COMPARATIVE PRIME-NUMBER THEORY. I

(INTRODUCTION)

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1. The investigations in the theory of distribution of primes in various residue-classes mod k point in two directions. The first — and prevailing — tendency intended to exhibit *uniformity* of the distribution. As its main result the relation

$$\lim_{x\to+\infty} \frac{\pi(x,k,l)}{\operatorname{Li} x} = \frac{1}{\varphi(k)} \tag{1.1}$$

can be considered, independently of l, where $(l,k)=1$ further $\pi(x,k,l)$ denotes the number of primes $p \leqq x$ with $p \equiv l \bmod k$ and

$$\operatorname{Li} x = \int_2^x \frac{dv}{\log v}.$$

The second tendency intends to exhibit *discrepancies* of this distribution. The first indication in this direction is due to Chebyshev as early as in 1853 (see CHEBYSHEV [1]). He asserted[1] inprecisely expressed that „there are more primes $\equiv 3 \bmod 4$ than $\equiv 1 \bmod 4$." The number of papers dealing with either trends on the theory was never very large (*not* due to the lack of interest); but as to this second trend, during the century which passed since Chebyshev's announcement, only three papers were written, all dealing with Chebyshev's above mentioned assertion. Chebyshev's assertion stated exactly that

$$\lim_{r\to+0} \sum_{p>2} (-1)^{\frac{p-1}{2}} e^{-pr} = -\infty \tag{1.2}$$

(p always reserved for primes), which if true, would allow only to assert that the preponderance of the primes $\equiv 3 \bmod 4$ holds only in „Abel-summation's sense" or shortly „in Abel's sense". None of these three papers decide the falsity or truth of (1.2). What they could prove, was its equivalence with the (very deep) fact that the function

$$L(s) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} \tag{1.3}$$

$$s = \sigma + it$$

[1] But never published his proof.

does vanish or not in the half-plane $\sigma > \frac{1}{2}$; necessity by Landau (see LANDAU [1]), the sufficiency by HARDY—LITTLEWOOD and simpler by Landau (see HARDY—LITTLEWOOD [1], resp. LANDAU [2]). As to the direct interpretation of Chebyshev's assertion, i. e. whether or not the function

$$\pi(x, 4, 1) - \pi(x, 4, 3) \tag{1.4}$$

is negative for all $x \geqq 2$, this was disproved by Littlewood (see HARDY—LITTLEWOOD [1]) who showed even that the function in (1. 4) has an infinity of sign-changes. Another, much more special assertion of Chebysev l. c. in this second trend, namely that for a suitable sequence $x_1 < x_2 < \ldots$

$$\frac{\pi(x_\nu, 4, 3) - \pi(x_\nu, 4, 1)}{\left(\frac{\sqrt{x_\nu}}{\log x_\nu}\right)} \to 1 \tag{1.5}$$

was proved by Phragmén (see PHRAGMÉN [1]) and then simpler by Landau (see LANDAU [3]) who extended it to general k; to this we shall return later.

2. It is an old recipe that if you cannot solve a problem, try to generalize it. The generalized problems seem very natural; one wonders, why they had not been formulated before. They all are elucidating various aspects of the central tendency, to compare the distribution of primes in various related forms from the point of view of *discrepancies;* their totality is what we call comparative prime-number theory. We shall enumerate some of its most plausible problems (without gradation of course).[2]

PROBLEM 1. For which (l_1, l_2)-pairs with $l_1 \neq l_2$ does the function

$$\pi(x, k, l_1) - \pi(x, k, l_2)$$

change its sign infinitely often?

If the answer is positive for all (l_1, l_2)-pairs (what is very probable) it is very plausible to raise the still deeper

PROBLEM 2. If $\varepsilon > 0$ and arbitrarily small, further $l_1 \neq l_2$ arbitrary, do there exist two sequences

$$x_1 < x_2 < \ldots \to +\infty$$

$$y_1 < y_2 < \ldots \to +\infty$$

such that

$$\pi(x_\nu, k, l_1) - \pi(x_\nu, k, l_2) > x_\nu^{\frac{1}{2} - \varepsilon}$$

and

$$\pi(y_\nu, k, l_1) - \pi(y_\nu, k, l_2) < -y_\nu^{\frac{1}{2} - \varepsilon} ?$$

[2] k is thought always fixed.

The use of the function $x^{\frac{1}{2}-\varepsilon}$ is motivated by the fact that if the so-called Riemann—Piltz-conjecture is true (see below) then the inequality

$$|\pi(x, k, l_1)-\pi(x, k, l_2)| = O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

holds (O-sign for $x \to +\infty$).

If the answer to Problem 2. is positive for all $(l_1\, l_2)$-pairs, then the oscillatory character of $\pi(x, k, l_1)-\pi(x, k, l_2)$ is much clearer if the following problem is solved.

