ON SIGN-CHANGES IN THE REMAINDER-TERM IN THE PRIME-NUMBER FORMULA

S. KNAPOWSKI

1. Let $\pi(x)$ stand, as usual, for the number of primes which do not exceed x. The relation

$$\pi(x) \sim \int_2^x \frac{du}{\log u} \quad \text{as } x \to \infty \tag{1.1}$$

is known as the prime-number theorem. Equivalent to it, in a well-established sense, is the relation

$$\psi(x) \sim x \quad \text{as } x \to \infty, \tag{1.2}$$

where
$$\psi(x) = \sum_{n \leqslant x} \Lambda(n) \equiv \sum_{p^m \leqslant x} \log p,$$

p^m running through prime powers. (1.1) and (1.2) may be put as follows:

$$\pi(x) = \operatorname{li} x + \mathscr{H}(x), \tag{1.3}$$

with

$$\operatorname{li} x = \lim_{\epsilon \to 0} \left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^x \frac{du}{\log u} \right), \quad \mathscr{H}(x) = o\left(\frac{x}{\log x}\right),$$

and
$$\psi(x) = x + \Delta(x) \tag{1.4}$$

with
$$\Delta(x) = o(x).$$

The orders of magnitude of the functions $\mathscr{H}(x)$, $\Delta(x)$ as $x \to \infty$ are closely connected with the distribution of the complex zeros of the zeta-function. In fact, on the one hand we have the relations

$$\mathscr{H}(x) = O(x^{\theta+\epsilon}), \quad \Delta(x) = O(x^{\theta+\epsilon}), \tag{1.5}$$

θ being the upper bound of the real parts of zeta-zeros and $\epsilon > 0$ arbitrary, and on the other hand

$$\mathscr{H}(x) = \Omega_\pm(x^{\theta-\epsilon}), \quad \Delta(x) = \Omega_\pm(x^{\theta-\epsilon}) \tag{1.6}$$

if $\theta > \frac{1}{2}$. In addition Littlewood proved in 1914 that

$$\mathscr{H}(x) = \Omega_\pm\left(x^{\frac{1}{2}} \frac{\log\log\log x}{\log x}\right), \quad \Delta(x) = \Omega_\pm(x^{\frac{1}{2}} \log\log\log x). \tag{1.7}$$

The latter inequalities complete (1.6) in case $\theta = \frac{1}{2}$, *i.e.* in case the Riemann hypothesis is true. So, in view of (1.6), it was permissible to assume the Riemann hypothesis in the course of proving (1.7).

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There is one serious drawback with the inequalities (1.6), (1.7): they are, so to speak, "pure existence results" and provide no information about the x-sequences in question. This failure becomes evident when studying the following problem: Riemann conjectured the difference $\pi(x)-\mathrm{li}\,x$ to be negative for all $x \geqslant 2$. It is in fact so for all values of $x \leqslant 10^7$ but the conjecture as a whole obviously breaks down in view of (1.3), (1.7). However, and here was the problem, it was very difficult to find a numerical bound below which the inequality $\pi(x) > \mathrm{li}\,x$ held for some value of $x \geqslant 2$. This question was answered only a few years ago by Skewes [2], who determined such a bound as $\exp\exp\exp\exp(7{\cdot}705)$. Skewes' argument, briefly, runs as follows:

Part I. Suppose that every zeta-zero $\rho = \beta + i\gamma$, for which

$$|\gamma| < X_1^3 \quad [X_1 = \exp\exp\exp(7{\cdot}703)],$$

is such that

$$\beta \leqslant \tfrac{1}{2} + X_1^{-3}\log^{-2} X_1.$$

In this case, the Riemann hypothesis being "almost" true in the range $|t| < X_1^3$, it was natural to follow Littlewood's way of proving (1.7). This, together with a new method due to Ingham and to Skewes himself, led to the inequality

$$\pi(x) > \mathrm{li}\,x \text{ for some } 2 \leqslant x < X_1.$$

Part II. Suppose that a zero $\rho_0 = \beta_0 + i\gamma_0$ exists satisfying

$$\left.\begin{aligned} \beta_0 &> \tfrac{1}{2} + X_1^{-3}\log^{-2} X_1, \\ 0 &< \gamma_0 < X_1^3. \end{aligned}\right\} \tag{1.8}$$

