ON THE DIFFERENCE $\pi(x) - \mathrm{li}\, x$ (II)

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Introduction

1. Let $\pi(x)$ denote, as usual, the number of primes less than or equal to x which we suppose always to be not less than 2, and let

$$\mathrm{li}\, x = \lim_{\epsilon \to 0} \left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^{x} \right) \frac{du}{\log u}.$$

The difference $d(x) = \pi(x) - \mathrm{li}\, x$ is negative for all values of x up to 10^7, and for all the special values of x for which $\pi(x)$ has been calculated (e.g. $d(x) = -1757$ for $x = 10^9$). Littlewood (**1**) proved in 1914, however, that $d(x)$ changes sign infinitely often, and in particular there exists an X such that $d(x) > 0$ for some $x < X$. This last result is our present subject. Littlewood's method depends on an 'explicit formula', as does all subsequent work, including the present paper.

If θ is the upper bound of the real parts of the zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function $\zeta(s)$, the 'Riemann hypothesis' [(RH) for short] is that $\theta = \frac{1}{2}$; if this is false, then $\frac{1}{2} < \theta \leqslant 1$. In this latter case it had long been known that, for each positive ϵ, $d(x)/x^{\theta-\epsilon}$ oscillates, as x tends to infinity, over a range including ± 1. In proving the mere existence of an X it is therefore permissible to assume (RH), and Littlewood naturally did this.

Littlewood's theorem is a 'pure existence theorem', and does not provide, even when (RH) is assumed, an explicit numerical X.

When we face the problem of a numerical X, free of hypotheses, the argument falls naturally into three stages.

(i) A new method is found which assumes (RH) and provides a numerical $X = X_1$. I gave such a method in 1933 (**3**). In the meantime Ingham (**4**) has developed an alternative method (which he applies to the more general problem of the infinity of changes of sign of $d(x)$). This, adapted to our more special case (one change of sign) and with some further modifications, gives a much better X_1 than my original paper did; the argument is given in full in Part I. One of the advantages of the new method is that we can largely eliminate the ρ's beyond a given point; we operate in fact with the 269 ρ's with $0 < \gamma < 500$, whose position is approximately known.

(ii) (This is easy.) The whole argument in (i) is based on the explicit formula for $\psi_0(x) = \frac{1}{2}\{\psi(x+0)+\psi(x-0)\}$ (in the usual notation of the subject) (**2**). This is

$$\psi_0(x)-x = -\sum_\rho \frac{x^\rho}{\rho}-\frac{\zeta'(0)}{\zeta(0)}-\tfrac{1}{2}\log\left(1-\frac{1}{x^2}\right) \quad \text{for } x > 1.$$

In the course of the proof the terms of the series $\sum x^\rho/\rho$ with $|\gamma| \geqslant G = X_1^3$ can (roughly) be rejected as negligible, (RH) *or not*. It is enough, in other words, to suppose, instead of (RH), only that $\beta = \frac{1}{2}$ for those γ's satisfying $|\gamma| < G$. This hypothesis can in turn be weakened; for $x < X_1$, the $|x^{\beta+i\gamma}|$ concerned differ from $|x^{\frac{1}{2}+i\gamma}|$ by something negligible, provided the β's concerned satisfy

$$b = \beta-\tfrac{1}{2} \leqslant B = X_1^{-3}\log^{-2}X_1.$$

With minor adjustments, then, the proof in (i) can be made to provide an X_1 [actual value $\exp\exp\exp(7{\cdot}703)$] subject only to the double modification of (RH) explained above. This modification, which we will call (H), is, to repeat,

(H) *Every zero $\rho = \beta+i\gamma$ for which $|\gamma| < G = X_1^3$ is such that*

$$b = \beta-\tfrac{1}{2} \leqslant B = X_1^{-3}\log^{-2}X_1.$$

(iii) Since (H) leads to an X_1, it remains only to show that (NH), the negation of (H), leads to an X_2, i.e. that $d(x) > 0$ for some $x < X_2$. Now (NH) asserts the existence of a $\rho = \rho_0 = \beta_0+i\gamma_0$ with

$$0 < \gamma_0 < G = X_1^3, \qquad b_0 = \beta_0-\tfrac{1}{2} > B,$$

where

$$B = X_1^{-3}\log^{-2}X_1;$$

that is, it provides a more or less given ρ to the right of $\sigma = \frac{1}{2}$. In particular, it asserts that $\theta > \frac{1}{2}+B$, in which case an X_2 certainly *exists* in virtue of the old theorem about $d(x) > x^{\theta-\epsilon}$. It is natural to expect further that the proof of that theorem could use the existence of the special ρ_0 to provide a numerical X_2. But this turns out not to be so; the proof in question is another 'pure existence' one. Some further idea is called for, and I am in fact indebted to Professor Littlewood for the sketch of a method for the simpler problem of finding an X such that, for a given $h > 0$ and for some $x < X$, $\psi(x)-x > h\sqrt{x}$.

There is now a last unexpected point. In the past it has always been possible to work with the function $\psi(x)$ and its simpler explicit formula, with only a last minute switch, on established lines, to $\pi(x)$. But with (NH) this is no longer possible, and it is necessary to work, in finding X_2, with $\Pi_0(x) = \frac{1}{2}\{\Pi(x+0)+\Pi(x-0)\}$, where

$$\Pi(x) = \sum_{p^m \leqslant x} \frac{1}{m} = \sum_{m=1}^{M} \frac{1}{m}\pi(x^{1/m}), \qquad M = [\log x/\log 2].$$

The explicit formula for $\Pi_0(x)$ is, for $x > 1$,

$$\Pi_0(x) = \operatorname{li} x - \sum_{\rho} \operatorname{li} x^{\rho} + \int_x^{\infty} \frac{du}{(u^2-1)u\log u} - \log 2.$$

In the actual working out of the paper stages (i) and (ii) are telescoped, and (RH) never appears. In Part I we assume (H) from the first, and arrive at an $X_1 = \exp\exp\exp(7{\cdot}703)$. Part II then assumes (NH) and arrives at an X_2 differing negligibly† in expression from e^{X_1}: a (just) permissible X_2 is

$$10^{10^{10^{10^3}}}.$$

I wish in conclusion to express my humble thanks to Professor Littlewood, but for whose patient profanity this paper could never have become fit for publication.

Part I

2. We begin by collecting, in Lemma 1, some results about the zeros ρ. The fundamental theorem underlying all its results is as follows (5).

Let $N(T)$ be the number of roots $\rho = \beta+i\gamma$ of the ζ-function satisfying $0 \leqslant \beta < 1$, $0 < \gamma \leqslant T$. Then

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + R(T), \tag{1}$$

where $$|R(T)| < (0{\cdot}137)\log T + (0{\cdot}443)\log\log T + 4{\cdot}350.$$

We make use also of the known values of $\gamma_1, \gamma_2, \ldots, \gamma_{29}$, that is, all the γ's satisfying $0 < \gamma \leqslant 100$. We have now

Lemma 1. *For all $T \geqslant \gamma_1 = 14{\cdot}13\ldots$,*

(i) $$\sum_{0<\gamma<T} \frac{1}{\gamma} < \frac{1}{4\pi}\log^2 T,$$

(ii) $$\sum_{\gamma \geqslant T} \frac{1}{\gamma^2} < \frac{1}{2\pi}\frac{\log T}{T},$$

(iii) $$\sum_{\gamma>0} \frac{1}{\gamma^2} < 0{\cdot}0233.$$

For $|h| < \frac{1}{2}T$,

(iv) $$|N(T+h)-N(T)| < \frac{1}{2\pi}(|h|+1{\cdot}77)\log T + 8{\cdot}7.$$

We suppress throughout the details of purely numerical calculations.

† X_1^{100} differs negligibly (in its top index) from X_1. For this and similar reasons some of our approximations can be very crude; only in those bearing on a top index is refinement called for.

We obtain (i) from (1) and the formula†

$$\sum_{0<\gamma_n<T} \frac{1}{\gamma_n} = \sum_{n=1}^{29} \frac{1}{\gamma_n} + \int\limits_{\gamma_{30}}^{T} \frac{N^*(x)}{x^2}\,dx + \frac{N^*(T)}{T},$$

where $N^*(T) = N(T)-29$. Since $\sum\limits_{n=1}^{29} \frac{1}{\gamma_n} < 0{\cdot}5925$, this leads by straightforward calculation to

$$\sum_{0<\gamma<T} \frac{1}{\gamma} = \frac{1}{4\pi}\log^2 T - \frac{\log 2\pi}{2\pi}\log T + R_1(T),$$

where $|R_1(T)| < 0{\cdot}312$. This leads at once to (i).