PROBLEM 3. If $\varepsilon > 0$ and arbitrarily small then for what $h_k(T) > 0$ can one assert that for each (l_1, l_2)-pairs with $l_1 \neq l_2$ and $T \geqq 1$ the inequalities

$$\max_{T \leqq x \leqq T+h_k(T)} \{\pi(x, k, l_1)-\pi(x, k, l_2)\} > T^{\frac{1}{2}-\varepsilon}$$

and

$$\min_{T \leqq x \leqq T+h_k(T)} \{\pi(x, k, l_1)-\pi(x, k, l_2)\} < -T^{\frac{1}{2}-\varepsilon}$$

hold?

An explicit $h_k(T)$ in the solution of Problem 3. would imply of course that all functions $\pi(x, k, l_1)-\pi(x, k, l_2)$ change sign in the intervals of the form

$$(T, T+h_k(T)). \tag{2. 1}$$

But this sign-change is obviously accompained by a „very large" oscillation; probably asking just for a sign-change the interval (2. 1) can be very much reduced. So arises the

PROBLEM 4. For which $g_k(T) > 0$ can we assert that for each (l_1, l_2)-pairs with $l_1 \neq l_2$ and $T \geqq 1$ all functions

$$\pi(x, k, l_1)-\pi(x, k, l_2) \tag{2. 2}$$

change sign at least once in every interval of the form[3]

$$T \leqq x \leqq T+g_k(T) \quad ?$$

In the case of a positive answer to Problem 4. one can get automatically an upper bound $a(k)$ depending only upon k such that for $1 \leqq x \leqq a(k)$ all functions in (2. 2) change their sign. This bound is most probably rather rough and direct approach is necessary to improve it. So we come to the

PROBLEM 5. For which $a(k)$ can we assert that for each (l_1, l_2)-pair with $l_1 \neq l_2$ all functions in (2. 2) vanish at least once in

$$1 \leqq x \leqq a(k) \ ?$$

The positive solution of problem 4. gives of course a lower bound for $W_k(T, l_1, l_2)$, for the number of sign-changes of

$$\pi(x, k, l_1)-\pi(x, k, l_2)$$

[3] A modified form of problems 3 and 4 arises when we wish to assure a sign-change in the interval (2.1) only for sufficiently large T's.

in the interval

$$0 < x \leqq T; \tag{2.3}$$

but this is most probably rather rough. So we come at once to the

PROBLEM 6. What is the asymptotical behaviour of $W_k(T, l_1, l_2)$ if $T \to +\infty$?

As told, LITTLEWOOD showed that the function in (1. 4) changes infinitely often its sign but Chebishev's assertion (1. 2) may be correct. One feels that Chebyshev's vague formulation could also be interpreted so as

$$\lim_{Y \to +\infty} \frac{N(Y)}{Y} = 0 \tag{2.4}$$

where $N(Y)$ denotes the number of the integer $m \leqq Y$ with the property

$$\pi(m, 4, 1) \geqq \pi(m, 4, 3). \tag{2.5}$$

Though numerical data are in such problems never too convincing, Shanks remarked (see SHANKS [1]) that (2. 5) is *not* fulfilled for

$$m \leqq 26860,$$

then it is fulfilled for $m = 26861$ and $m = 26862$ and again false for

$$26863 \leqq m \leqq 616768,$$

which indicates a strong preponderance for those m's for which $\pi(m, 4, 3) > \pi(m, 4, 1)$. Generalization of this leads at once to

PROBLEM 7. For fixed (l_1, l_2)-pair, $l_1 \neq l_2$, what is the asymptotical behaviour of $N_{l_1 l_2}(Y)$ for $Y \to +\infty$, where $N_{l_1 l_2}(Y)$ denotes the number of integers $m \leqq Y$ with

$$\pi(m, k, l_1) \geqq \pi(m, k, l_2) \ ? \tag{2.6}$$

Shanks's paper leads to a further interesting and plausible problem. To illustrate the problem for $k = 8$, say, let us consider the game, played by four players, called „1", „3", „5" and „7", the player „j" scoring a point when by the enumeration of all primes[4] a prime $\equiv j \bmod 8$ occurs. According to the calculations of Shanks the player „1" plays rather poorly, being on the last place after the first $\pi(10^6)$ steps. Will this always be the case? If not, will the player „1" infinitely often take the lead? In a generalized form we assert the

PROBLEM 8. (Race-problem of SHANKS-RÉNYI). For each permutations

$$l_1, l_2, \ldots, l_{\varphi(k)}$$

of the reduced set of residue classes mod k does there exist an infinity of integer m's with

$$\pi(m, k, l_1) < \pi(m, k, l_2) < \ldots < \pi(m, k, l_{\varphi(k)}) \quad ?$$

[4] In increasing order.