For this case Skewes has developed another method, sketched by Littlewood for the simpler problem of finding an $X = X(h)$ ($h > 0$ arbitrary) with

$$\max_{1 \leqslant x \leqslant X} \big(\psi(x) - x\big) > h \,.\, X^{\frac{1}{2}}$$

which enabled him to arrive at a numerical X_2 $\big(= \exp\exp\exp\exp(7{\cdot}705)\big)$ such that

$$\pi(x) > \mathrm{li}\,x \text{ for some } 2 \leqslant x < X_2.$$

Skewes in fact worked with the function

$$\Pi(x) = \sum_{p^m \leqslant x} \frac{1}{m} = \pi(x) + \tfrac{1}{2}\pi(x^{\frac{1}{2}}) + \tfrac{1}{3}\pi(x^{\frac{1}{3}}) + \ldots,$$

this being easier to handle than $\pi(x)$. The whole procedure of Part II is substantially nothing but the seeking of a lower bound for the difference

$$\Pi(x) - \mathrm{li}\,x$$

as x ranges over a certain (numerical) interval. This particular question and *eo ipso* all Part II, would obviously be settled if we succeeded in establishing the following general result:

Let $\rho_0=\beta_0+i\gamma_0$ be an arbitrary zero of $\zeta(s)$. Then for arbitrary $\epsilon>0$ we have

$$\max_{2\leqslant x\leqslant T}\{\Pi(x)-\operatorname{li}x\}>T^{\beta_0-\epsilon} \text{ for } T>c(\rho_0,\epsilon), \tag{1.9}$$

the latter function being explicit.

For (1.8) together with (1.9) would give the inequality

$$\max_{2\leqslant x\leqslant T_0}\{\Pi(x)-\operatorname{li}x\}>T_1^{\frac{1}{2}+\eta},$$

valid with certain (sufficiently large) T_1 and some $\eta>0$, and a quick switch to the function $\pi(x)$ would provide the desired numerical X_2. In view of this, and apart from anything else, it is of importance to estimate explicitly the expressions $\max_{2\leqslant x\leqslant T}\{\Pi(x)-\operatorname{li}x\}$ [from below as in (1.9)] and $\min_{2\leqslant x\leqslant T}\{\Pi(x)-\operatorname{li}x\}$ (similarly from above). Such one-sided estimations, and analogous ones for the remainder-term $\Delta(x)$ in (1.4), will be the subject of the present paper.

The case of an explicit estimation of $|\Delta(x)|$ (from below) has been settled by P. Turán [4] (see also [5]). The result, established by means of his new method in Diophantine Approximation, reads as follows.

Let $\rho_0=\beta_0+i\gamma_0$, $\beta_0\geqslant\frac{1}{2}$ be an arbitrary ζ-zero. Then we have for

$$T>\max\left(c_1, \exp\exp(60\log^2|\rho_0|)\right)\dagger$$

$$\max_{1\leqslant x\leqslant T}|\Delta(x)|>T^{\beta_0}\exp\left(-21\frac{\log T}{\sqrt{(\log\log T)}}\right) \tag{1.10}$$

(c_1 being explicitly calculable).

What is essential and underlies (1.10) is the following theorem (a) particular case of Turán's second main theorem; see [5], p. 52).

Let $z_1, z_2, \ldots, z_M$ be complex numbers such that

$$|z_1|\geqslant|z_2|\geqslant\cdots\geqslant|z_M|,\quad |z_1|\geqslant 1;$$

then, if m is positive and $N\geqslant M$, we have

$$\max_{m\leqslant\nu\leqslant m+N}|z_1^\nu+z_2^\nu+\cdots+z_M^\nu|\geqslant\left(\frac{1}{48e^2}\cdot\frac{N}{2N+m}\right)^N, \tag{1.11}$$

where ν runs through integers.

This theorem enables one to arrive at the estimate (1.10). In order to get estimates of the type (1.9) we must start from a one-sided analogue of (1.11). Here is the one which will enable us to carry out the task‡:

† Throughout this paper $c_1, c_2, \ldots$ stand for positive numerical constants.