We obtain (ii) similarly, from (1) and the formula†

$$\sum_{\gamma \geqslant T} \frac{1}{\gamma^2} = \int\limits_{T}^{\infty} \frac{2N(x)}{x^3}\,dx - \frac{N(T)}{T^2}.$$

Of the remaining results (iii) is known, and (iv) follows at once from (1).

3. Lemma 2. *Let $\psi_0(x)$ be defined, as usual, by*

$$\psi_0(x) = \tfrac{1}{2}\{\psi(x+0)+\psi(x-0)\},$$

where $\psi(x) = \sum\limits_{n\leqslant x} \Lambda(n)$. For $x > 1$, $\psi_0(x)$ is known to possess the explicit formula‡

$$\psi_0(x)-x = -\sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \tfrac{1}{2}\log\left(1-\frac{1}{x^2}\right), \tag{2}$$

where $\sum\limits_{\rho} \frac{x^\rho}{\rho}$ is defined as the limit of $\sum\limits_{|\gamma|\leqslant T} \frac{x^\rho}{\rho}$ as $T \to \infty$. If

$$\sum_{\rho} \frac{x^\rho}{\rho} = \sum_{|\gamma|\leqslant T} \frac{x^\rho}{\rho} + R(x,T),$$

then

(i) $|R(x,T)| < 1000\dfrac{x^3}{x-1}\dfrac{\log^2 T}{T} + 3\log x \quad (x \geqslant e,\ T \geqslant 9)$;

(ii) $|R(x,T)| < (0{\cdot}0001)x^{\frac{1}{4}} \quad (x \geqslant \exp(10^4),\ T \geqslant x^2)$;

(iii) $\left|\sum\limits_{\rho} \dfrac{x^\rho}{\rho}\right| < 3x\log x \quad (x \geqslant e)$.

The proof of (i) proceeds by straightforward calculation on the lines of the proof of (2); (ii) follows from (i); and (iii) follows from (2) and the definitions of $\psi_0(x)$ and $\psi(x)$, since

$$\sum_{n\leqslant x} \Lambda(n) \leqslant x\log x \quad \text{and} \quad |\psi_0(x)-\psi(x)| \leqslant \tfrac{1}{2}\log x.$$

† (2), 18, Theorem A. ‡ (2), 77, Theorem 29.

4. We shall for the present assume the following hypothesis, which we call (H), about the zeros $\rho = \beta + i\gamma$.

(H) *Let* $X_1 = \exp\exp\exp(7{\cdot}703)$, $G = X_1^3$, $B = X_1^{-3}\log^{-2}X_1$. *Then for every zero* ρ *such that* $|\gamma| < G$, β *satisfies*

$$b = \beta - \tfrac{1}{2} \leqslant B.$$

For reference we shall prefix (H) to those results which depend on the hypothesis (H).

(H) LEMMA 3. *Let* $\psi_1(x)$ *be defined, as usual, by*

$$\psi_1(x) = \int_1^x \psi(u)\,du = {\sum_{n\leqslant x}}'(x-n)\Lambda(n).$$

Then, on hypothesis (H),

(i) $|\psi_1(x) - \tfrac{1}{2}x^2| < 8x \quad (2 < x \leqslant e^8)$;

(ii) $|\psi_1(x) - \tfrac{1}{2}x^2| < \tfrac{1}{4}x^{\frac{3}{2}} \quad (e^8 < x < X_1)$.

For $x \geqslant 1$ we have the formula†

$$\psi_1(x) - \tfrac{1}{2}x^2 = -\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - x\frac{\zeta'(0)}{\zeta(0)} + \frac{\zeta'(-1)}{\zeta(-1)} - \sum_{r=1}^{\infty} \frac{x^{1-2r}}{2r(2r-1)}. \tag{3}$$

From Lemma 1 (ii) and (iii), and assuming (H), we have, for $x < X_1$,

$$x^{-\frac{3}{2}}\left|\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)}\right| \leqslant \sum_{|\gamma|<G}\left|\frac{x^{\rho-\frac{1}{2}}}{\rho(\rho+1)}\right| + \sum_{|\gamma|\geqslant G}\left|\frac{x^{\rho-\frac{1}{2}}}{\rho(\rho+1)}\right|$$

$$\leqslant X_1^B \sum_{|\gamma|<G} \frac{1}{\gamma^2} + X_1^{\frac{1}{2}} \frac{1}{\pi} \frac{\log G}{G}$$

$$\leqslant \tfrac{1}{10}.$$

Substituting in (3) and noting that

$$\zeta'(0)/\zeta(0) = \log 2\pi \quad \text{and} \quad |\zeta'(-1)/\zeta(-1)| < 1,$$

we obtain both (i) and (ii).

5. LEMMA 4.

(i) $\operatorname{li} u < (1{\cdot}0004)u/\log u \quad (u \geqslant \exp(4.10^3))$;

(ii) $\operatorname{li} u < 2u/\log u \quad (u \geqslant 2)$.

The value of li 2 is 1·04.... For $u > u_0 \geqslant e^2$, say,

$$\frac{1}{\log u} < \frac{d}{du}\left(\frac{u}{\log u}\right) \Big/ \left(1 - \frac{1}{\log u_0}\right).$$

† (2), 73, Theorem 28.

Hence
$$\mathrm{li}\,u = \mathrm{li}\,u_0 + \int_{u_0}^{u} \frac{dv}{\log v}$$
$$< u_0 + \frac{1}{1-1/\log u_0}\left[\frac{v}{\log v}\right]_{u_0}^{u},$$
and the result (i) follows by taking $u_0 = \exp(3.10^3)$. By taking $u_0 = e^2$, we find that (ii) is valid for $u \geqslant 8$, say, and, for $2 \leqslant u < 8$, (ii) is trivial.

6. We define $\Pi(x)$, as usual, by
$$\Pi(x) = \sum_{m=1}^{M} \frac{1}{m}\pi(x^{1/m}), \qquad M = [\log x/\log 2].$$

LEMMA 5. *For $x > \exp(4.10^3)$, either $\pi(\xi) > \mathrm{li}\,\xi$ for some ξ of $2 \leqslant \xi \leqslant x^{\frac{1}{2}}$, or else*
$$0 < \Pi(x)-\pi(x) < (1{\cdot}0005)x^{\frac{1}{2}}/\log x.$$

Supposing the former alternative to be false, we apply Lemma 4 (i) to the first term on the right-hand side of
$$\Pi(x)-\pi(x) = \tfrac{1}{2}\pi(x^{\frac{1}{2}}) + \sum_{m=3}^{M} \frac{1}{m}\pi(x^{1/m}),$$
and Lemma 4 (ii) to the remainder. Then
$$\Pi(x)-\pi(x) \leqslant \tfrac{1}{2}\mathrm{li}\,x^{\frac{1}{2}} + 2\sum_{m=3}^{M} x^{1/m}/\log x$$
$$< (1{\cdot}0004)x^{\frac{1}{2}}/\log x + (0{\cdot}0001)x^{\frac{1}{2}}/\log x,$$
and the desired result follows.

7. (H) LEMMA 6. *Let $P(x)$ be defined by*
$$P(x) = (\Pi(x)-\mathrm{li}\,x)-(\psi(x)-x)/\log x.$$
Then, on hypothesis (H),
$$|P(x)| < (0{\cdot}0005)x^{\frac{1}{2}}/\log x \quad (\exp(10^4) \leqslant x < X_1).$$

We have [(**2**), 64]
$$P(x) = \int_{2}^{x} \frac{\psi(u)-u}{u\log^2 u}\,du + \frac{2}{\log 2} - \mathrm{li}\,2,$$

and therefore, after integrating by parts,
$$|P(x)| \leqslant \left|\frac{\psi_1(x)-\frac{1}{2}x^2}{x\log^2 x} - \frac{\psi_1(2)-2}{2\log^2 2} + \frac{2}{\log 2} - \mathrm{li}\,2\right| + |J|, \tag{4}$$
where
$$J = \int_{2}^{x} \{\psi_1(u)-\tfrac{1}{2}u^2\}\, d\left\{\frac{1}{u\log^2 u}\right\}.$$

Now
$$|J| \leqslant \int\limits_2^x |\psi_1(u)-\tfrac{1}{2}u^2| \frac{4}{u^2\log^2 u}\,du = \int\limits_2^{e^8} + \int\limits_{e^8}^x.$$

Now apply Lemma 3. We have, on (H),
$$|J| < \int\limits_2^{e^8} \frac{32}{u\log^2 u}\,du + \int\limits_{e^8}^x \left(\frac{u^{\frac{1}{4}}}{\log^2 u}\right)\frac{1}{u^{\frac{3}{4}}}\,du.$$

Since $u^{\frac{1}{4}}/\log^2 u$ increases with u for $u > e^8$, it follows that, for
$$\exp(10^4) \leqslant x < X_1,$$
$$|J| < \frac{32}{\log 2} + \frac{x^{\frac{1}{4}}}{\log^2 x}4x^{\frac{1}{4}} < 48 + 0{\cdot}0004x^{\frac{1}{2}}/\log x.$$

Substituting this inequality in (4) and applying Lemma 3 (ii) to
$$(\psi_1(x)-\tfrac{1}{2}x^2)/x\log^2 x,$$
we obtain the required inequality.