As well-known LITTLEWOOD proved — in contrary to an assertion of Riemann — that for a suitable sequence

$$x_1' < x_2' < \dots$$

of integers the inequality

$$\pi(x_\nu') > \operatorname{Li} x_\nu'$$

holds. An interesting generalization would be to prove that even the inequalities

$$\text{(2.7)} \qquad \begin{aligned} \pi(x_\nu', 4, 1) &> \frac{1}{2} \operatorname{Li} x_\nu' \\ \pi(x_\nu', 4, 3) &> \frac{1}{2} \operatorname{Li} x_\nu' \end{aligned}$$

hold simultaneously. More generally we state the

PROBLEM 9. Does there exist an infinity of integer m_ν's such that for $j=1, 2, \dots$ $\dots, \varphi(k)$ simultaneously

$$\text{(2.8)} \qquad \pi(m_\nu, k, l_j) > \frac{1}{\varphi(k)} \operatorname{Li} m_\nu \quad ?$$

A natural continuation is the

PROBLEM 10. If the answer to Problem 9 is positive (which is very probable) what are the distribution-properties of the m_ν-sequence? In particular what upper bound can be given for the smallest m_ν with (2.8)?

3. The above problems referred to $\pi(x, k, l)$. Chebyshev's assertion gives the impression that turning to „Abel-type" theorems the deviations become more significant. One can expect namely generally that there are „more" primes in the residue-class $l_1 \bmod k$ than $l_2 \bmod k$ if and only if the number of incongruent solutions of the congruence

$$\text{(3.1)} \qquad x^2 \equiv l_1 \bmod k$$

is less than that of the congruence

$$\text{(3.2)} \qquad x^2 \equiv l_2 \bmod k.$$

Hence all the previous problems remain (or even become more) interesting, replacing $\pi(x, k, l)$ by

$$\text{(3.3)} \qquad \sum_{p \equiv l \bmod k} e^{-pr}$$

and $\operatorname{Li} x$ by

$$\text{(3.4)} \qquad \sum_{n=2}^{\infty} \frac{e^{-nr}}{\log n} \quad \text{or} \quad \int_2^{\infty} \frac{e^{-rv}}{\log v}\, dv$$

so we get the problems 11—20. Of course the wording of these problems must be understood mutatis mutandis; e. g.

PROBLEM 12. If $\varepsilon > 0$ and arbitrarily small, further the number of solutions of (3. 1) and (3. 2) are equal, do there exist two sequences

$$r_1' > r_2' > \dots \to 0$$

$$r_1'' > r_2'' > \dots \to 0$$

such that

$$\sum_{p \equiv l_1 \bmod k} e^{-pr_\nu'} - \sum_{p \equiv l_2 \bmod k} e^{-pr_\nu'} > \left(\frac{1}{r_\nu'}\right)^{\frac{1}{2}-\varepsilon}$$

and

$$\sum_{p \equiv l_1 \bmod k} e^{-pr_\nu''} - \sum_{p \equiv l_2 \bmod k} - \left(\frac{1}{r_\nu''}\right)^{\frac{1}{2}-\varepsilon} \quad ?$$

Though the Abel-summation seems to be a more suitable tool for exhibiting deviations in the distribution of primes in residueclasses mod k, working with it causes a lot of numerical difficulties which increases rapidly with k. Hence it is still more interesting to find the „proper" summation-method which can show up the deviations with the least trouble. We shall not formulate this point of view explicitely in problems.

As well-known in the theory of primes, the investigations are much more directly carried out on

$$\psi(x, k, l) \overset{\text{def}}{=} \sum_{\substack{n \leq x \\ n \equiv l \bmod k}} \Lambda(n) \tag{3. 5}$$

or

$$\Pi(x, k, l) \overset{\text{def}}{=} \sum_{\substack{n \leq x \\ n \equiv l \bmod k}} \frac{\Lambda(n)}{\log n}. \tag{3. 6}$$

The corresponding problems we label as problems 21—40. resp. 41—60. Their investigation runs generally parallel but there are problems which we can solve at present *only* for $\psi(x, k, l)$.

We shall not enumerate the analogous problems concerning the distribution of primes in binary quadratic forms with fixed discriminant or of the prime ideals of a fixed field in various idealclasses.

4. Previously we spoke about the results explicitely existing in the literature; now we review some of these which could have been deduced by previously known methods. The main tool is the following classical theorem of LANDAU. If $A(x)$ is real,

$$g(s) = \int_1^\infty \frac{A(x)}{x^s}\, dx \tag{4. 1}$$

is regular for $\sigma > 1$, $A(x)$ does not change its sign for $x > x_0$ and $g(s)$ is regular on the segment $s > \gamma$ (< 1), then $g(s)$ is regular in the half-plane $\sigma > \gamma$. As to problem 21 one could start from the formula[5] (valid for $r > 1$)

$$(4.2)\qquad \int_1^\infty \frac{\psi(x,k,l_1)-\psi(x,k,l_2)\pm c_1 x^{\frac{1}{2}-\varepsilon}}{x}\cdot\frac{dx}{x^s} =$$

$$= \frac{1}{\varphi(k)s}\sum_{\chi\neq\chi_0}\left(\bar\chi(l_2)-\bar\chi(l_1)\right)\frac{L'}{L}(s,\chi)\pm\frac{c_1}{s-\frac{1}{2}+\varepsilon}.$$