‡ I quote this one-sided theorem thanks to Professor P. Turán's kindness and namely from his letter dated 29 November, 1959. The theorem is due to appear in *Acta Mathematica Acad. Scient. Hungaricae.*

TURAN'S ONE-SIDED THEOREM. *Let $z_1, z_2, \ldots, z_n$ be complex numbers such that*

$$|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n|, \quad |z_1| \geqslant 1;$$

if for a κ with $0 < \kappa \leqslant \pi/2$ we have

$$\kappa \leqslant |\arg z_j| \leqslant \pi \quad (j = 1, 2, \ldots, n), \tag{1.12}$$

and if m is a positive integer, then we have

$$\max_{m+1 \leqslant \nu \leqslant m+10n/\kappa} \Re \sum_{j=1}^{n} z_j^{\nu} \geqslant \left(\frac{1}{81(m+n)}\right)^{10n^3/\kappa} \tag{1.13}$$

and

$$\min_{m+1 \leqslant \nu \leqslant m+10n/\kappa} \Re \sum_{j=1}^{n} z_j^{\nu} \leqslant -\left(\frac{1}{81(m+n)}\right)^{10n^3/\kappa} \tag{1.14}$$

where ν runs through integers.

Applying this to our question we shall prove the following

THEOREM. *Let $\rho_0 = \beta_0 + i\gamma_0$, $\beta_0 \geqslant \frac{1}{2}$, $\gamma_0 > 0$ be an arbitrary ζ-zero. Then for*

$$T > \max\left(c_2, \exp\exp(\log^2 \gamma_0)\right) \tag{1.15}$$

we have the inequalities

$$\max_{1 \leqslant t \leqslant T} \{\psi(t) - t\} > T^{\beta_0} \exp\left(-15 \frac{\log T}{\sqrt{(\log\log T)}}\right), \tag{1.16}$$

$$\min_{1 \leqslant t \leqslant T} \{\psi(t) - t\} < -T^{\beta_0} \exp\left(-15 \frac{\log T}{\sqrt{(\log\log T)}}\right), \tag{1.17}$$

and also

$$\max_{2 \leqslant t \leqslant T} \{\Pi(t) - \mathrm{li}\, t\} > T^{\beta_0} \exp\left(-15 \frac{\log T}{\sqrt{(\log\log T)}}\right), \tag{1.18}$$

$$\min_{2 \leqslant t \leqslant T} \{\Pi(t) - \mathrm{li}\, t\} < -T^{\beta_0} \exp\left(-15 \frac{\log T}{\sqrt{(\log\log T)}}\right). \tag{1.19}$$

Remark. Note that the inequality

$$\max_{2 \leqslant x \leqslant T} |\pi(x) - \mathrm{li}\, x| > T^{\beta_0} \exp\left(-15 \frac{\log T}{\sqrt{(\log\log T)}}\right)$$

(under the assumptions of the theorem) is an obvious consequence of (1.19).

It does not seem unlikely that one might improve the Skewes constant expexpexpexp(7·705) by working along the lines used in this paper. It would probably only require a careful choice of the parameters which occur in the proof.

Another point where the estimates (1.16)-(1.19) prove useful is the following: let $W(n)$, $V(n)$ stand, respectively, for the number of changes of sign in the sequences

$$\psi(1)-1,\ \psi(2)-2,\ \ldots,\ \psi(n)-n;$$

$$\pi(2)-\mathrm{li}\,2,\ \pi(3)-\mathrm{li}\,3,\ \ldots,\ \pi(n)-\mathrm{li}\,n.$$

It has been proved by Ingham [1] that

$$\varliminf_{n\to\infty}\frac{V(n)}{\log n}>0$$

if the following is true:

there exists a ζ-zero $\rho_0=\sigma_0+it_0$ such that $\zeta(s)\neq 0$ in the half-plane $\sigma>\sigma_0$.

Estimates (1.16) and (1.17) give for all sufficiently large T

$$\max_{T^{\frac{1}{3}}<n\leqslant T}\{\psi(n)-n\}>\tfrac{1}{2}T^{\frac{1}{2}}\exp\left(-15\frac{\log T}{\sqrt{(\log\log T)}}\right)>0$$

and

$$\min_{T^{\frac{1}{3}}<n\leqslant T}\{\psi(n)-n\}<-\tfrac{1}{2}T^{\frac{1}{2}}\exp\left(-15\frac{\log T}{\sqrt{(\log\log T)}}\right)<0,$$

which means that there is at least one sign-change of $\psi(n)-n$ between $T^{\frac{1}{3}}$ and T. Hence without any conjecture

$$W(n)>\frac{\log\log n}{\log 3}+O(1).$$

Of course we can state the same for sign-changes of the difference

$$\Pi(x)-\mathrm{li}\,x.$$

To the related question concerning $\pi(x)-\mathrm{li}\,x$ I shall return in another paper.†