(H) Lemma 7. *Assume hypothesis* (H). *Then for any given x satisfying* $\exp(10^4) \leqslant x < X_1$, *either*

(i) $\pi(\xi)-\mathrm{li}\,\xi > 0$ *for some ξ of* $2 \leqslant \xi \leqslant x^{\frac{1}{2}}$,

or else

(ii) '$\psi_0(x)-x > (1{\cdot}001)x^{\frac{1}{2}}$' *implies* '$\pi(x)-\mathrm{li}\,x > 0$'.

(i) is the first alternative of Lemma 5, and (ii) follows from the second one and Lemma 6, since
$$\begin{aligned}(\pi(x)-\mathrm{li}\,x)\log x &= \{\psi(x)-\psi_0(x)\}+\{\psi_0(x)-x\}-\\ &\quad -\{\psi_0(x)-x-(\Pi(x)-\mathrm{li}\,x)\log x\}-\{(\Pi(x)-\pi(x))\log x\}\\ &> 0+(1{\cdot}001)x^{\frac{1}{2}}-(0{\cdot}0005)x^{\frac{1}{2}}-(1{\cdot}0005)x^{\frac{1}{2}} = 0.\end{aligned}$$

8. (H) Lemma 8. *On hypothesis* (H),
$$\sum_{|\gamma|<G} \left|\frac{x^{\rho-\frac{1}{2}}}{\rho} - \frac{x^{i\gamma}}{i\gamma}\right| < 0{\cdot}0234 \quad (\exp(10^4) \leqslant x < X_1).$$

For brevity write the series on the left as $S(x)$, and let, as usual,
$$\rho-\tfrac{1}{2} = \beta-\tfrac{1}{2}+i\gamma = b+i\gamma.$$

Then
$$S(x) = \sum_{|\gamma|<G} \left|\frac{x^{b+i\gamma}}{\beta+i\gamma} - \frac{x^{i\gamma}}{i\gamma}\right| \leqslant \left(\sum_{|\gamma|<G}\frac{1}{\gamma^2}\right)\max_{|\gamma|<G}\{|\gamma(x^b-1)|+\beta\}.$$

Applying (H) and Lemma 1 (iii), we have therefore
$$\begin{aligned}S(x) &< (0{\cdot}0466)[G(2B\log X_1)+\tfrac{1}{2}+B]\\ &< 0{\cdot}0234.\end{aligned}$$

9. We now develop a modification of Ingham's argument. Consider the formula (see (4), 204 (6))

$$\int_a^b \chi(x)(\psi_0(x)-x)\,dx$$
$$= -\sum_\rho \frac{1}{\rho}\int_a^b \chi(x)x^\rho\,dx + \int_a^b \chi(x)\{\tfrac{1}{2}\log(1-x^{-2})^{-1}-\zeta'(0)/\zeta(0)\}\,dx, \quad (5)$$

where $1 < a < b < \infty$, and $\chi(x)$ is any function integrable in the sense of Lebesgue. Let

$$K(y) = \left(\frac{\sin\frac{1}{2}y}{\frac{1}{2}y}\right)^2,$$

so that, for real α,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} K(y)e^{i\alpha y}\,dy = \begin{cases} 1-|\alpha| & (|\alpha| \leqslant 1), \\ 0 & (|\alpha| > 1). \end{cases} \quad (6)$$

Let $T = 500$ and ω be any number satisfying $\omega > 2.10^4$. In (5) substitute

$$x = e^u, \qquad \chi(e^u) = e^{-\frac{1}{2}u}TK\{T(u-\omega)\}, \qquad a = e^{\frac{1}{2}\omega}, \qquad b = e^{\frac{3}{2}\omega}.$$

Then, writing for brevity

$$F(u) = \{\psi_0(e^u)-e^u\}e^{-\frac{1}{2}u}, \quad (7)$$

we have

$$\int_{\frac{1}{2}\omega}^{\frac{3}{2}\omega} TK\{T(u-\omega)\}F(u)\,du = -\sum_\rho \frac{1}{\rho}\int_{\frac{1}{2}\omega}^{\frac{3}{2}\omega} TK\{T(u-\omega)\}e^{(\rho-\frac{1}{2})u}\,du + R, \quad (8)$$

where, if we define $r(u)$ by

$$r(u) = e^{-\frac{1}{2}u}\{\tfrac{1}{2}\log(1-e^{-2u})^{-1}-\zeta'(0)/\zeta(0)\},$$

R is given by
$$R = \int_{\frac{1}{2}\omega}^{\frac{3}{2}\omega} TK\{T(u-\omega)\}r(u)\,du.$$

Since $\frac{1}{2}\omega \leqslant u \leqslant \frac{3}{2}\omega$ and $\omega > 2.10^4$, we have

$$|r(u)| < 2e^{-\frac{1}{2}\omega} < 0{\cdot}00001;$$

hence [in virtue of (6), with $\alpha = 0$]

$$|R| < (0{\cdot}00001)\int_{\frac{1}{2}\omega}^{\frac{3}{2}\omega} TK\{T(u-\omega)\}\,du$$
$$= (0{\cdot}00001)\int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y)\,dy < (0{\cdot}00001)2\pi. \quad (9)$$

Substituting $u = \omega + y/T$ in (8), we have from (8) and (9)

$$\int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y)F(\omega+y/T)\,dy = -\sum_{\rho}\frac{1}{\rho}\int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y)e^{(\rho-\frac{1}{2})(\omega+y/T)}\,dy + R, \quad (10)$$

where $|R| < (0{\cdot}00002)\pi$.

For the infinite series on the right-hand side of (10) we shall substitute the finite series

$$\sum_{|\gamma|<G}\frac{e^{i\gamma\omega}}{i\gamma}\int_{-\infty}^{\infty} K(y)e^{i\gamma y/T}\,dy,$$

where G is the number defined in hypothesis (H), § 4. The total error introduced will be the sum of three errors e_1, e_2, e_3, where e_1 comes from discarding those terms for which $|\gamma| \geqslant G$, e_2 from replacing $(e^{(\rho-\frac{1}{2})(\omega+y/T)})/\rho$ by $(e^{i\gamma(\omega+y/T)})/i\gamma$, and e_3 from extending the limits of integration from $\pm\frac{1}{2}T\omega$ to $\pm\infty$. We shall deal with these errors in separate lemmas.

10. Lemma 9. *For* $2.10^4 \leqslant \omega \leqslant \frac{1}{3}\log G$, *the error*

$$e_1 = \sum_{|\gamma|\geqslant G}\frac{1}{\rho}\int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y)e^{(\rho-\frac{1}{2})(\omega+y/T)}\,dy$$

satisfies $$|e_1| < (0{\cdot}0002)\pi.$$

Since $-\frac{1}{2}T\omega \leqslant y \leqslant \frac{1}{2}T\omega$, we have $\frac{1}{2}\omega \leqslant \omega+y/T \leqslant \frac{3}{2}\omega$. Let

$$M = \sup\left|\sum_{|\gamma|\geqslant G}\frac{1}{\rho}e^{(\rho-\frac{1}{2})m}\right| \quad \text{for } \tfrac{1}{2}\omega \leqslant m \leqslant \tfrac{3}{2}\omega.$$

Then
$$|e_1| = \left|\int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y)\left(\sum_{|\gamma|\geqslant G}\frac{1}{\rho}e^{(\rho-\frac{1}{2})(\omega+y/T)}\right)dy\right|$$
$$\leqslant M\int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y)\,dy \leqslant 2\pi M,$$

and this is less than $(0{\cdot}0001)2\pi$ by Lemma 2 (ii) applied to $e^{-\frac{1}{2}m}R(e^m, G)$, since $(\frac{1}{2}\omega, \frac{3}{2}\omega)$ is included in the appropriate range.