Let k be such that for all L-functions mod k

$$(4.3)\qquad L(s,\chi)\neq 0 \quad\text{on the segment}\quad 0<s<1.$$

For $k|24$ e. g. this is the case since for $k < 227$ Rosser proved (see ROSSER [1]) that no $L(s, \chi)$ with real characters vanishes on this segment and for $k|24$ there are only real characters mod k. In this case the function on the right of (4. 2) is regular on the real segment $\frac{1}{2}-\varepsilon<s<1$; hence if for an (l_1, l_2)-pair and a c_1 for $x > x_0$ the inequality

$$(4.4)\qquad \psi(x,k,l_1)-\psi(x,k,l_2)\pm c_1 x^{\frac{1}{2}-\varepsilon} > 0$$

would be true then Landau's theorem in (4. 1) would result that the function

$$(4.5)\qquad \sum_{\chi\neq\chi_0}\left(\bar\chi(l_2)-\bar\chi(l_1)\right)\frac{L'}{L}(s,\chi)$$

is regular for $\sigma > \frac{1}{2}-\varepsilon$.

Let first $l_1 = 1$. Since owing to the functional-equation each $L(s, \chi)$ has zeros in the half-plane

$$(4.6)\qquad \sigma \geqq \frac{1}{2},$$

choosing χ^* so that $\chi^*(l_2)\neq 1$ each zero of $L(s, \chi^*)$ in the half-plane (4. 6) is obviously a pole of the function in (4. 5). This being in contradiction to (4. 5) we obtained that under supposition of (4. 3), chosing c_1 arbitrarily we have infinitely often (x_ν, y_ν positive integers, $l \neq 1$)

$$(4.7)\qquad \psi(x_\nu,k,1)-\psi(x_\nu,k,l) < -c_1 x_\nu^{\frac{1}{2}-\varepsilon}$$

and analogously

$$(4.7)\qquad \psi(y_\nu,k,1)-\psi(y_\nu,k,l) > c_1 y_\nu^{\frac{1}{2}-\varepsilon}.$$

[5] By $c_1, c_2, \ldots$ we denote always positive explicitely calculable, numerical constants. By $a_1, a_2, \ldots$ we shall denote quantities depending exclusively upon k.

In the general case however this reasoning is functioning only — even supposing (4. 3) — when there is a ϱ with the Re $\varrho \geqq \frac{1}{2}$ and

$$\sum_{\chi}{}'(\bar{\chi}(l_1)-\bar{\chi}(l_2)m_\varrho(\chi)\neq 0 \tag{4. 8}$$

where the dash in (4. 8) means that the summation in (4. 8) is extended over all characters χ with[6]

$$L(\varrho, \chi)=0 \tag{4. 9}$$

and $m_\varrho(\chi)$ stands for the multiplicity of the ϱ-zero. However simple looking we could not ascertain the existence of such a ϱ for *general k*, even supposing the truth of the conjecture of Riemann—Piltz, according to which no $L(s, \chi)$-function vanishes in the half-plane

$$\sigma > \frac{1}{2}. \tag{4. 11}$$

For *special k*-values, such as for $k|24$ for example, owing to a communication of DR. P. C. HASELGROVE there is such a ϱ with the property[7] (4. 8)—(4. 9); hence for $k|24$ the inequalities (4. 7) hold without any conjectures (and even (4. 10)). As to $\Pi(x, k, l)$ the analoga of all results, deduced in this 4. for $\psi(x, k, l)$, can be proved *with the exception of* (4. 10). A glance at the above proofs shows at once this principially cannot give finer details of the oscillatory character, nothing concerning, say, the deeper problems 23, 25 or 43, 45, resp.

A modification of Landau's theorem in (4. 1), due to Pólya[8] (see PÓLYA [1]) would give, denoting by $U_k(T, l_1, l_2)$ the number of sign-changes of

$$\psi(x, k, l_1)-\psi(x, k, l_2) \tag{4. 12}$$

for $1 \leqq x \leqq T$, the inequality

$$\overline{\lim_{T\to+\infty}} \frac{1}{\log T} U_k(T, l_1, l_2) > 0 \tag{4. 13}$$

supposing (4. 8)—(4. 9) and also

$$L(s, \chi)\neq 0 \tag{4. 14}$$

for

$$0<\sigma<1, \qquad |t| \leqq A(k)$$

[6] One could also prove by a proper modification that for an infinity of integers x_ν, y_ν even the inequalities

$$\psi(x_\nu, k, 1)-\psi(x_\nu, k, l) > a_1 \sqrt{x_\nu} \tag{4. 10}$$

and

$$\psi(y_\nu, k, 1)-\psi(y_\nu, k, l) < -a_1 \sqrt{x_\nu} \tag{4. 10}$$

with a suitably small a_1 hold, supposing only (4. 3). In the general case the analogon of (4. 10) hold supposing (4. 3) and the inconvenient (4. 8)—(4. 9).

[7] Even *each* $L(s, \chi)$-function mod k with $\chi\neq\chi_0$ has zeros, not common with *any* other $L(s, \chi)$ mod k.