2. Before passing on to the proof of the theorem we give the following

LEMMA. *Let $T>c_3$, $\rho=\beta+i\gamma$ run through non-trivial zeros of $\zeta(s)$. Then there exists an $\alpha_0=\alpha_0(T)$, $10\leqslant\alpha_0\leqslant 12$, such that writing*

$$x_0=\frac{\log T}{\alpha_0(\log T)/(\log\log T)+(\log T)^{4/5}},\tag{2.1}$$

we have

$$\left|\arg\left(\frac{e^{i\gamma x_0}}{\beta+i\gamma}\right)\right|\geqslant\frac{c_4}{|\gamma|^5\log|\gamma|}\tag{2.2}$$

for all $\rho=\beta+i\gamma$.

† *Added in proof:* I have proved in the meantime

$$V(n)\geqslant e^{-25}\log\log\log\log n,$$

for

$$n\geqslant\exp\exp\exp\exp\exp 25.$$

This result will be published in *Acta Arithmetica.*

Proof. On letting α_0 range in [10, 12], the numbers (2.1) will certainly fill the interval

$$I = \left(\frac{\log\log T}{12}, \frac{\log\log T}{10\cdot 1}\right).$$

We are seeking now for the measure of those $x \in I$ for which the inequality

$$\left|\tan(x\gamma) - \frac{\gamma}{\beta}\right| \leqslant \frac{\epsilon}{\gamma^2}, \tag{2.3}$$

holds for a fixed $\rho = \beta + i\gamma$ and some $\epsilon > 0$.

The set in question will be denoted by $\underset{x \in I}{\mathsf{E}}\{x\colon (2.3) \text{ with a } \rho = \beta + i\gamma\}$ For a fixed $\rho = \beta + i\gamma$ and integer k, let $x_k(\rho)$ be the number from I satisfying the following conditions, if such a number exists:

1. $(k - \frac{1}{2})\pi \leqslant x_k(\rho) . \gamma < (k + \frac{1}{2})\pi$,
2. $\tan\left(x_k(\rho).\gamma\right) = \frac{\gamma}{\beta}$.

The number of k's for which $x_k(\rho)$ exists is obviously

$$\leqslant c_5 |\gamma| \log\log T.$$

We have then (always with one fixed $\rho = \beta + i\gamma$)

$$\underset{(k-\frac{1}{2})\pi \leqslant x\gamma < (k+\frac{1}{2})\pi}{\mathsf{E}} \left\{x\colon \left|\tan(x\gamma) - \tan\left(x_k(\rho).\gamma\right)\right| \leqslant \frac{\epsilon}{\gamma^2}\right\}$$

$$\subset \underset{(k-\frac{1}{2})\pi \leqslant x\gamma < (k+\frac{1}{2})\pi}{\mathsf{E}} \left\{x\colon |x - x_k(\rho)| \leqslant \frac{\epsilon}{|\gamma|^3}\right\},$$

whence

$$m \underset{(k-\frac{1}{2})\pi \leqslant x\gamma < (k+\frac{1}{2})\pi}{\mathsf{E}} \left\{x\colon \left|\tan(x\gamma) - \frac{\gamma}{\beta}\right| \leqslant \frac{\epsilon}{\gamma^2}\right\} \leqslant \frac{2\epsilon}{|\gamma|^3};$$

consequently

$$m \underset{x \in I}{\mathsf{E}}\{x\colon (2.3) \text{ with a } \rho = \beta + i\gamma\} \leqslant \frac{2\epsilon}{\gamma^2} . c_5 \log\log T.$$

Finally, on choosing a sufficiently small numerical $\epsilon > 0$, we obtain

$$m \underset{x \in I}{\mathsf{E}} \left\{x\colon \left|\tan(x\gamma) - \frac{\gamma}{\beta}\right| > \frac{\epsilon}{\gamma^2} \text{ for all } \rho = \beta + i\gamma\right\}$$

$$> mI - \sum_{\gamma} \frac{2\epsilon}{\gamma^2} c_5 \log\log T > mI - \frac{\log\log T}{200} > 0.$$

Hence there exists a number (2.1) such that the inequality

$$\left|\tan(x_0\gamma) - \frac{\gamma}{\beta}\right| > \frac{c_6}{\gamma^2} \tag{2.4}$$

holds for all $\rho = \beta + i\gamma$.

