\chi)}\left(1-\frac{|t_\varrho|}{\tau}\right)\cdot\frac{e^{-i\frac{t_\varrho\pi}{2\tau}}}{it_\varrho}<-\frac{1}{32\pi}\log\tau. \tag{18.10}$$

19. Next we determine γ so that

$$\left\|\frac{t_\varrho}{2\pi}\gamma\right\|<\frac{1}{q} \tag{19.1}$$

with the q defined in (14. 7) should hold for all $1\leq t_\varrho\leq\tau$ of *all* $L(s,\chi)$-functions belonging to mod k; here $||x||$ denotes as usual the distance of x from the next integer. Owing to (18. 5) and (5. 3) (if c_{11} is sufficiently large) their total number is at most

$$\frac{3\varphi(k)}{2\pi}\tau\log\tau$$

and hence Dirichlet's theorem assures the existence of a γ with (19. 1) and

$$q^{\frac{\varphi(k)}{2\pi}\tau\log\tau} \leqq \gamma \leqq q^{\frac{2\varphi(k)}{\pi}\tau\log\tau}. \tag{19.2}$$

Then (18. 9), (18. 5), (14. 7), (14. 6) and (19. 1) result

$$Z_5(\chi) \overset{\text{def}}{=} \sum_{\substack{\varrho(\chi)\\ 1<|t_\varrho|\leqq\tau}} \left(1-\frac{|t_\varrho|}{\tau}\right)\frac{e^{it_\varrho\left(\frac{\pi}{2\tau}+\gamma\right)}}{it_\varrho} \geqq \frac{1}{32\pi}\log\tau -$$

$$-\left|\sum_{\substack{\varrho(\chi)\\ 1<|t_\varrho|\leqq\tau}}\left(1-\frac{|t_\varrho|}{\tau}\right)\frac{e^{i\frac{t_\varrho\pi}{2\tau}}}{it_\varrho}\left(e^{2\pi i\frac{t_\varrho\gamma'}{2\pi}}-1\right)\right| \geqq \frac{\log\tau}{32\pi}-\frac{4\pi}{q}\sum_{1<|t_\varrho|\leqq\tau}\frac{1}{|t_\varrho|} >$$

$$> \frac{\log\tau}{32\pi} - \frac{4\pi}{c_{28}\log_5 T}c_{53}\log k\tau\cdot\log\tau > \frac{\log\tau}{64\pi}$$

if only c_{28} is sufficiently large (in dependence upon c_{11}). Hence

$$\frac{1}{\varphi(k)}\sum_\chi(1-\bar\chi(l))\sum_{\substack{\varrho(\chi)\\ 1<|t_\varrho|\leqq\tau}}\left(1-\frac{|t_\varrho|}{\tau}\right)\frac{e^{it_\varrho\left(\frac{\pi}{2\tau}+\gamma\right)}}{it_\varrho} > \frac{\log\tau}{64\pi} \tag{19.3}$$

and

$$\frac{1}{\varphi(k)}\sum_\chi(1-\bar\chi(l))\sum_{\substack{\varrho(\chi)\\ 1<|t_\varrho|\leqq\tau}}\left(1-\frac{|t_\varrho|}{\tau}\right)\frac{e^{it_\varrho\left(-\frac{\pi}{2\tau}+\gamma\right)}}{it_\varrho} < -\frac{\log\tau}{64\pi}. \tag{19.4}$$

Owing to (19. 2), (14. 6), (14. 7) and (5. 3) we see at once that (16. 4) is satisfied and hence $\gamma\pm\frac{\pi}{2\tau}$ can be used as ω. Hence from (19. 3), (19. 4), (17. 7) and (5. 3) we get

$$\frac{1}{2\pi}\int\limits_{\frac{3}{4}\left(\gamma+\frac{\pi}{2\tau}\right)}^{\frac{5}{4}\left(\gamma+\frac{\pi}{2\tau}\right)} g(r)G_{\gamma+\frac{\pi}{2\tau}}(r)\,dr < -\frac{\log\tau}{70\pi} \tag{19.5}$$

$$\frac{1}{2\pi}\int\limits_{\frac{3}{4}\left(\gamma-\frac{\pi}{2\tau}\right)}^{\frac{5}{4}\left(\gamma-\frac{\pi}{2\tau}\right)} g(r)G_{\gamma-\frac{\pi}{2\tau}}(r)\,dr > \frac{\log\tau}{70\pi}. \tag{19.6}$$