[8] According to a remark of PÓLYA (see INGHAM [1], p. 202.) the proof contains a gap, easy to fill out. As far as I know, this was performed so far by nobody.

i. e. for $k|24$ unconditionally. This again gives principially nothing for the above-mentioned problems. This method does not work for $\Pi(x, k, l)$ and gives nothing for $V_k(T, l_1, l_2)$, for the number of sign-changes of $\Pi(x, k, l_1) - \Pi(x, k, l_2)$ in $1 \leqq x \leqq T$.

The only method which can be adopted to the study of the finer details of the oscillation of $\psi(x, k, l_1) - \psi(x, k, l_2)$ is due to Ingham (see INGHAM [1]), which he applied to the study of sign-changes of $\pi(x) - \mathrm{Li}\, x$. His method (an improvement of Littlewood's proof) would result[9]

(4. 15)
$$\max_{T \leqq x \leqq a_2 T} \{\psi(x, k, 1) - \psi(x, k, l)\} > \sqrt{T} \log_3 T$$

and
$$\min_{T \leqq x \leqq a_2 T} \{\psi(x, k, 1) - \psi(x, k, l)\} < -\sqrt{T} \log_3 T$$

if only $T > a_3$, (4. 14) holds and moreover all zeros of all L-functions mod k are in the half-plane $\sigma \leqq \Theta$ (< 1) *with attained equality-sign* (e. g. if Riemann-Piltz conjecture is true); similar result could be deduced for $\Pi(x, k, 1) - \Pi(x, k, l)$. We do not see how this method can give any results even supposing the truth of the Riemann—Piltz conjecture, for the case when none of l_1 and l_2 are 1.

5. So far we sketched the results to be found implicitly in the literature concerning our problems with $\psi(x, k, l)$ and $\Pi(x, k, l)$. What is the situation concerning $\pi(x, k, l)$? Since we shall need it often in the sequel, we shall denote throughout this series the number of incongruent solutions of the congruence

(5. 1)
$$x^2 \equiv l \bmod k$$

by[10] $N_k(l)$. Now first we remark that if

(5. 2)
$$N_k(l_1) = N_k(l_2) = 0$$

then we have

(5. 3)
$$\Pi(x, k, l_1) - \Pi(x, k, l_2) = \pi(x, k, l_1) - \pi(x, k, l_2) + O(x^{\frac{1}{3}})$$

and hence one would think, this gives a smooth passage from results concerning $\Pi(x, k, l_1) - \Pi(x, k, l_2)$ to $\pi(x, k, l_1) - \pi(x, k, l_2)$. However as far as we see, the classical methods furnish only the following theorems.

Supposing (4. 3), further with some a_4

(5. 4)
$$L(s, \chi) \neq 0 \quad \text{for} \quad \sigma > 1 - a_4$$

and also $N_k(1) = N_k(l)$, we have for an infinity of positive integers x_ν and y_ν the inequalities

(5. 5)
$$\pi(x_\nu, k, 1) - \pi(x_\nu, k, l) > x_\nu^{\frac{1}{2} - \varepsilon}$$

[9] We shall denote always $e_1(x) = e^x$ and $e_\nu(x) = e_{\nu-1}(e_1(x))$ further $\log_1 x = \log x$ and $\log_\nu x = \log_{\nu-1}(\log_1 x)$.

[10] We remark the well-known fact that the value of $N_k(l)$ is either 0 or $N_k(1)$ if only $(l, k) = 1$.

and

$$\pi(y_\nu, k, 1) - \pi(y_\nu, k, l) < -y_\nu^{\frac{1}{2}-\varepsilon}. \tag{5.6}$$

The corresponding inequalities hold for the case of general (l_1, l_2) but we have to require now

$$N(l_1) = N(l_2)$$

and also the horrible requirement (4. 8)—(4. 9); however for the important case

$$N_k(l_1) = N_k(l_2) = 0 \tag{5.7}$$

the requirement (5. 4) can be dropped.

This theorem with all of its ramifications follows from the Π-analogon of (4. 7) at once; we could not use (4. 10) for a similar purpose (which would make the restriction (5. 4) superfluous). For $k|24$ and for the case (5. 7) the theorem (i. e. the analoga of (5. 5) and (5. 6)) holds without conjectures.

The method of LITTLEWOOD—INGHAM would allow to prove the following theorem.

Supposing (4. 14) and also the existence of an $\frac{1}{2} \leqq a_5 < 1$ such that

$$L(s, \chi) \neq 0 \quad \text{for} \quad \sigma > a_5 \tag{5.8}$$

but for a suitable real t_0

$$L(a_5 + it_0) = 0, \tag{5.9}$$

then for $T > a_6$, all functions

$$\pi(x, k, 1) - \pi(x, k, l)$$

change their sign in the interval

$$T \leqq x \leqq a_7 T. \tag{5.10}$$

Recently Skewes (see SKEWES [1]) solved, using also the ideas of LITTLEWOOD and INGHAM, the problem (related to ours) to give a numerical upper bound for the first sign-change of $\pi(x) - \mathrm{Li}\, x$.[11] It is not impossible that this method can also lead to some results in the investigations of the finer oscillatory properties of $\pi(x, k, 1) - \pi(x, k, l)$ (but certainly to none in the general case).

