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ON THE ZEROS OF THE DIRICHLET L-FUNCTIONS

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1. Let m be a positive integer, $\chi = \chi(n)$ a character modulo m and $L(s, \chi)$ the corresponding Dirichlet L-function. Landau, Littlewood and, more recently, Paley[1] observed a remarkable analogy between the behavior of Riemann's $\zeta(s)$, for variable $s = \sigma + it$ and $t \to \infty$, and that of $L(s, \chi)$, for variable χ and $m \to \infty$. Their results are concerned with estimates and averages of the absolute value of these functions. In the present paper we shall develop an analogy to some known theorems about the distribution of the zeros of $\zeta(s)$.

Suppose $m > 15$ and introduce the abbreviations

$$\log m = m_1, \qquad \log\log m = m_2, \qquad \log\log\log m = m_3,$$

so that $m_3 > 0$. The number of characters χ, for any given m, equals

$$h = \varphi(m) = m \prod_{p \mid m} (1 - p^{-1}).$$

It is well known that the product

$$P(s) = P_m(s) = (s - 1) \prod_{\chi} L(s, \chi) \tag{1}$$

is an entire function of s which has all its zeros in the half-plane $\sigma < 1$. Throughout the whole paper, T_0 denotes an arbitrarily large fixed positive constant; moreover, $c_1, \cdots, c_8$ are certain positive numbers which depend only upon the choice of T_0.

THEOREM I: *If $m_2^{-1} < \delta < \frac{1}{2}$, then the number of zeros of $P_m(s)$ in the rectangle $\frac{1}{2} + \delta < \sigma < 1$, $-T_0 < t < T_0$ is less than $c_1\delta^{-1}m_1^{-2\delta}h$.*

As an immediate consequence of Theorem I we have

THEOREM II: *If $m > c_2$, then at least one of the h functions $L(s, \chi)$ has no zero in the rectangle $\frac{1}{2} + \frac{1}{2}m_2^{-1}m_3 < \sigma < 1$, $-T_0 < t < T_0$.*

The two preceding propositions deal with rectangles in the half-plane $\sigma > \frac{1}{2}$. In the following results the rectangles contain a segment of the critical line $\sigma = \frac{1}{2}$.

THEOREM III: *Let $0 < T < T_0$ and denote by $A(T)$ the number of zeros of $P_m(s)$ in the rectangle $0 < \sigma < 1$, $0 \leqq t < T$; then*

$$\left| A(T) - \frac{1}{2\pi} m_1 h T \right| < c_3 m_1^{2/3} h.$$

Since $\overline{P(s)} = P(\bar{s})$, we infer from Theorems I, III and the functional equation of $L(s, \chi)$ that at least one of the h functions $L(s, \chi)$ has a zero in the rectangle $\frac{1}{2} \leqq \sigma \leqq \frac{1}{2} + m_2^{-1}$, $T_1 < t < T_2$, provided $T_2 - T_1 > c_4 m_1^{-\frac{1}{3}}$ and $-T_0 < T_1 < T_2 < T_0$. Consequently every point of the critical line $\sigma = \frac{1}{2}$ is a limit point for the set of the zeros of all $L(s, \chi)$, with variable m and χ.

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[1] R. E. A. C. PALEY, *On the k-analogues of some theorems in the theory of the Riemann ζ-function*, Proc. London Math. Soc. (2) 32, pp. 273–311 (1931).

THEOREM IV: *Let* $-T_0 < T_1 < T_2 < T_0$, $T_2 - T_1 > 4m_3^{-1}$ *and* $m > c_5$; *then each function* $L(s, \chi)$ *has a zero in the rectangle* $\frac{1}{2} \leqq \sigma < 1$, $T_1 < t < T_2$.

It follows from Theorems II and IV that there exists a subset of the functions $L(s, \chi)$, for variable m and χ, whose zeros cluster exactly towards all points of the critical line.

Our propositions estimate the number of zeros of the h functions $L(s, \chi)$, as $m \to \infty$, in certain rectangles lying in a fixed bounded domain of the s-plane; they are the counterparts of known theorems concerning the Riemann zeta-function, where $T_0 \to \infty$. In particular, Theorem I corresponds to results of Bohr and Landau, Carlson, Littlewood, and Theorem III to a formula of Riemann and von Mangoldt; Theorem IV is the analogue of a theorem of Littlewood[2] and Hoheisel.[3]

It is clear that it suffices to prove the theorems under the further assumption $T_0 > 1$.