(H) Lemma 10. *On the hypothesis* (H) *and for* ω *subject to*

$$2.10^4 \leqslant \omega \leqslant \tfrac{2}{9}\log G,$$

the error

$$e_2 = \sum_{|\gamma|<G}\frac{1}{\rho}\int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y)e^{(\rho-\frac{1}{2})(\omega+y/T)}\,dy - \sum_{|\gamma|<G}\frac{1}{i\gamma}\int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y)e^{i\gamma(\omega+y/T)}\,dy$$

satisfies $$|e_2| < (0{\cdot}0468)\pi.$$

As in Lemma 9, we have $\frac{1}{2}\omega \leqslant \omega+y/T \leqslant \frac{3}{2}\omega$. Suppose here that m is that value of $\omega+y/T$ for which the value of

$$\sum_{|\gamma|<G} \left|\frac{1}{\rho} e^{(\rho-\frac{1}{2})(\omega+y/T)} - \frac{1}{i\gamma} e^{i\gamma(\omega+y/T)}\right|$$

is greatest. On (H) and for $2.10^4 \leqslant \omega \leqslant \frac{2}{9}\log G$, the error e_2 satisfies

$$|e_2| \leqslant \int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y) \sum_{|\gamma|<G} \left|\frac{1}{\rho} e^{(\rho-\frac{1}{2})(\omega+y/T)} - \frac{1}{i\gamma} e^{i\gamma(\omega+y/T)}\right| dy$$

$$\leqslant \sum_{|\gamma|<G} \left|\frac{1}{\rho} e^{(\rho-\frac{1}{2})m} - \frac{1}{i\gamma} e^{i\gamma m}\right| \int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y)\, dy$$

$$< (0{\cdot}0234)2\pi$$

by Lemma 8, since m again lies in the relevant range.

LEMMA 11. *For $\omega \geqslant 2.10^4$ and $T = 500$, the error*

$$e_3 = \sum_{|\gamma|<G} \frac{e^{i\gamma\omega}}{i\gamma}\left(\int_{-\infty}^{-\frac{1}{2}T\omega} + \int_{\frac{1}{2}T\omega}^{\infty}\right) K(y) e^{i\gamma y/T}\, dy$$

satisfies $\qquad |e_3| < (0{\cdot}00002)\pi.$

Since $K(y)$ is an even function of y and the γ's are symmetrically distributed, we have

$$|e_3| \leqslant 4 \sum_{0<\gamma<G} \frac{1}{\gamma} \left|\int_{\frac{1}{2}T\omega}^{\infty} K(y) e^{i\gamma y/T}\, dy\right|$$

$$= 4 \sum_{0<\gamma<T} + 4 \sum_{T\leqslant\gamma<G}.$$

Now we have the two inequalities†

$$\left|\int_{\frac{1}{2}T\omega}^{\infty} K(y) e^{i\gamma y/T}\, dy\right| \begin{cases} \leqslant \displaystyle\int_{\frac{1}{2}T\omega}^{\infty} 4y^{-2}\, dy = \frac{8}{T\omega}, \\ = \left|\displaystyle\int_{\frac{1}{2}T\omega}^{\infty} \frac{T}{i\gamma}\{e^{i\gamma y/T} - e^{\frac{1}{2}i\gamma\omega}\} K'(y)\, dy\right| < \dfrac{2T}{|\gamma|} \dfrac{4}{\frac{1}{2}T\omega}. \end{cases}$$

Using the former inequality in $\sum_{0<\gamma<T}$ and the latter in $\sum_{T\leqslant\gamma<G}$, we have, from Lemma 1 (i) and (ii),

$$|e_3| \leqslant \frac{32}{T\omega} \sum_{0<\gamma<T} \frac{1}{\gamma} + \frac{64}{\omega} \sum_{T\leqslant\gamma<G} \frac{1}{\gamma^2} < \frac{32}{T\omega} \frac{1}{4\pi} \log^2 T + \frac{64}{\omega} \frac{1}{2\pi} \frac{\log T}{T}.$$

Since $\omega > 2.10^4$ and $T = 500$ the required result follows.

† We have $K'(y) = 2\sin y/y^2 - 8\sin^2 \frac{1}{2}y/y^3$ and $|K'(y)| < 4/y^2$ in the range concerned.

11. From Lemmas 9, 10, and 11 we may now replace (10), subject to the condition

$$2.10^4 \leqslant \omega < \tfrac{2}{9}\log G, \tag{11}$$

by

$$\int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y)F(\omega+y/T)\,dy = -\sum_{|\gamma|<G} \frac{e^{i\gamma\omega}}{i\gamma} \int_{-\infty}^{\infty} K(y)e^{i\gamma y/T}\,dy + E, \tag{12}$$

where

$$|E| = |R-e_1-e_2+e_3| < (0{\cdot}0471)\pi.$$

Applying (6) to the series on the right-hand side of (12), we have, still subject to (11),

$$\int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y)F(\omega+y/T)\,dy = -2\sum_{0<\gamma<T} 2\pi\frac{\sin\gamma\omega}{\gamma}\left(1-\frac{\gamma}{T}\right)+E$$

$$> -2\sum_{0<\gamma<T} 2\pi\frac{\sin\gamma\omega}{\gamma}\left(1-\frac{\gamma}{T}\right)-(0{\cdot}0471)\pi. \tag{13}$$

Now let $F_M = F_M(\omega)$ be the upper bound of $F(\omega+y/T)$ for the range $-\frac{1}{2}T\omega \leqslant y \leqslant \frac{1}{2}T\omega$. Since $K(y) \geqslant 0$, (13) gives

$$F_M J = F_M \frac{1}{2\pi}\int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y)\,dy \geqslant \frac{1}{2\pi}\int_{-\frac{1}{2}T\omega}^{\frac{1}{2}T\omega} K(y)F(\omega+y/T)\,dy$$

$$> -2\sum_{0<\gamma<T} \frac{\sin\gamma\omega}{\gamma}\left(1-\frac{\gamma}{T}\right)-0{\cdot}0236. \tag{14}$$

Now by the definition (7) of F we have

$$F_M = \textit{upper bound of } (\psi_0(x)-x)x^{-\frac{1}{2}} \quad \textit{for } e^{\frac{1}{2}\omega} \leqslant x \leqslant e^{\frac{3}{2}\omega}. \tag{15}$$

We are therefore in a position to establish the following lemma.

(H) LEMMA 12. *On the hypothesis* (H) *a sufficient condition that*

$$\pi(x)-\mathrm{li}\,x > 0,$$

for some x satisfying $2 \leqslant x < X_1$,† *is that, for some ω subject to the condition* (11),

$$-\sum_{0<\gamma<500} \frac{\sin\gamma\omega}{\gamma}\left(1-\frac{\gamma}{500}\right) > 0{\cdot}5123. \tag{16}$$

When (16) is true we have, by (14)‡ (and the fact that $T = 500$), $F_M J > 1{\cdot}001$, and *a fortiori* $F_M > 1{\cdot}001$ since $J < \frac{1}{2\pi}\int_{-\infty}^{\infty} = 1$. Lemma 12 then follows from Lemma 7.

† We recall that X_1 is the number concerned in (H), § 4, namely exp exp exp(7·703).

‡ Which is valid subject to (11).

12. Our problem is now to find a suitable ω. It must be chosen so that the sines in (16) are predominantly negative, and such a choice is made as follows.

The number N of terms in the series on the left-hand side of (16) is equal to the number of γ's satisfying $0 < \gamma < 500$; this is known to be

$$N = 269. \tag{17}$$

Let ω_0 and q be the numbers

$$\omega_0 = 2.10^4+1, \qquad q = 3600. \tag{18}$$

By Dirichlet's theorem there is a number ω' satisfying

$$\omega_0 \leqslant \omega' \leqslant \omega_0 q^N \tag{19}$$

such that
$$\left|\frac{\gamma_n \omega'}{2\pi} - r_n\right| < \frac{1}{q} \quad (n = 1, 2,\ldots, N), \tag{20}$$

where r_n is an integer. Now let

$$\omega = \omega' - k, \tag{21}$$

where
$$k = \tfrac{3}{400}. \tag{22}$$

Then, from (20) and (21),

$$\sin \gamma_n \omega = -\sin(k\gamma_n - \phi_n),$$

where $|\phi_n| < 2\pi/q = 0° \, 6'$. Now from (17), (18), (19), and (22),

$$2.10^4 < \omega < \omega_0 q^N = (2.10^4+1)3600^{269} = \exp\exp(7{\cdot}7021\ldots) < \tfrac{2}{9}\log G.$$

The condition (11) is therefore satisfied. Hence, by Lemma 12, we shall have $\pi(x) - \operatorname{li} x > 0$ for some x satisfying

$$2 \leqslant x < X_1$$

provided that

$$S = \sum p(\gamma_n) = \sum_{n=1}^{269} \frac{\sin(k\gamma_n - \phi_n)}{\gamma_n}\left(1 - \frac{\gamma_n}{500}\right) > 0{\cdot}5123. \tag{23}$$

13. The inequality (23) is actually true. The right-hand side is what determines the top index of X_1 and it is here that we try to refine. It will suffice to sketch the numerical considerations involved.