One has still to remark that the fact that for the finer distribution-properties of the primes in residue-classes the numbers $N_k(l)$ have a significance, was observed by LANDAU (a hint to it was made in LANDAU [1]). He proved namely (to mention only the most elegant case) that supposing (4. 3) we have to an arbitrarily small $\varepsilon > 0$ an infinity of positive integers x_ν such that

$$\left| \frac{\pi(x_\nu, k, l_1) - \pi(x_\nu, k, l_2)}{\left(\dfrac{\sqrt{x_\nu}}{\log x_\nu}\right)} - \frac{N_k(l_2) - N_k(l_1)}{\varphi(k)} \right| < \varepsilon. \tag{5.11}$$

[11] For an alternative simple proof see the forthcoming paper of the first of us in *Journal of London Math. Soc.*

As the case $k=4$ shows, the really deep result would be if $N_k(l_2)-N_k(l_1)$ could be replaced by $N_k(l_1)-N_k(l_2)$ in (5. 11).

6. In the last years the second of us observed that a certain extension of the classical theory of diophantine approximation leads to a large number of applications in the analysis and in the analytical theory of numbers (see TURÁN [1]). The applicability of these methods and in particular of what is called second main theorem, in the comparative prime-number theory was shown by the first of us (see KNAPOWSKI [1]). In a lecture in Göttingen in 1957 the second of us risked the assertion that it is possible to find „onesided" forms of the main theorems which would enlarge considerably the number of the numerous applications and in particular applicable to Problem 5 of this paper. Such one-sided theorems have been proved since (see TURÁN [2], [3]); their applicability for the prime-number theory was made possible by a lemma of the first of us (see his forthcoming paper [2]). This series will deal with applications to the comparative prime-number theory and will consist of 8 papers with the following sub-titles:[12]

II. Comparison of the progressions $\equiv 1$ and $\equiv l \bmod k$.
III. Continuation.
IV. Paradigma to the general case ($k=5$ and $k=8$)
V. Some theorems concerning the general case.
VI. Continuation.
VII. The first sign-change.
VIII. Chebyshev's problem for $k=8$.

Generally speaking we intended to make the papers self-contained; only II—III resp. V—VI are to a certain extent dependent of each other. This will explain a few short repetitions.

7. The present state of the comparative prime number theory, as it was sketched in 5, is rather rudimentary. This implies that also conditional theorems are of interest. Beside the conjecture of Riemann—Piltz we shall prove some theorems supposing only a weaker form of it, namely that for a sufficiently large[13] c_2 for $\chi \neq \chi_0$

$$(7.1) \qquad L(s,\chi)\neq 0 \quad \text{for} \quad \sigma>\frac{1}{2}, \quad |t|\leqq c_2 k^{10}.$$

A still weaker assumption, we shall use sometimes, is that made in (4. 14); we shall call it shortly Haselgrove-condition since he was the first who determined *explicit* values of $A(k)$ for $k|24$. It would be very interesting to prove at least that the number of k's satisfying Haselgrove's condition is infinite. Of course there is no *principal* difficulty to determine $A(k)$ for any fixed k (for which $A(k)$ exist at all). As we know from a written communication of MR. HASELGROVE, MR. D. DAVIES has determined the corresponding values for $k=5, 7, 11$ and 19; the interest of these cases is, that

[12] We gave an account of the main results in two lectures; one at the Mathematical Congress in Leningrad on 4 July 1961 and one at the DMV-Tagung in Halle, on 21 Sept. 1961.

[13] It would be of importance to decide whether an a_8 can exist with the property that the truth of the full Riemann—Piltz conjecture follows if no $L(s,\chi)$ mod k vanishes for $\sigma>\frac{1}{2}$, $|t|\leqq a_8$. If yes, then it is still very unprobable that $a_8=c_2k^{10}$ can be chosen; so, that (7.1) seems to be weaker than Riemann—Piltz conjecture indeed.

now we have complex characters too. In paper II we shall prove among others that if for a k Haselgrove's condition (4. 14) is fulfilled and

$$T > \max\left(e_5(c_3 k), e_2\left(\frac{1}{A(k)^3}\right)\right) \tag{7. 2}$$

with a sufficiently large c_3 then for all $(l, k)=1$ the inequalities

$$\max_{e_1(\log_3^{\frac{1}{130}} T) \leqq x \leqq T} \frac{\pi(x, k, 1)-\pi(x, k, l)}{\left(\frac{\sqrt{x}}{\log x}\right)} > \frac{1}{100} \log_5 T \tag{7. 3}$$

and

$$\min_{e_1(\log_3^{\frac{1}{130}} T) \leqq x \leqq T} \frac{\pi(x, k, 1)-\pi(x, k, l)}{\left(\frac{\sqrt{x}}{\log x}\right)} < -\frac{1}{100} \log_5 T \tag{7. 4}$$

hold (i. e. *for* $k|24$ *or for the Davies-values unconditionally*). This means in a little weakened form that for T's with (7. 2) the interval