We shall now derive (2.2) from (2.4). First of all we have

$$\Im\left(\frac{e^{i\gamma x_0}}{\beta+i\gamma}\right)=\frac{\beta\sin(\gamma x_0)-\gamma\cos(\gamma x_0)}{\beta^2+\gamma^2}.$$

Let us distinguish two cases:

(a) $|\cos(\gamma x_0)|\leqslant c_7\gamma^{-2}$ with a certain (sufficiently small) c_7.

Then we get

$$\left|\Im\left(\frac{e^{i\gamma x_0}}{\beta+i\gamma}\right)\right|\geqslant\frac{\beta|\sin(x_0\gamma)|-c_7/|\gamma|}{\beta^2+\gamma^2}>\frac{\frac{1}{2}\beta-c_7/|\gamma|}{\beta^2+\gamma^2}.$$

As is well known, we have (see *e.g.* [**3**], p. 53)

$$\beta>\frac{c_8}{\log|\gamma|},$$

whence

$$\left|\Im\left(\frac{e^{i\gamma x_0}}{\beta+i\gamma}\right)\right|>\frac{c_9}{\gamma^2\log|\gamma|}.$$

This clearly gives (2.2).

(b) $|\cos(\gamma x_0)|>c_7\gamma^{-2}$.

Here we have by (2.4)

$$|\beta\sin(x_0\gamma)-\gamma\cos(x_0\gamma)|>\frac{\beta\,.\,c_{10}}{\gamma^4}>\frac{c_{11}}{\gamma^4\log|\gamma|},$$

whence

$$\left|\Im\left(\frac{e^{i\gamma x_0}}{\beta+i\gamma}\right)\right|>\frac{c_{12}}{\gamma^6\log|\gamma|}$$

and (2.2) follows.

3. Before anything else we shall investigate the sum

$$S(T,\nu)=\sum_{|\gamma|\leqslant 2\log^{\frac{1}{10}}T}\left(\frac{e^{(\rho-\beta_0)x_0}}{\rho/|\rho_0|}\right)^{\nu} \tag{3.1}$$

with x_0 from the lemma, $\rho=\beta+i\gamma$ being ζ-zeros, ρ_0 as formulated in the theorem and ν being an integer.

The number n of terms in (3.1) is, as is well known,

$$\leqslant c_{13}\log^{\frac{1}{10}}T(\log\log T).$$

We wish to estimate $S(T,\nu)$ by means of Turán's one-sided theorem. Choose

$$m=\left[\frac{\alpha_0\log T}{\log\log T}\right]$$

where α_0 is as in the lemma. By (2.2) we may put

$$\kappa=\frac{c_{14}}{(\log T)^{\frac{4}{5}}(\log\log T)}.$$

We obtain then from (1.14):

there exists an integer ν_1 such that

$$\alpha_0 \frac{\log T}{\log\log T} \leqslant \nu_1 \leqslant \alpha_0 \frac{\log T}{\log\log T} + (\log T)^{4/5} \tag{3.2}$$

and
$$\Re S(T, \nu_1) < -\exp\left(-c_{15}\frac{\log T}{\log\log T}\right). \tag{3.3}$$

Similarly we have from (1.13):

there exists an integer ν_2 such that

$$\alpha_0 \frac{\log T}{\log\log T} \leqslant \nu_2 \leqslant \alpha_0 \frac{\log T}{\log\log T} + (\log T)^{4/5} \tag{3.4}$$

and
$$\Re S(T, \nu_2) > \exp\left(-c_{15}\frac{\log T}{\log\log T}\right). \tag{3.5}$$

4. *Proof of the theorem.* We start from the integral

$$J(T) = \frac{1}{2\pi i}\int_{(2)} \left(\frac{e^{x_0 s}}{s}\right)^{\nu_1} \left(-\frac{\zeta'}{\zeta}(s) - \zeta(s)\right) ds.$$

On the one hand we obtain by standard methods

$$J(T) = \sum_{n \leqslant e^{x_0 \nu_1}} \{\Lambda(n) - 1\} \frac{\log^{\nu_0 - 1}(e^{x_0 \nu_0}/n)}{(\nu_1 - 1)!},$$

and on the other by Cauchy's theorem of residues

$$J(T) = -\sum_{\rho} \left(\frac{e^{\rho x_0}}{\rho}\right)^{\nu_1} + \operatorname*{Res}_{s=0} \frac{e^{x_0 \nu_1 s}}{s^{\nu_1}} \left(-\frac{\zeta'}{\zeta}(s) - \zeta(s)\right)$$
$$+ \frac{1}{2\pi i}\int_{(-3/2)} \left(\frac{e^{x_0 s}}{s}\right)^{\nu_1} \left(-\frac{\zeta'}{\zeta}(s) - \zeta(s)\right) ds.$$