The angles $k\gamma_n - \phi_n$ range from 6° to 215°, and the first 213 sines are positive. In the case of the remaining negative terms, for which

$$180° < k\gamma_n - \phi_n < 215°,$$

the γ's satisfy
$$420 < \gamma_n < 500.$$

Hence, in addition to the fact that $1/\gamma_n$ is small, either the absolute value of the sine or the factor $(1-\gamma_n/500)$ is small, and these negative terms prove to be of little importance. For the rest, sufficient is known about the values of the γ's† to enable us to obtain a lower bound to S by straightforward calculation.

In this the first 29 terms are calculated separately, the remainder are grouped in intervals (of the values of the γ's) of 10. For example, the first group contains the 4 terms for which $100 < \gamma \leqslant 110$, and the last group contains the 7 terms for which $490 < \gamma \leqslant 500$. We obtain a lower bound to the contribution of each term, or group of terms, by making use of the fact that the function $p(\gamma_n)$, defined in (23), satisfies (whatever the particular values of ϕ_n, ϕ_{n+1}) $p(\gamma_n) < p(\gamma_{n+1})$ for $\gamma_n < 457$ (approximately), and thereafter satisfies $p(\gamma_n) > p(\gamma_{n+1})$. We may replace each of the first 29 γ's by the upper bound to the interval in which it is known to lie, and for those groups for which $\gamma \leqslant 450$ we replace each of the γ's in the group by the upper bound of the interval in which it lies. The same replacement applies for the subgroup 450–457. For the subgroup 457–460 and the remaining groups for which $\gamma > 460$, since $p(\gamma_n)$ is now increasing, we replace the γ's in each group by the lower bound of the interval concerned. For example, each of the 4 γ's between 100 and 110 is replaced by 110, while each of the 7 γ's between 470 and 480 is replaced by 470. ϕ_n is replaced by $+6'$ or $-6'$ according as $\gamma_n k < 90°$ or $\gamma_n k > 90°$.

We find that $$S > 0{\cdot}5131 > 0{\cdot}5123.$$

$\pi(x) - \operatorname{li} x$ is therefore positive for some x satisfying

$$2 \leqslant x < X_1 = \exp\exp\exp(7{\cdot}703).$$

Part II

14. Before we can begin developing the consequences of (NH), the negation of (H), we need a number of preliminary results about the function

$$\Pi_0(x) - \operatorname{li} x = \tfrac{1}{2}\{\Pi(x+0)+\Pi(x-0)\} - \operatorname{li} x,$$

where $\Pi(x)$ is defined as in § 6. For $x > 1$ we have ((2), 81–82)

$$\Pi_0(x) - \operatorname{li} x = -\sum_\rho \operatorname{li} x^\rho + \int_x^\infty \frac{du}{(u^2-1)u\log u} - \log 2, \tag{24}$$

the series being boundedly convergent in any finite interval $1 < a \leqslant x \leqslant b$. The li function for a complex argument is defined by

$$\operatorname{li} x^\rho = \operatorname{li} e^{\rho \log x}, \tag{25}$$

and, for $w = u+vi$ where $v \neq 0$,

$$\operatorname{li} e^w = \int_{-\infty+vi}^{u+vi} \frac{e^z}{z}\,dz.$$

† (6), (7), (8), (9). In addition I have used some calculations performed by Dr. Comrie, kindly lent to me by Professor Titchmarsh.

We define the function $L(t)$ for $t>0$ by the series

$$L(t) = -e^{-\frac{1}{2}t}\sum_{\rho}\mathrm{li}\,e^{\rho t}. \tag{26}$$

From (25) and (26) we have, for $t>0$,

$$-L(t) = e^{-\frac{1}{2}t}\sum_{\rho}\int\limits_{-\infty+i\gamma t}^{(\beta+i\gamma)t}\frac{e^{z}}{z}\,dz = \sum_{\rho}e^{(\rho-\frac{1}{2})t}\int\limits_{0}^{\infty}\frac{e^{-v}}{\rho t-v}\,dv$$

$$= \sum e^{(\rho-\frac{1}{2})t}\left[\frac{1}{\rho t}+\int\limits_{0}^{\infty}\frac{e^{-v}}{(\rho t-v)^{2}}\,dv\right]$$

$$= \sum\frac{e^{(\rho-\frac{1}{2})t}}{\rho t}+\sum e^{(\rho-\frac{1}{2})t}\int\limits_{0}^{\infty}\frac{e^{-v}\,dv}{(\rho t-v)^{2}},$$

both series being boundedly convergent in any interval of type

$$0<a'\leqslant t\leqslant b'$$

since the first is. So

$$-L(t) = \sum_{\rho}u_{1}(\rho,t)+\sum_{\rho}u_{2}(\rho,t), \tag{27}$$

$$u_{1} = \frac{e^{(\rho-\frac{1}{2})t}}{\rho t}, \qquad u_{2} = e^{(\rho-\frac{1}{2})t}\int\limits_{0}^{\infty}\frac{e^{-v}\,dv}{(\rho t-v)^{2}}. \tag{28}$$

15. LEMMA 13. $\quad |L(t)| \leqslant 4e^{\frac{1}{4}t} \quad (t\geqslant 1).$

By Lemma 2 (iii), if $\tau = e^{t}\geqslant e$,

$$|\sum u_{1}(\rho,t)| = |\tau^{-\frac{1}{2}}(\log\tau)^{-1}\sum\tau^{\rho}/\rho| < 3\tau^{\frac{1}{4}}.$$

In $u_{2}(\rho,t)$ we have $|\rho t-v|^{2}\geqslant|i\gamma t|^{2}=\gamma^{2}t^{2}$,

$$|u_{2}|\leqslant\tau^{\frac{1}{2}}t^{-2}\gamma^{-2}, \qquad |\sum u_{2}|\leqslant e^{\frac{1}{2}t}t^{-2}\,2\sum_{\gamma>0}\gamma^{-2}<e^{\frac{1}{4}t},$$

since $\sum\gamma^{-2}<0{\cdot}05$. The result follows.

16. LEMMA 14. *A sufficient condition that* $\pi(x)-\mathrm{li}\,x>0$ *for some* x *of* $2\leqslant x\leqslant X$ *is that, for some* y *satisfying* $10^{4}\leqslant y\leqslant\log X$,

$$L(y)\geqslant 1. \tag{29}$$

Suppose the condition of the lemma is satisfied for a certain y, and let $x=e^{y}$. Then, by Lemma 5, either $\pi(\xi)-\mathrm{li}\,\xi>0$ for some ξ of $2\leqslant\xi\leqslant x^{\frac{1}{2}}$, or else

$$\Pi_{0}(x)-\pi(x)\leqslant\Pi(x)-\pi(x)<(1{\cdot}0005)x^{\frac{1}{2}}/\log x<2x^{\frac{1}{2}}/\log x.$$

In the first alternative, we have what we want at once, and we have only to consider the second. Now, from (24) and (26), the integral in (24) being positive,

$$\Pi_{0}(x)-\mathrm{li}\,x>x^{\frac{1}{2}}L(y)-\log 2\geqslant x^{\frac{1}{2}}-\log 2,$$

and so, from the second alternative,

$$\pi(x)-\operatorname{li} x > x^{\frac{1}{2}}-\log 2-2x^{\frac{1}{2}}/\log x > 0,$$

as desired.

17. Let X_1 and G be, as in § 4,

$$X_1 = \exp\exp\exp(7{\cdot}703), \qquad G = X_1^3.$$

Since we cannot use O's in connexion with numerical bounds, we shall use ϑ's (ϑ', ϑ_1, etc.) to denote numbers, possibly complex, satisfying $|\vartheta| \leqslant 1$. They will in general not be the same from one occurrence to the next, but where more than one occurs in the same expression we distinguish them.

Let $y \geqslant G$, and let λ be any real number satisfying†

$$|\lambda| \leqslant G \; (\leqslant y).$$

Consider‡ the function $F(y,\lambda)$ defined by

$$F(y,\lambda) = \int\limits_{\frac{1}{2}y}^{\infty} [-L(t)]te^{E}\,dt, \tag{30}$$

$$E = E(t,y,\lambda) = -\lambda it-\tfrac{1}{2}(t-y)^2/y. \tag{31}$$

We have the following result.

LEMMA 15. *Write* $b = \beta-\frac{1}{2}$, $r = \rho-\frac{1}{2}-i\lambda = b+i(\gamma-\lambda)$. *Then, subject to* $y \geqslant G \geqslant |\lambda|$,

$$F(y,\lambda) = \sum_{\rho} U(\rho), \tag{32}$$

where

$$U(\rho) = (2\pi y)^{\frac{1}{2}}e^{(r+\frac{1}{2}r^2)y}\left(\frac{1}{\rho}+\frac{400\vartheta}{\gamma^2 y}\right)+\vartheta'\frac{e^{-\frac{1}{10}y}}{\gamma^2}. \tag{33}$$

The proof of this is rather long, and we break it up into two subsidiary lemmas, A and B, and a short final deduction from them. We have among other things to show that the series (27) for L can be integrated term by term in (30): this involves a limit-process $T \to \infty$ for fixed y, λ. The parts of Lemmas A, B dealing with this use O's, which are accordingly uniform in the ρ (or γ), but not in the 'fixed' y, λ; the K's similarly are independent of ρ, γ, but not of y, λ.

