the matrix $C^{\alpha}_{ij}\lambda^i$ is $n-1$ for arbitrary λ^i. In addition to equations (1) the constants of structure of G are shown by Cartan ([3], pp. 49-50) to satisfy

$$C^{\alpha}_{ij} = k\, C^{j}_{\alpha i},$$

where k is a non-zero constant. It follows that the rank of the matrix $C^{i}_{j\alpha}\lambda^j$ is $n-1$. Hence there exist $n-1$ linearly independent infinitesimal generators of A_H, and the proof is complete.

References.

1. É. Cartan, " Les groupes réels simples finis et continus ", *Ann. Ec. Norm.*, 31 (1914), 263–355.
2. ———, " Sur certaines formes riemanniennes remarquables des géométries à groupe fondamental simple ", *Ann. Éc. Norm.*, t.44 (1927), 345–467.
3. ———, *La théorie des groupes finis et continus et l'analysis situs, Mémorial Sc. Math.* 62 (Gauthier-Villars, 1930).
4. C. Chevalley, *Theory of Lie Groups* (Princeton, 1946).
5. E. T. Copson and H. S. Ruse, " Harmonic Riemannian spaces ", *Proc. Roy. Soc. Edinburgh*, 60 (1940), 117–133.
6. L. P. Eisenhart, *Continuous groups of transformations* (Princeton, 1933).
7. H. Hopf and W. Rinow, " Über den Begriff der vollständigen differentialgeometrischen Fläche ", *Comm. Math. Helv.*, 3 (1931), 209–225.
8. A. J. Ledger, " Harmonic homogeneous spaces of Lie groups ", *Journal London Math. Soc.*, 29 (1954), 345–347.
9. A. Lichnerowicz, " Sur les espaces riemanniens complètement harmoniques ", *Bull. Soc. Math. de France*, 72 (1944), 146–168.
10. A. G. Walker, " Symmetric harmonic spaces ", *Journal London Math. Soc.*, 21 (1946), 47–57.
11. H. C. Wang, " Two-point homogeneous spaces ", *Annals of Mathematics*, 55 (1952), 177–191.

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NOTE ON THE DISTRIBUTION OF PRIME NUMBERS

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I have recently carried out some numerical studies of the distribution of prime numbers into certain arithmetic progressions, on which I hope to report fully in due course. Some of the results are so surprising, however, as to be worth making the subject of a separate note.

Let $\pi_1(x)$, $\pi_3(x)$ be the numbers of primes $1 < p \leqslant x$ of the forms $4n+1$ and $4n+3$ respectively. Littlewood's method† shows that the difference

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† See A. E. Ingham, *The distribution of prime numbers* (Cambridge, 1932), Ch. 5.

$\pi_3(x)-\pi_1(x)$ changes sign infinitely often for sufficiently large x, but it gives no indication of how large the least such x may be. In the comparable case of $\pi(x)-\operatorname{li} x$, Skewes* has obtained limits below which a change of sign must occur; these are far beyond the limits of present or projected evaluation of $\pi(x)$, and it is likely that no value of x for which $\pi(x) > \operatorname{li} x$ will ever be exhibited. In contrast, however, I have found values of x for which $\pi_1(x) > \pi_3(x)$. The smallest of these is $x = 26{,}861$, for which $\pi_1(x) = 1473$, $\pi_3(x) = 1472$. This value of x is isolated. There are a number of others in the range 616,000 to 634,000; the greatest difference is for $x = 623{,}681$, for which $\pi_1(x) = 25{,}444$, $\pi_3(x) = 25{,}436$. There are no other such values of x below 3,000,000, which is the present limit of my studies.