$$e_1(\log_3^{\frac{1}{130}} T) \leqq x \leqq T \tag{7. 5}$$

contains certainly values x' and x'' with

$$\pi(x', k, 1)-\pi(x', k, l) > \frac{1}{100} \frac{\sqrt{x'}}{\log x'} \log_5 x' \tag{7. 6}$$

and

$$\pi(x'', k, 1)-\pi(x'', k, l) < -\frac{1}{100} \frac{\sqrt{x''}}{\log x''} \log_5 x'' \tag{7. 7}$$

which are — at least when the Riemann—Piltz-conjecture is true — not „very far" from the best-possible inequality

$$|\pi(x, k, 1)-\pi(x, k, l)| = O(\sqrt{x} \log x) \tag{7. 8}$$

and — what is the most essential — with the localization (7. 5). For $k|24$ — i. e. also in the Chebyshevian case $k=4$ — we emphasize that a theorem of this type is proved here for the first time without any conjectures; the result (5. 10) indicates however that probably the interval in (7. 7) can be replaced by $(T, a_9 T)$.

8. As to the case $k=4$ we make a slight digression with respect to Chebyshev's assertion (1. 2). We may write it in the form

$$\lim_{r \to +0} (1-e^{-r}) \sum_{n=2}^{\infty} \{(\pi(n, 4, 1)-\pi(n, 4, 3)\} e^{-nr} = -\infty. \tag{8. 1}$$

If we could find a proper sequence

$$r_1 > r_2 > \ldots \to 0$$

with

(8.2) $$\lim_{\nu\to\infty}(1-e^{-r_\nu})\sum_{n=2}^{\infty}\{\pi(n,4,1)-\pi(n,4,3)\}e^{-nr_\nu}>-c_4$$

this would disprove owing to Hardy—Littlewood the Riemann—Piltz conjecture for the function (1. 3). Now we suppose (8. 1) would be true. Then according to Landau's mentioned theorem the zeros of (1. 3) are in $\sigma\leqq\frac{1}{2}$ and hence (7. 8) holds. Now let T be large and we choose as r_ν the $\frac{1}{x'}$ (x' from (7. 6)); then the „essential" part of the sum in (8. 1) is that with $n\ll x'\log x'$. Now owing to (7. 6) the terms with an n-index in a „large" neighbourhood of $n=[x']$ have a „large positive" contribution and the further terms which can spoil it, are stipulated by (7. 8), further these „large positive" contributions are owing to (7. 5) „not too far" from each other, one could think choosing r this way, the corresponding r_ν-sequence could prove (8. 2) even with $c_4=0$. As P. ERDŐS remarked by an example this reasoning *alone* cannot lead to a disproof of (8. 1), even if the interval (7. 5) could be replaced by (T, c_5T), but a still deeper study of the oscillatory character of $\pi(n,4,1)-\pi(n,4,3)$ could perhaps increase considerably the „positive contribution" of the terms.

9. Returning to the preliminary discussion of some of our results we remark that (7. 5) gives only a very weak lower bound for $W_k(T,1,l)$, defined in (2. 3). In paper III we prove the inequality

(9.1) $$W_k(T,1,l)>k^{-c_5}\log_4 T$$

for T's in (7. 2) and all l's, if only the Haselgrove-condition is fulfilled (i. e. e. g. for $k|24$ unconditionally). A similar estimation holds for the number of sign-changes of

(9.2) $$\pi(x,k,1)-\frac{1}{\varphi(k)}\pi(x).$$

As to theorem (7. 3)—(7. 4) we prove that in the case when

(9.3) $$N_k(1)=N_k(l),$$

we have for

(9.4) $$T>\max\left(c_6, e_2(k), e_2\left(\frac{1}{A(k)^3}\right)\right)$$

the much sharper inequalities

(9.5) $$\max_{T^{\frac{1}{3}}\leqq x\leqq T}\{\pi(x,k,1)-\pi(x,k,l)\}>\sqrt{T}\,e_1\left(-43\frac{\log T\log_3 T}{\log_2 T}\right)$$

and

(9.6) $$\min_{T^{\frac{1}{3}}\leqq x\leqq T}\{\pi(x,k,1)-\pi(x,k,l)\}<-\sqrt{T}\,e_1\left(-43\frac{\log T\log_3 T}{\log_2 T}\right),$$

if only Haselgrove-condition holds for k. This shows of course at once that in the case (9. 3) — which occurs very often, e. g. when k is prime — in (9. 1) one can replace $\log_4 T$ by $\log_2 T$.