LEMMA A. *For* $u_1(\rho,t)$, *defined by* (28), *we have*

$$\int\limits_{T}^{\infty} u_1\, te^{E}\,dt = O\left(\frac{e^{-KT^2}}{\gamma^2}\right), \tag{34}$$

$$\int\limits_{\frac{1}{2}y}^{\infty} u_1\, te^{E}\,dt = \frac{(2\pi y)^{\frac{1}{2}}}{\rho}\,e^{(r+\frac{1}{2}r^2)y}+\frac{2\vartheta e^{-\frac{1}{8}y}}{\gamma^2}. \tag{35}$$

† These conditions hold throughout the rest of the paper. Note that G is so large that any inequalities like $100y^{10}e^{-y/8} < e^{-y/10}$ that occur in the run of our argument will be true when they are 'true for large y'.

‡ The introduction of $F(y,\lambda)$ is the idea given me by Professor J. E. Littlewood.

LEMMA B. *For $u_2(\rho,t)$, defined by* (28), *we have*

$$\int_T^\infty u_2 te^E\,dt = O\left(\frac{e^{-KT^2}}{\gamma^2}\right), \tag{36}$$

$$\int_{\frac{1}{4}y}^\infty u_2 te^E\,dt = \frac{324\vartheta}{y}(2\pi y)^{\frac{1}{2}}\frac{e^{(r+\frac{1}{2}r^2)y}}{\gamma^2}+\frac{9\vartheta'}{\gamma^2}e^{-\frac{1}{8}y}. \tag{37}$$

18. In Lemmas A, B we may, by symmetry (since λ can take either sign), suppose without loss of generality that $\gamma > 0$.

Proof of Lemma A. We have

$$u_1 te^E = \frac{1}{\rho}e^{\gamma it}f(t), \qquad f(t) = e^{(\beta-\frac{1}{2}-i\lambda)t-\frac{1}{2}(t-y)^2/y}. \tag{38}$$

For $t \geqslant T$,

$$\int_T^t u_1 te^E\,dt = \frac{1}{\rho}\left[\frac{e^{\gamma it}}{\gamma i}f(t)\right]_T^t-\frac{1}{\rho}\int_T^t \frac{e^{\gamma it}}{\gamma i}f'(t)\,dt.$$

As $T\to\infty$ we have, uniformly in $t \geqslant T$, $f(t), f'(t) = O(e^{-KT^2})$. It follows that $\int_T^\infty u_1 te^E\,dt$ exists, and that it is $O(\gamma^{-2}e^{-KT^2})$; and this is (34).

Next, $$\int_{-\infty}^\infty u_1 te^E\,dt = \frac{1}{\rho}\int_{-\infty}^\infty e^{rt-\frac{1}{2}(t-y)^2/y}\,dt = \frac{(2\pi y)^{\frac{1}{2}}}{\rho}e^{(r+\frac{1}{2}r^2)y}. \tag{39}$$

Again,

$$\int_{-\infty}^{\frac{1}{4}y} u_1 te^E\,dt = \frac{1}{\rho}\left[\frac{e^{\gamma it}}{\gamma i}f(t)\right]_{-\infty}^{\frac{1}{4}y}-\frac{1}{\rho}\int_{-\infty}^{\frac{1}{4}y}\frac{e^{\gamma it}}{\gamma i}f'(t)\,dt = J_1+J_2,\text{ say.} \tag{40}$$

We have $f(-\infty) = 0$ and $|f(\frac{1}{4}y)| \leqslant e^{|b|\frac{1}{4}y-\frac{9}{32}y} < e^{-\frac{1}{8}y}$, so that

$$J_1 = \vartheta\gamma^{-2}e^{-\frac{1}{8}y}.$$

Also, for $-\infty < t \leqslant \frac{1}{4}y$,

$$\begin{aligned}|f'(t)| &= |(\beta-\tfrac{1}{2}-i\lambda)-(t-y)/y|e^{bt-\frac{1}{2}(t-y)^2/y}\\ &\leqslant (\tfrac{1}{2}+|\lambda|+|t-y|)e^{bt-\frac{1}{2}(t-y)^2/y}.\end{aligned}$$

Writing $u = |t-y| = y-t$, and observing that $u \geqslant \frac{3}{4}y$ and

$$\tfrac{1}{2}+|\lambda| < 2y \leqslant \tfrac{8}{3}u,$$

we have $$|f'| \leqslant 4ue^{b(y-u)-\frac{1}{2}u^2/y} \leqslant 12(by+u)e^{by}e^{-bu-\frac{1}{2}u^2/y},$$

$$\begin{aligned}|J_2| &\leqslant 12\gamma^{-2}e^{by}\int_{\frac{3}{4}y}^\infty (by+u)e^{-bu-\frac{1}{2}u^2/y}\,du = 12\gamma^{-2}ye^{(\frac{1}{4}b-\frac{9}{32})y}\\ &\leqslant 12\gamma^{-2}ye^{-\frac{5}{32}y} < \gamma^{-2}e^{-\frac{1}{8}y}.\end{aligned}$$

So $J_1+J_2 = 2\vartheta\gamma^{-2}e^{-\frac{1}{4}y}$, which, combined with (39) and (40), gives (35) and completes the proof of Lemma A.

19. *Proof of Lemma B.* Let $t \geqslant T$ and $T \to \infty$. We have $|\rho t-v|^2 \geqslant \gamma^2 t^2$, and so, from (28),

$$\left|\int_T^t u_2 te^E\,dt\right| \leqslant \int_0^\infty e^{-v}\,dv\,.\,\gamma^{-2}\int_T^t t^{-2}te^{bt-\frac{1}{2}(t-y)^2/v}dt$$
$$= O(\gamma^{-2}e^{-KT^2}),$$

and $\int_T^\infty$ exists and satisfies (36).

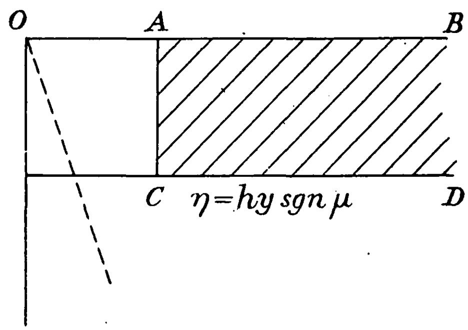

FIG. 1.

Next we have
$$\int_{\frac{1}{4}y}^\infty u_2 te^E\,dt = \int_0^\infty e^{-v}H(\rho,v)\,dv,$$

$$H = H(\rho,v) = \int_{\frac{1}{4}y}^\infty e^{rt-\frac{1}{2}(t-y)^2/y}\frac{t\,dt}{(\rho t-v)^2}. \tag{41}$$

We prove (37) of Lemma B by showing that for each v of $(0,\infty)$ H is of the form of the right-hand side of (37) (noting that $\int_0^\infty e^{-v}\,dv = 1$).

We deform the t-contour $\frac{1}{4}y$ to ∞, or AB, in a manner independent of v, as follows. Let $\mu = \gamma-\lambda$, $h = \min(2, |\mu|)$. With $t = \xi+i\eta$ we take a line $\eta = hy\,\mathrm{sgn}\,\mu$ $(= \pm hy)$, and replace the original path AB by ACD of the figure (drawn for the worst case, namely $\mathrm{sgn}\,\mu = -1$). First, the pole $t = v/\rho$ is outside the shaded area, so that† $H = \int_{ACD}$. For the pole is on a line (dotted in the figure) whose slope (tangent) is $-\gamma/\beta$; this is downward‡ and steeper absolutely than $\gamma_1 > 14$, steeper, therefore, than OC in the unfavourable case (of the figure) when C is below A.

† The integrand is uniformly $O(e^{-K\xi^2})$ as $t \to \infty$ in the shaded area.

‡ Recall that in this proof we have $\gamma > 0$.