2. Let q be a positive integer, $(m, q) = 1$ and define

$$g_k = g_k(s) = \sum_{\substack{n=1 \\ (k+nm,q)=1}}^{\infty} (k + nm)^{-s} \qquad (k = 1, \cdots, m; \sigma > 1). \tag{2}$$

The Möbius function $\mu(l)$ satisfies the equations

$$\sum_{d \mid l} \mu(d) = \begin{cases} 1 & (l = 1) \\ 0 & (l > 1); \end{cases}$$

hence

$$g_k = \sum_{\substack{d \mid (k+nm,\, q) \\ n>0}} \mu(d)(k + nm)^{-s} = \sum_{\substack{d \mid q \\ m<d n\equiv k (\mathrm{mod}\, m)}} \mu(d)(dn)^{-s} = \sum_{d \mid q} \mu(d) d^{-s} h_r,$$

where

$$h_r = \sum_{n=0}^{\infty} (r + nm)^{-s}$$

and $r = r(d)$ is determined by the conditions

$$\frac{m}{d} < r \leqq \frac{m}{d} + m, \qquad dr \equiv k (\mathrm{mod}\ m).$$

In virtue of the simplest case of Euler's summation formula we have

$$h_r - \frac{m^{-s}}{s-1} = \tfrac{1}{2} r^{-s} + m^{-1} \int_r^m x^{-s}\, dx - s \int_0^{\infty} (r + x)^{-s-1} \left(\frac{x}{m} - \left[\frac{x}{m} \right] - \tfrac{1}{2} \right) dx.$$

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[2] J. E. LITTLEWOOD, *Two notes on the Riemann Zeta-function*, Proc. Cambridge Phil. 22, pp. 234–242 (1925).

[3] GUIDO HOHEISEL, *Der Wertevorrat der ζ-Funktion in der Nähe der kritischen Geraden*, Jahresbericht der Schlesischen Gesellschaft für vaterländische Kultur 99, pp. 1–11 (1926).

The right-hand member is regular in the half-plane $\sigma > 0$. Since $0 < r \leqq 2m$, we obtain the estimate

$$\left| h_r - \frac{m^{-s}}{s-1} \right| \leqq \tfrac{1}{2} r^{-\sigma} + m^{-1} \left| \int_r^m x^{-\sigma}\, dx \right| + \tfrac{1}{2} |s| \int_0^\infty (r+x)^{-\sigma-1}\, dx$$

$$\leqq \tfrac{1}{2} r^{-\sigma} + \frac{|m-r|}{2m} (r^{-\sigma} + m^{-\sigma}) + \tfrac{1}{2}(\sigma + |t|)\sigma^{-1} r^{-\sigma}$$

$$\leqq \tfrac{1}{2}(3 + \sigma^{-1}|t|) r^{-\sigma} + \tfrac{1}{2} m^{-\sigma} \qquad (\sigma > 0).$$

It follows that the function

$$(3) \quad f_k = g_k - \frac{m^{-s}}{s-1} \sum_{d|q} \mu(d) d^{-s} = \sum_{d|q} \mu(d) d^{-s} \left(h_r - \frac{m^{-s}}{s-1} \right) \quad (k = 1, \cdots, m)$$

is regular in the half-plane $\sigma > 0$ and satisfies there the inequality

$$(4) \quad \begin{aligned} |f_k| &\leqq \sum_{d|q} |\mu(d)| d^{-\sigma} \left\{ \tfrac{1}{2}(3 + \sigma^{-1}|t|) \left(\frac{m}{d}\right)^{-\sigma} + \tfrac{1}{2} m^{-\sigma} \right\} \\ &\leqq \left(2 + \frac{|t|}{2\sigma} \right) m^{-\sigma} \sum_{d|q} |\mu(d)|. \end{aligned}$$

It remains to determine a simple upper bound of the last sum considered as a function of q.