The latter integral is $O(1)$. Furthermore we have

$$\operatorname*{Res}_{s=0} \frac{e^{x_0 \nu_1 s}}{s^{\nu_1}} \left(-\frac{\zeta'}{\zeta}(s) - \zeta(s)\right) = \frac{1}{2\pi i}\int_{|s| = 1/x_0} \frac{e^{x_0 \nu_1 s}}{s^{\nu_1}} \left(-\frac{\zeta'}{\zeta}(s) - \zeta(s)\right) ds$$
$$= O\left(\exp\left(c_{16}\frac{\log T}{\log\log T} \log\log\log T\right)\right).$$

Putting all together we obtain

$$\sum_{n \leqslant e^{x_0 \nu_1}} \{\Lambda(n) - 1\} \frac{\log^{\nu_1 - 1}(e^{x_0 \nu_1}/n)}{(\nu_1 - 1)!}$$
$$= -\sum_{\rho} \left(\frac{e^{x_0 \rho}}{\rho}\right) + O\left(\exp\left(c_{16}\frac{\log T}{\log\log T} \log\log\log T\right)\right). \tag{4.1}$$

Further we have

$$\sum_{|\gamma| > 2\log^{\frac{1}{10}} T} \left(\frac{e^{x_0 \rho}}{\rho}\right)^{\nu_1} = O\left(e^{x_0 \nu_1} \sum_{\gamma > 2\log^{\frac{1}{10}} T} \frac{1}{\gamma^{\nu_1}}\right) = O\left(T\int_{\log^{\frac{1}{10}} T}^{\infty} \frac{\log x}{x^{\nu_1}} dx\right) = O(1).$$

Hence, and from (4.1), we get

$$\sum_{n \leqslant e^{x_1} \nu_0} \{\Lambda(n)-1\} \frac{\log^{\nu_1-1}(e^{x_0}\nu_1/n)}{(\nu_1-1)!} = -\frac{e^{\nu_1 \beta_0 x_0}}{|\rho_0|^{\nu_1}} S(T, \nu_1) + O(T^{\frac{1}{2}}). \qquad (4.2)$$

Taking real parts in (4.2),

$$\sum_{n \leqslant e^{x_0} \nu_1} \{\Lambda(n)-1\} \frac{\log^{\nu_1-1}(e^{x_0}\nu_1/n)}{(\nu_1-1)!} = -\frac{e^{\nu_1 \beta_0 x_0}}{|\rho_0|^{\nu_1}} \Re S(T, \nu_1) + O(T^{\frac{1}{2}}). \quad (4.3)$$

Similarly

$$\sum_{n \leqslant e^{x_0} \nu_2} \{\Lambda(n)-1\} \frac{\log^{\nu_2-1}(e^{x_0}\nu_2/n)}{(\nu_2-1)!} = -\frac{e^{\nu_2 \beta_0 x_0}}{|\rho_0|^{\nu_2}} \Re S(T, \nu_2) + O(T^{\frac{1}{2}}). \quad (4.4)$$

Let us notice that (2.1) and (3.2) give

$$T \exp\Big(-\tfrac{1}{10}\log^{4/5} T(\log\log T)\Big) \leqslant e^{x_0 \nu_1} \leqslant T,$$

whence and by (1.15), (3.3) the right-hand side of (4.3) is

$$\geqslant \frac{T^{\beta_0}\exp\Big(-\frac{1}{10}\log^{4/5} T(\log\log T)\Big)}{\Big(2\exp\sqrt{(\log\log T)}\Big)^{13(\log T)/(\log\log T)}} \cdot \exp\left(-c_{15}\frac{\log T}{\log\log T}\right) + O(T^{\frac{1}{2}})$$

$$> T^{\beta_0}\exp\left(-14\frac{\log T}{\sqrt{(\log\log T)}}\right). \qquad (4.5)$$