Taking the integral for H along ACD, then, we have

$$|\rho t-v| \geqslant |\operatorname{im}(\rho t-v)| = |\gamma\xi+\beta\eta|$$
$$\geqslant \gamma\xi-1.2y \geqslant \tfrac{1}{3}\gamma\xi$$

(since $\gamma > 14$, $\xi \geqslant \frac{1}{4}y$). So

$$|H| \leqslant 9\gamma^{-2} \int_{ACD} |e^{E_1}\xi^{-2}t\,dt|, \tag{42}$$

where
$$E_1 = rt-\tfrac{1}{2}(t-y)^2/y, \qquad r = b+i\mu, \tag{43}$$

and, as alternative forms,

$$E_1 = -\tfrac{1}{2}y+(r+1)t-\tfrac{1}{2}t^2/y = (r+\tfrac{1}{2}r^2)y-\tfrac{1}{2}[t-(r+1)y]^2/y, \tag{44}$$
$$\operatorname{re} E_1 = -\tfrac{1}{2}y+(b+1)\xi-\mu\eta-\tfrac{1}{2}\xi^2/y+\tfrac{1}{2}\eta^2/y. \tag{45}$$

On AC we have $\xi = \frac{1}{4}y$, $\eta = \sigma hy \operatorname{sgn}\mu$, $0 \leqslant \sigma \leqslant 1$,

$$\operatorname{re} E_1 = [-\tfrac{1}{2}+\tfrac{1}{4}(b+1)]y-\tfrac{1}{32}y-\sigma hy(|\mu|-\tfrac{1}{2}\sigma h),$$

and since the last term is non-positive and $b+1 < \frac{3}{2}$, $\operatorname{re} E_1 < -\frac{1}{7}y$, and

$$\int_{AC} |e^{E_1}\xi^{-2}t\,dt| \leqslant e^{-\frac{1}{7}y}(\tfrac{1}{4}y)^{-2}OC.AC < \tfrac{1}{2}e^{-\frac{1}{8}y}. \tag{46}$$

For CD we have two cases.

Case (i). $|\mu| \leqslant 2$. Here $\eta = \mu y$ $(= \operatorname{im} r.y)$,

$$E_1 = (r+\tfrac{1}{2}r^2)y-\tfrac{1}{2}[\xi-(b+1)y]^2/y,$$

$$\int_{CD} |e^{E_1}\xi^{-2}t\,dt| \leqslant |e^{(r+\frac{1}{2}r^2)y}| \int_{\frac{1}{4}y}^{\infty} e^{-\frac{1}{2}[\xi-(b+1)y]^2/y}\{\xi^{-2}(2y+\xi)\}\,d\xi.$$

The curly bracket is greatest for $\xi = \frac{1}{4}y$, and it is then $36y^{-1}$. Taking this outside, and then the integral from $-\infty$ to ∞, we find that $\int_{CD}$ is at most $(2\pi y)^{\frac{1}{2}}|e^{(r+\frac{1}{2}r^2)y}|36y^{-1}$.

Combining this with (42) and (46) we have, in case (i),

$$|H| \leqslant 9\gamma^{-2}(e^{-\frac{1}{8}y}+(2\pi y)^{\frac{1}{2}}|e^{(r+\frac{1}{2}r^2)y}|36y^{-1}). \tag{47}$$

Case (ii). $|\mu| > 2$. Here CD has $\eta = 2y \operatorname{sgn}\mu$. We have from (45)

$$\begin{aligned}\operatorname{re} E_1 &= -\tfrac{1}{2}y+(b+1)\xi-2|\mu|y-\tfrac{1}{2}\xi^2/y+2y\\ &\leqslant -\tfrac{5}{2}y+\tfrac{3}{2}\xi-\tfrac{1}{2}\xi^2/y, \quad \text{since } |\mu| > 2,\\ &= -\tfrac{1}{4}y-\tfrac{1}{4}\xi^2/y-\tfrac{1}{4}(\xi-3y)^2/y \leqslant -\tfrac{1}{4}y-\tfrac{1}{4}\xi^2/y.\end{aligned}$$

As before, $|\xi^{-2}t| \leqslant 36y^{-1}$, and so

$$\int_{CD} |e^{E_1}\xi^{-2}t\,dt| \leqslant 36y^{-1} \int_{-\infty}^{\infty} e^{-\frac{1}{4}y-\frac{1}{4}\xi^2/y}\,d\xi \leqslant \tfrac{1}{2}e^{-\frac{1}{8}y}.$$

From this and (46), $|H| \leqslant 9\gamma^{-2}e^{-\frac{1}{4}y}$, and (47) is true also in case (ii). *A fortiori* H is of the form of the right-hand side of (37), and, as we observed above, this proves (37). This completes the proof of Lemma B.

20. We now have Lemmas A and B (in which γ now is not restricted to be positive), and can take up Lemma 15. By Lemma 13 and (30) we have

$$F = \lim_{T\to\infty} \int_{\frac{1}{4}y}^{T} [-L(t)]te^{E}\,dt,$$

since $\mathrm{re}E = -\frac{1}{2}(t-y)^2/y < -Kt^2$ as $t \to \infty$. In $\int_{\frac{1}{4}y}^{T}$ we may substitute $-L(t) = \sum u_1 + \sum u_2$ from (27) and integrate term by term, since the two series are boundedly convergent. If we then replace T by ∞ in each term, the error is

$$\sum \int_{T}^{\infty} u_1\,te^{E}\,dt + \sum \int_{T}^{\infty} u_2\,te^{E}\,dt = O(e^{-KT^2}\sum \gamma^{-2}),$$

by (34) from Lemma A and (36) from Lemma B, and this tends to 0 as $T \to \infty$. Hence

$$F(y,\lambda) = \sum_{\rho} U(\rho),$$

where

$$U(\rho) = \int_{\frac{1}{4}y}^{\infty} u_1\,te^{E}\,dt + \int_{\frac{1}{4}y}^{\infty} u_2\,te^{E}\,dt,$$

and when we substitute from (35) and (37) (and make a couple of small adjustments) we arrive at Lemma 15.

21. LEMMA 16.† *For $y \geqslant G \geqslant \lambda \geqslant 0$ we have*

$$F(y,\lambda) = \sum_{|\gamma-\lambda|\leqslant 2} \frac{1}{\rho}(2\pi y)^{\frac{1}{2}}e^{(r+\frac{1}{2}r^2)y}\left(1+\frac{30\vartheta}{y}\right)+\vartheta'.$$

Since $400|\rho|/(\gamma^2 y) < 30/y < 1$, Lemma 15 shows that F is equal to something of the form of the $\sum$ in the lemma, *plus*

$$2\vartheta_1 \sum_{|\gamma-\lambda|>2} \frac{(2\pi y)^{\frac{1}{2}}|e^{r+\frac{1}{2}r^2)y}|}{|\gamma|} + \vartheta_2\,e^{-\frac{1}{16}y}\sum \frac{1}{\gamma^2}. \tag{48}$$

When $|\gamma-\lambda| > 2$ we have

$$\mathrm{re}(r+\tfrac{1}{2}r^2) = b+\tfrac{1}{2}b^2-\tfrac{1}{2}(\gamma-\lambda)^2 < -\tfrac{1}{4}(\gamma-\lambda)^2-\tfrac{3}{8},$$

† (i) From now on λ is non-negative (we normalized in the *proof* above to $\gamma > 0$ and λ of both signs). (ii) The ϑ, of course, varies with the term it occurs in.

and also $|\gamma/(\gamma-\lambda)| \leqslant 1+\lambda < 2y$. The first term in (48) is therefore

$$\vartheta \sum_{\rho} \frac{4y}{\gamma^2}(2\pi y)^{\frac{1}{2}}\{|\gamma-\lambda|e^{-\frac{1}{4}(\gamma-\lambda)^2}\}e^{-\frac{3}{8}y} = \tfrac{1}{2}\vartheta, \tag{49}$$

since the curly bracket is less than (say) 10. Lemma 16 follows.

22. We are now in a position to develop the consequences of (NH), the negation of the hypothesis (H). To assume (NH) is to assume that a zero $\beta_0+i\gamma_0$ exists (with γ_0 positive, by the symmetry) satisfying

$$\text{(NH)}\quad \begin{cases} b_0 = \beta_0-\frac{1}{2} > X_1^{-3}\log^{-2}X_1 = B, \\ 0 < \gamma_0 < X_1^3 = G. \end{cases}$$

We begin by supposing that (for an undetermined Y) the relation '$L(y) \geqslant 1$ for some y' occurring in Lemma 14 is *not* satisfied for the range $G \leqslant y \leqslant 4Y$; that is, we suppose that†

$$L(\eta) < 1 \quad \text{for } G \leqslant \eta \leqslant 4Y. \tag{50}$$

By arguing from the pair of hypotheses (NH) and (50) we find ourselves able to produce a Y_0 (actually G^{10}) such that, if the Y of (50) is Y_0, there is a contradiction. Then (NH) implies (i) that (50) is false for $Y = Y_0$; so (ii) that for *some* y of the range $G \leqslant y \leqslant 4Y_0$ we must have $L(y) \geqslant 1$, when Lemma 14 (with $4Y_0$ for $\log X$) gives $\pi(x)-\operatorname{li} x > 0$ for some x of

$$2 \leqslant x < X = \exp(4Y_0)\ [= \exp(4G^{10})].$$

This, then, is what results from (NH), and since the X is greater than the X_1 derived from (H), it is our final number.