A closely comparable result relates to the number of primes $a+bi$ of the Gaussian number field. Let $\pi_i(x)$ be the number of primes $a+bi$ with $a \geqslant 1$, $b \geqslant 0$, $1 < a^2+b^2 \leqslant x$. Then $\pi_i(x) \sim \operatorname{li} x$, but $\pi_i(x)-\operatorname{li} x$ is predominantly negative, like $\pi(x)-\operatorname{li} x$. We may express $\pi_i(x)$ in the form

$$\pi_i(x) = 2\pi_1(x)+\pi_3(\sqrt{x})+1,$$

since each rational prime $p_1 = a^2+b^2 \leqslant x$ gives two Gaussian primes in the range, each rational prime $p_3 = 4n+3 \leqslant \sqrt{x}$ is a Gaussian prime in the range, and the prime $1+i$ is also to be included. The dominant term in this expression is $2\pi_1(x)$, and where $\pi_1(x)$ is unusually large, for example where $\pi_1(x) > \pi_3(x)$, we may expect a change of sign of $\pi_i(x)-\operatorname{li} x$. I have in fact found values of x in the range 615,000 to 626,000 for which $\pi_i(x) > \operatorname{li} x$; the greatest difference being for $x = 617{,}537$, for which $\pi_i(x) = 50{,}509 = \operatorname{li} x+19{\cdot}5$.

The dominant terms in the explicit formula for $\pi_3(x)-\pi_1(x)$ are

$$\tfrac{1}{2}\operatorname{li}\sqrt{x}+\sum_{\rho}\operatorname{li} x^{\rho},$$

where the summation is over all complex zeros ρ of the Dirichlet function

$$L(s, \chi) = 1-3^{-s}+5^{-s}-7^{-s}+\dots.$$

I find that of the first 20 pairs of zeros, which range from $\frac{1}{2}\pm 6{\cdot}020948i$ to $\frac{1}{2}\pm 49{\cdot}723129i$, 16 pairs give negative contributions to the sum in the explicit formula for $x = 620{,}000$, 12 of the pairs of terms having phase angles between $\frac{2}{3}\pi$ and $\frac{4}{3}\pi$. Pairs of zeros after the 20th give contributions of fairly uniformly mixed signs. This predominant negative bias of the first 20 pairs of zeros, which is unexpected for x so small, clearly correlates with the changes of sign reported above.

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* S. Skewes, "On the difference $\pi(x)-\operatorname{li} x$" (I), *Journal London Math. Soc.*, 8 (1933), 277–283; (II) *Proc. London Math. Soc.* (3), 5 (1955), 48–70.

The counts of primes were carried out on the EDSAC at the University Mathematical Laboratory, Cambridge, using tapes of differences between consecutive primes constructed by J. C. P. Miller on the EDSAC, and the values and zeros of the Dirichlet function were computed by C. B. Haselgrove on the EDSAC. This work is still unpublished.

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GENERAL INTEGRABILITY THEOREMS FOR POWER SERIES

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1. Suppose that

$$f(x) = \sum_{0}^{\infty} c_n x^n \quad (0 \leqslant x < 1;\ c_n \geqslant 0). \tag{1.1}$$

Heywood ([1], Theorem 1) has proved that *if* $\gamma < 1$, *then*

$$(1-x)^{-\gamma} f(x) \in L(0, 1)$$

if and only if $\Sigma n^{\gamma-1} c_n$ *is convergent.* The purpose of this note is to prove simple general theorems which contain this result as a special case. Thus the case $0 \leqslant \gamma < 1$ is included in the following theorem.

THEOREM 1. *Let* $f(x)$ *be given by* (1.1), *and suppose that* $\phi(x)$ *is non-negative and non-decreasing in* (0, 1), *and that* $\phi(x) \in L(0, 1)$. *Let*

$$\Phi(n) = \int_{1-1/n}^{1} \phi(x)\, dx.$$

Then $f(x)\phi(x) \in L(0, 1)$ *if and only if*

$$\Sigma c_n \Phi(n) \tag{1.2}$$

is convergent.

To achieve a similar generalization of Heywood's result for $\gamma < 0$, we are compelled to impose more exacting restrictions on $\phi(x)$, as described in the following theorem.

THEOREM 2. *Suppose that there is an integer* $p \geqslant 1$ *such that* $\phi(x)$, $\phi'(x)$, ..., $\phi^{(p-1)}(x)$ *are absolutely continuous for* $0 \leqslant x \leqslant 1$ *and vanish at* $x = 1$. *Suppose also that* $\phi^{(p)}(x)$ *has constant sign, and* $|\phi^{(p)}(x)|$ *is non-decreasing, in the set in* (0, 1) *where* $\phi^{(p)}(x)$ *exists (which set is, a fortiori, of*

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