But from (19. 2), (14. 6), (14. 7) and (5. 3) we get

$$\frac{5}{4}\left(\gamma+\frac{\pi}{2\tau}\right) < \frac{5}{4}\left(1+q^{\frac{2\varphi(k)}{\pi}\tau\log\tau}\right) < (\log_3 T)^{\frac{1}{10}},$$

further

$$\frac{3}{4}\left(\gamma-\frac{\pi}{2\tau}\right) > \frac{3}{4}\left(-1+q^{\frac{\varphi(k)}{2\pi}\tau\log\tau}\right) > (\log_3 T)^{\frac{1}{130}};$$

hence from this, (19. 5), (19. 6), (14. 6) and (5. 3) we obtain

$$\max_{(\log_3 T)^{\frac{1}{130}} \leqq r \leqq (\log_3 T)^{\frac{1}{10}}} g(r) > \frac{1}{80\pi} \log_5 T$$

and

$$\min_{(\log_3 T)^{\frac{1}{130}} \leqq r \leqq (\log_3 T)^{\frac{1}{10}}} g(r) < -\frac{1}{80\pi} \log_5 T.$$

Going back to the definition of $g(r)$ we obtain at once

$$\max_{e_1(\log_3^{\frac{1}{130}} T) \leqq x \leqq e_1(\log_3^{\frac{1}{10}} T)} x^{-\frac{1}{2}} \big(\psi(x, k, 1) - \psi(x, k, l)\big) > \frac{1}{80\pi} \log_5 T \tag{19. 7}$$

and

$$\min_{e_1(\log_3^{\frac{1}{130}} T) \leqq x \leqq e_1(\log_3^{\frac{1}{10}} T)} x^{-\frac{1}{2}} \big(\psi(x, k, 1) - \psi(x, k, l)\big) < -\frac{1}{80\pi} \log_5 T. \tag{19. 8}$$

20. In order to complete the proof of Theorem 5. 2 we need the

LEMMA IV. *Putting for* $l_1 \not\equiv l_2$ mod k

$$\Psi_{l_1 l_2}(\mu, k) \overset{\text{def}}{=} \int_2^{\mu} \{\psi(v, k, l_1) - \psi(v, k, l_2)\}\, dv$$

we have for $2 \leqq \mu \leqq e_1(\sqrt{\log_3 T})$ *the estimation*

$$\Psi_{l_1 l_2}(\mu, k) < c_{54} \mu^{\frac{3}{2}} \log_6 T.$$

Namely the „exact" formula of Riemann—Landau gives after integration

$$\left| \Psi_{l_1 l_2}(\mu, k) - \frac{1}{\varphi(k)} \sum_{\chi} \big(\bar{\chi}(l_2) - \bar{\chi}(l_1)\big) \sum_{\varrho(\chi)} \frac{\mu^{\varrho+1}}{\varrho(\varrho+1)} \right| < c_{55} \mu \log \mu \log k \tag{20. 1}$$

taking in account (15. 3). We split the sum in (20. 1) according to

$$|t_\varrho| \leqq 1, \quad 1 < |t_\varrho| \leqq \frac{1}{40} \log_2 T, \quad |t_\varrho| > \frac{1}{40} \log_2 T.$$

The first sum is owing to (14. 5) and (5. 3) and the functional-equation absolutely

$$\leqq c_{56} \mu^{\frac{3}{2} + 42 \frac{\log_3 T}{\log_2 T}} \log k < c_{57} \mu^{\frac{3}{2}} \log_6 T. \tag{20. 2}$$