23. Lemma 17. *If [in accordance with* (50)*]* $L(\eta) < 1$ *for* $G \leqslant \eta \leqslant 4Y$, *then for* $4G \leqslant y \leqslant Y$, $0 \leqslant \lambda \leqslant G$, *we have*

$$|F(y,0)| \leqslant 1, \tag{51}$$

$$|F(y,\lambda)| < 6Y^{\frac{3}{2}}+4. \tag{52}$$

When $\lambda = 0$, the condition $|\gamma-\lambda| \leqslant 2$ is vacuous, and (51) is a case of Lemma 16.

Next, since $L(t)$ is real for $t > 0$, we have, for λ of $0 \leqslant \lambda \leqslant G$, by (30) and (31),

$$\begin{aligned} -F(y,\lambda) &= \int_{\frac{1}{4}y}^{\infty} t\,L(t)(\cos\lambda t - i\sin\lambda t)e^{-\frac{1}{2}(t-y)^2/y}\,dt \\ &= \mathscr{R}-i\mathscr{I}, \text{ say.} \end{aligned} \tag{53}$$

† This means 'for *all* η of the range', and similar interpretations are intended wherever we do not explicitly have 'some'. This being the usual interpretation, we may seem to be labouring the obvious, but the distinctions of 'all' and 'some' are very vital, and complicated by ranges (those in Lemma 17) that 'look' alike, but are not quite so.

Consider the four expressions

$$-F(y,0)\,\begin{matrix}\pm\mathscr{R}\\ \pm\mathscr{I}\end{matrix} = \left(\int\limits_{\frac{1}{4}y}^{4Y} + \int\limits_{4Y}^{\infty}\right) tL(t)\left\{1\,\begin{matrix}\pm\cos\lambda t\\ \pm\sin\lambda t\end{matrix}\right\} e^{-\frac{1}{2}(t-y)^2/y}\,dt$$

$$= J_1+J_2, \text{ say.} \tag{54}$$

In J_2 we substitute $|L(t)| \leqslant 4e^{\frac{1}{2}t}$ from Lemma 13, and, remembering that $4G \leqslant y \leqslant Y$, we obtain

$$|J_2| \leqslant \int\limits_{4y}^{\infty} t\,.\,4e^{\frac{1}{2}t}\,.\,2e^{-\frac{1}{2}(t-y)^2/y}\,dt$$

$$= 8e^{-\frac{1}{2}y}\int\limits_{4y}^{\infty} te^{-\frac{1}{2}(t-4y)-\frac{1}{2}(t-2y)^2/y}\,dt < 1. \tag{55}$$

In J_1 we have $G \leqslant t \leqslant 4Y$, and so $L(t) < 1$ by the hypothesis (50); hence, *the curly bracket in* (54) *being in all four cases non-negative*, we have, *algebraically*,

$$J_1 \leqslant \int\limits_{\frac{1}{4}y}^{4Y} 2t\,e^{-\frac{1}{2}(t-y)^2/y}\,dt \leqslant \int\limits_{-\infty}^{\infty} (2y+2|t-y|)e^{-\frac{1}{2}(t-y)^2/y}\,dt$$

$$= 2y(2\pi y)^{\frac{1}{2}}+8y < 6y^{\frac{3}{2}} \leqslant 6Y^{\frac{3}{2}}. \tag{56}$$

Since $|F(y,\lambda)| \leqslant |\mathscr{R}|+|\mathscr{I}|$, from (53), *and since* $|\mathscr{R}|+|\mathscr{I}|$ *is, for each* y, *one* (*varying with* y) *of the four combinations* $\pm\mathscr{R}\pm\mathscr{I}$, we have, from (54) to (56),

$$|F(y,\lambda)| \leqslant 6Y^{\frac{3}{2}}+1+2|F(y,0)| < 6Y^{\frac{3}{2}}+4,$$

the desired result.

24. We now combine Lemmas 16 and 17, and take $Y = G^{10}$ (Y has this meaning from now on). The upshot is that, subject to (NH), and to the further 'hypothesis'

$$(\mathrm{H}_1) \qquad L(\eta) < 1 \quad (G \leqslant \eta \leqslant 4Y),$$

we have, for λ, y satisfying

$$0 \leqslant \lambda \leqslant G, \tag{57}$$

$$4G \leqslant y \leqslant Y, \tag{58}$$

and for *some* set of ϑ's,

$$\left|\sum_{|\gamma-\lambda|\leqslant 2} \frac{1}{\rho}e^{(r+\frac{1}{2}r^2)y}\left(1+\frac{30\vartheta}{y}\right)\right| < (2\pi y)^{-\frac{1}{2}}(6Y^{\frac{3}{2}}+4+1) < \tfrac{1}{2}Y^{\frac{3}{2}}, \tag{59}$$

where $r = b+i(\gamma-\lambda)$.†

We now take $\lambda = \gamma_0$, where γ_0 is the number in (NH), § 22. [λ duly

† We go on to derive a contradiction from this state of things, as a result of which one of (NH) and (H_1) must be false.

satisfies (57).] So from (59) with $\lambda = \gamma_0$ and so $r = b+i(\gamma-\gamma_0)$,

$$\mathrm{re}\sum_{|\gamma-\gamma_0|\leqslant 2}\frac{i}{\rho}\left(1+\frac{30\vartheta}{y}\right)\exp[(b+\tfrac{1}{2}b^2)y-\tfrac{1}{2}(\gamma-\gamma_0)^2y+i(\gamma-\gamma_0)(1+b)y] < \tfrac{1}{2}Y^{\frac{3}{5}}. \tag{60}$$

We need to know an upper bound for the number N of terms in the sum; Lemma 1 (iv) with $h = 4$, $T = \gamma_0-2$, gives

$$N < \frac{6}{2\pi}\log\gamma_0+8{\cdot}7 < \log G. \tag{61}$$

We proceed to choose, without violating (58), a y ($= y_0$) for which the real parts of the terms in the sum in (60) are all positive. In the first place, since $\gamma > 14$, the argument of any factor i/ρ lies between $\pm 5°$, and that of any $1+30\vartheta/y$ between $\pm 1°$. Now by Dirichlet's theorem there exists a y_0 satisfying

$$Y^{\frac{1}{2}} \leqslant y_0 \leqslant Y^{\frac{1}{2}}5^N, \tag{62}$$

and such that, for each of the N γ's satisfying $|\gamma-\gamma_0| \leqslant 2$,

$$|(\gamma-\gamma_0)(1+b)y_0-2\pi k| < \tfrac{1}{5}.2\pi,$$

where k is an integer. Further, with $Y = G^{10}$ and N satisfying (61), $y = y_0$ [satisfying (62)] duly satisfies (58). With $y = y_0$ the arguments of all the terms in the sum in (60) lie between $\pm 80°$; hence the real parts of all the terms are positive, and the sum of them is at least as great as any one term. Choosing the one term to be that with $\gamma = \gamma_0$, we have

$$\frac{\gamma_0}{\gamma_0^2+\beta_0^2}e^{(b_0+\frac{1}{2}b_0^2)y_0}\left\{1-\frac{30}{y_0}\right\} < \tfrac{1}{2}Y^{\frac{3}{5}},$$

$$e^{b_0y_0} < \tfrac{1}{2}(\gamma_0+1/\gamma_0)\{1-30/y_0\}^{-1}Y^{\frac{3}{5}} < \gamma_0 Y^{\frac{3}{5}} \leqslant GY^{\frac{3}{5}},$$

$$e^{By_0} < Y^{\frac{3}{5}}G = G^{16}.$$

With $B = X_1^{-3}\log^{-2}X_1$, $X_1^3 = G$, this contradicts $y_0 \geqslant Y^{\frac{1}{2}} = G^5$ of (62).†
So either (NH) is false [and (H) true] or (H_1) is false. In the first case $\pi(x)-\mathrm{li}\,x > 0$ for an $x < X_1$; in the second it happens for an x of

$$2 \leqslant x < X_2 = \exp(4Y) = \exp(4G^{10}) = \exp(4X_1^{30}).$$

Since $X_2 > X_1 = \exp\exp\exp(7{\cdot}703)$, we conclude finally that $\pi(x)-\mathrm{li}\,x > 0$ for some $x < X$, where

$$X = \exp\exp\exp\exp(7{\cdot}705) < 10^{10^{10^{10^3}}}$$

† There is a great deal to spare at this point: see the footnote on p. 50.

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