10. The case, when none of the l's are 1, is, as told, much more difficult. This case is dealt in the papers IV, V, VI and VII. In paper IV which is a sort of preludium to the difficult papers V, VI we round off the case $k=8$ completely without any conjectures by showing that for $T>c_7$ we have

$$(10.1)\qquad \max_{T^{\frac{1}{3}}\leqq x\leqq T}\{\pi(x,8,l_1)-\pi(x,8,l_2)\}>\sqrt{T}\,e_1\left(-23\frac{\log T\log_3 T}{\log_2 T}\right)$$

if only[14] $l_1\neq l_2\neq 1$. In paper V we prove among others for

$$(10.2)\qquad T>\max\left\{e_2(c_8k^{20}),\ e_1\left(2e_1\left(\frac{1}{A(k)^3}\right)+c_8k^{20}\right)\right\}$$

for all $l_1\neq l_2$ the inequality

$$(10.3)\qquad \max_{T^{\frac{1}{3}}\leqq x\leqq T}\{\Pi(x,k,l_1)-\Pi(x,k,l_2)\}>\sqrt{T}\,e_1\left(-22\frac{\log T\log_3 T}{\log_2 T}\right)$$

if only *both* Haselgrove-condition and (7. 1) hold for k. Analogous theorem holds for $\psi(x,k,l)$ too. If $N_k(l_1)=N_k(l_2)=0$ then this gives at once the corresponding theorem for $\pi(x,k,l)$ too; nevertheless in paper VI we prove the same theorem if only

$$(10.4)\qquad N_k(l_1)=N_k(l_2).$$

This is perhaps the deepest among our theorems, together with the theorem of the paper VI which asserts, as partial response to the Göttingen-problem 5 that at fixed k *all* functions

$$\psi(x,k,l_1)-\psi(x,k,l_2)$$

change sign in the interval

$$(10.5)\qquad 1\leqq x\leqq \max\left(e_2(k^{c_9}),\ e_2\left(\frac{2}{A(k)^3}\right)\right),$$

supposing only the truth of Haselgrove-condition. Finally in paper VIII we resume Chebyshev's problem for $k=8$ in a slightly modified form and settle it in the sense of Hardy—Littlewood—Landau. No new ideas are necessary in proving that the truth of the relation

$$(10.6)\qquad \lim_{r\to+0}\{\sum_{p\equiv 1 \bmod 8}\log p\cdot e^{-px}-\sum_{p\equiv l \bmod 8}\log p\cdot e^{-px}\}=-\infty$$

for all $l\neq 1$ is equivalent to the non-vanishing of all $L(s,\chi)$ functions mod 8 with $\chi\neq\chi_0$ for $\sigma>\frac{1}{2}$; for the principially new case (for which there is no analogon if

[14] Since none of l_1 and l_2 are distinguished to the other, changing them we get automatically the corresponding negative upper bound for $\min_{T^{1/3}\leqq x\leqq T}\{\pi(x,8,l_1)-\pi(x,8,l_2)\}$.

$k=4$) we prove without any conjectures that for $0<\delta<c_{10}$ and $l_1 \neq l_2$ among 3, 5, 7 the inequality

$$\max_{\delta \leq x \leq \delta^{\frac{1}{3}}} \{ \sum_{p \equiv l_1 \bmod k} \log p \cdot e^{-px} - \sum_{p \equiv l_2 \bmod k} \log p \cdot e^{-px} \} > \tag{10.7}$$

$$> \frac{1}{\sqrt{\delta}} e_1 \left(-22 \frac{\log \frac{1}{\delta} \log_3 \frac{1}{\delta}}{\log_2 \frac{1}{\delta}} \right)$$

holds.

11. One can observe that all of our results relating to general k needs some assumptions, the weakest among them being Haselgrove's condition. Is this a necessity or only a defect of the method? One can prove easily that $\psi(x, k, l_1) - \psi(x, k, l_2)$ is approximately

$$\frac{1}{\varphi(k)} \sum_{\chi \neq \chi_0} (\bar{\chi}(l_2) - \bar{\chi}(l_2)) \sum_{|t_\varrho| \leq x^3} {}_{\varrho(\chi)} \frac{x^\varrho}{\varrho}. \tag{11.1}$$

Now let us suppose e. g. that there is a positive $\vartheta_0 > \frac{1}{2}$ which is a zero of an $L(s, \chi')$ mod k with $\chi' \neq \chi_0$ and with a $\delta > 0$ all other zeros of all L-functions mod k are in the half-plane $\sigma \leq \vartheta_0 - \delta$ (a possibility which is not excluded so far). This χ' is obviously real. If l_1' and l_2' are such that $\chi'(l_1') \neq \chi'(l_2')$, then the expression in (11. 1) is, as easy to see,

$$\frac{1}{\varphi(k)} (\chi'(l_2') - \chi'(l_1')) \frac{x^{\vartheta_0}}{\vartheta_0} + O(x^{\vartheta_0 - \delta} \log^2 x),$$

which means that $\psi(x, k, l') - \psi(x, k, l'')$ does *not* change sign if x sufficiently large. This shows at least that the existence of real roots of L-functions is intrinsically connected with the oscillatory character of $\psi(x, k, l_1) - \psi(x, k, l_2)$.

Finally, we remark that the above sketched problems are only a part of a theory which may be called a comparative theory of numbers; the similar questions have an interest also for squarefree numbers, for numbers with a fixed number of prime-factors, for more general number-theoretical functions, for the n's with $\sum_{\nu \leq n} \frac{\mu(\nu)}{\nu} > 0$ etc.

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