On the other hand the left-hand side of (4.3) is

$$\int_{\frac{1}{2}}^{e^{x_0}\nu_2} \frac{\log^{\nu_1-1}(e^{x_0}\nu_1/t)}{(\nu_1-1)!} d\Big(\psi(t)-[t]\Big)$$

$$= \int_1^{e^{x_0}\nu_1} \frac{\{\psi(t)-[t]\}}{(\nu_1-1)!} d\Big(-\log^{\nu_1-1}\Big(\frac{e^{x_0}\nu_1}{t}\Big)\Big)$$

$$\leqslant \max_{1\leqslant t\leqslant T}\{\psi(t)-[t]\}\frac{\log^{\nu_1-1} T}{(\nu_1-1)!}$$

$$\leqslant \max_{1\leqslant t\leqslant T}\{\psi(t)-[t]\} \cdot \exp\Big(c_{17}\frac{\log T}{\log\log T}\log\log\log T\Big)$$

This and (4.5) gives (1.16).

(1.17) follows in a similar way. The left-hand side of (4.4) is

$$\int_1^{e^{x_0}\nu_2} \frac{\{\psi(t)-[t]\}}{(\nu_2-1)!} d\Big(-\log^{\nu_2-1}\Big(\frac{e^{x_0}\nu_0}{t}\Big)\Big)$$

$$\geqslant \min_{1\leqslant t\leqslant T}\{\psi(t)-[t])\} \cdot \exp\left(c_{17}\frac{\log T}{\log\log T}\log\log\log T\right),$$

and its right-hand side, owing to (3.4), (3.5), is

$$< -T^{\beta_0} \exp\left(-14\frac{\log T}{\sqrt{(\log\log T)}}\right).$$

The proof of (1.18) and (1.19) is now a matter of a few lines. We put (4.3) in the form

$$\sum_{2\leqslant n\leqslant e^{x_0}\nu_1}\left(\frac{\Lambda(n)}{\log n}-\frac{1}{\log n}\right)\frac{\log^{\nu_1-1}(e^{x_0}\nu_1/n)}{(\nu_1-1)!}\log n = \frac{-e^{\nu_1\beta_0 x_0}}{|\rho_0|^{\nu_1}}\Re S(T,\nu_1)+O(T^{\frac{1}{4}}),$$

and have further

$$\int_1^{e^{x_0}\nu_1}\left\{\Pi(t)-\sum_{2\leqslant n\leqslant t}\frac{1}{\log n}\right\}d\left(-\log t\frac{\log^{\nu_1-1}(e^{x_0}\nu_1/t)}{(\nu_1-1)!}\right)$$

$$\geqslant T^{\beta_0}\exp\left(-14\frac{\log T}{\sqrt{(\log\log T)}}\right).$$

Noting that the derivative

$$\left(-\log t\frac{\log^{\nu_1-1}(e^{x_0}\nu_1/t)}{(\nu_1-1)!}\right)'=\frac{\nu_1\log^{\nu_1-2}(e^{x_0}\nu_1/t)}{t(\nu_1-1)!}\log\frac{t}{e^{x_0}}$$

is positive for $e^{x_0}<t<e^{x_0}\nu_1$, we obtain

$$\max_{e^{x_0}\leqslant t\leqslant e^{x_0}\nu_1}\left\{\Pi(t)-\sum_{2\leqslant n\leqslant t}\frac{1}{\log n}\right\}.\int_{e^{x_0}}^{e^{x_0}\nu_1}d\left(-\log t\frac{\log^{\nu_1-1}(e^{x_0}\nu_1/t)}{(\nu_1-1)!}\right)$$

$$\geqslant T^{\beta_0}\exp\left(-14\frac{\log T}{\sqrt{(\log\log T)}}\right)+O\left(\exp\left(c_{18}\frac{\log T}{\log\log T}\log\log\log T\right)\right),$$

whence (1.18). And (1.19) gives no more trouble.

References.

1. A. E. Ingham, "A note on the distribution of primes", *Acta Arithmetica*, 1 (1936), 201–211.
2. S. Skewes, "On the difference $\pi(x)-\text{li}\,x$ (II)", *Proc. London Math. Soc.* (3), 5 (1955), 48–70.
3. E. C. Titchmarsh, *The theory of the zeta-function of Riemann* (Oxford, 1951).
4. P. Turán, "On the remainder-term of the prime-number formula (I)", *Acta. Math. Hung.*, 1 (1950), 48–63.
5. ———, *Eine neue Methode in der Analysis und deren Anwendungen* (Budapest, 1953).

A. Mickiewicz University,
Poznań, Poland.