The second sum is owing to (14. 5) absolutely

$$< c_{58} \mu^{\frac{3}{2} + 42 \frac{\log_3 T}{\log_2 T}} \max_{\chi} \sum_{\substack{\varrho(\chi) \\ |t_\varrho| \leqq 1}} \frac{1}{|\varrho|^2} < c_{59} \mu^{\frac{3}{2}} \log k < c_{59} \mu^{\frac{3}{2}} \log_6 T. \tag{20. 3}$$

Finally the third sum is using (5. 3)

$$(20.4) \qquad 2\mu^2 \max_x \sum_{t_\varrho > \frac{1}{40}\log_2 T} \frac{1}{t_\varrho^2} < c_{60}\mu^2 \frac{\log k + \log_3 T}{\log_2 T} < c_{61}.$$

Lemma IV follows from (20. 1), (20. 2), (20. 3), (20. 4) and (5. 3).

21. Next we start from the identity

$$\int_2^x \Psi_{l_1 l_2}(\mu, k) \frac{\log^2 \mu + 2\log \mu}{\mu^2 \log^4 \mu} d\mu = \int_2^x \Psi_{l_1 l_2}(\mu, k)\, d\left(-\frac{1}{\mu \log^2 \mu}\right) =$$

$$= -\frac{\Psi_{l_1 l_2}(x, k)}{x \log^2 x} + \int_2^x \frac{\psi(\mu, k, l_1) - \psi(\mu, k, l_2)}{\mu \log^2 \mu} d\mu = -\frac{\Psi_{l_1 l_2}(x, k)}{x \log^2 x} +$$

$$+ \int_2^x \{\psi(\mu, k, l_1) - \psi(\mu, k, l_2)\}\, d\left(-\frac{1}{\log \mu}\right) = -\frac{\Psi_{l_1 l_2}(x, k)}{x \log^2 x} -$$

$$-\frac{\psi(x, k, l_1) - \psi(x, k, l_2)}{\log x} + \frac{\psi(2, k, l_1) - \psi(2, k, l_2)}{\log 2} +$$

$$+ \Pi(x, k, l_1) - \Pi(x, k, l_2) - (\Pi(2, k, l_1) - \Pi(2, k, l_2))$$

valid for $x \geqq 2$. This gives, using Lemma IV, for $2 \leqq x \leqq e_1(\sqrt{\log_3 T})$ the inequality

$$(21.1) \qquad \left| \Pi(x, k, 1) - \Pi(x, k, l) - \frac{\psi(x, k, 1) - \psi(x, k, l)}{\log x} \right| <$$

$$< 1 + \frac{|\Psi_{1l}(x, k)|}{x \log^2 x} + 4 \int_2^x \frac{|\Psi_{1l}(\mu, k)|}{\mu^2 \log^2 \mu} d\mu < c_{62} \frac{\sqrt{x}}{\log^2 x} \log_6 T.$$

But then we get for $x \geqq 2$

$$\left| \pi(x, k, 1) - \pi(x, k, l) - \frac{\psi(x, k, 1) - \psi(x, k, l)}{\log x} \right| < c_{62} \frac{\sqrt{x}}{\log^2 x} \log_6 T +$$

$$+ \pi(\sqrt{x}) + 2x^{\frac{1}{3}} \log x < c_{63} \frac{\sqrt{x}}{\log x} \log_6 T$$

roughly. From this, (19. 7) and (19. 8) the proof of the second case is complete.

22. As to Theorem 5. 4 we have again to consider the cases I and II. In the case I instead of $\Pi(x, k, 1) - \Pi(x, k, l)$ we have to consider

$$\Pi(x, k, 1) - \frac{1}{\varphi(k) - 1} \sum_{\substack{(l,k)=1 \\ l \not\equiv 1}} {}_l\, \Pi(x, k, l)$$

and the proof can be achieved on the line of the proof of (3. 4)—(3. 5) starting instead of $F(s)$ in (11. 1) with the function

$$F^*(s) \overset{\text{def}}{=} -\frac{1}{\varphi(k)} \sum_{\chi \neq \chi_0} \frac{L'}{L}(s, \chi).$$

Similar remark holds on case II.

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