For a later purpose we investigate the more general expression

$$\lambda_q(\rho) = \sum_{d|q} |\mu(d)| d^{-\rho} = \prod_{p|q} (1 + p^{-\rho}) \qquad (\rho \geqq 0).$$

Let $\nu = \nu(q)$ be the number of different prime factors of q, and denote by p_1, p_2, $\cdots$ the prime numbers in their natural order. Then

$$(5) \quad \log q \geqq \sum_{l=1}^{\nu} \log p_l = \sum_{p \leqq p_\nu} \log p \sim p_\nu \sim \nu \log \nu \qquad (\nu \to \infty)$$

and

$$\log \lambda_q(\rho) \leqq \sum_{l=1}^{\nu} \log (1 + p_l^{-\rho}) \leqq \sum_{p \leqq p_\nu} p^{-\rho};$$

hence

$$\log \lambda_q(1) \leqq \sum_{p \leqq p_\nu} p^{-1} = \log \log p_\nu + O(1) \qquad (\nu \to \infty),$$

$$(6) \quad \lambda_q(1) = O(\log \log q) \qquad (q \to \infty)$$

and, for $\rho \geqq \frac{1}{2}$,

$$\log \lambda_q(\rho) \leqq \sum_{n=1}^{p_\nu} \frac{\pi(n) - \pi(n-1)}{n^{\frac{1}{2}}} = \nu p_\nu^{-\frac{1}{2}} + \sum_{n=1}^{p_\nu - 1} \pi(n) \{ n^{-\frac{1}{2}} - (n+1)^{-\frac{1}{2}} \}$$

$$= \nu p_\nu^{-\frac{1}{2}} + O\left(\sum_{n=2}^{p_\nu} \frac{1}{\log n} n^{-\frac{1}{2}} \right) = \nu p_\nu^{-\frac{1}{2}} + O\left(\frac{p_\nu^{\frac{1}{2}}}{\log p_\nu} \right) = O\left(\frac{p_\nu^{\frac{1}{2}}}{\log p_\nu} \right) \quad (\nu \to \infty)$$

$$(7) \quad \log \lambda_q(\rho) = O\left(\frac{\log^{\frac{1}{2}} q}{\log \log q} \right) \qquad (q \to \infty;\ \rho \geqq \tfrac{1}{2}).$$

Furthermore,

$$\log \lambda_q(0) = \nu \log 2 \sim \frac{p_\nu}{\log p_\nu} \log 2 \qquad (\nu \to \infty)$$

(8) $$\sum_{d|q} |\mu(d)| = \lambda_q(0) = O(e^{\gamma_1 \log q/\log\log q}) \qquad (q \to \infty),$$

for any given constant $\gamma_1 > \log 2$.

In all following estimates the symbol O refers to the passage to the limit $m \to \infty$, and these estimates hold uniformly with respect to all variable parameters. We define

$$\gamma_l = \log 2 + \frac{l}{4}(1 - \log 2) \qquad (l = 1, 2, 3),$$

so that $\log 2 < \gamma_1 < \gamma_2 < \gamma_3 < 1$.

In virtue of (4), (8) we have the formula

(9) $$|f_k| = (1 + \sigma^{-1}|t|)e^{\gamma_1 \log q/\log\log q} O(m^{-\sigma}) \quad (k = 1, \cdots, m; \sigma > 0).$$

3. Let $\epsilon_k = 1$ in case $(k, q) = 1$ and $\epsilon_k = 0$ otherwise, then the functions

(10) $$G_k = \epsilon_k k^{-s} + f_k \qquad (k = 1, \cdots, m)$$

fulfill the inequality

$$\left(\sum_{k=1}^{m} |G_k|^2\right)^{\frac{1}{2}} \leqq \left(\sum \epsilon_k k^{-2\sigma}\right)^{\frac{1}{2}} + \left(\sum |f_k|^2\right)^{\frac{1}{2}}$$

(11) $$= \left(\sum_{(k,q)=1} k^{-2\sigma}\right)^{\frac{1}{2}} + (1 + \sigma^{-1}|t|)e^{\gamma_1 \log q/\log\log q} O(m^{\frac{1}{2}-\sigma}) \quad (\sigma > 0)$$

by (9).

On the other hand, let $\chi = \chi(n)$ be a character modulo m and

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_p (1 - \chi(p)p^{-s})^{-1} \qquad (\sigma > 1)$$

the corresponding L-function; plainly, $L(s, \chi)$ is related to the more general function

(12) $$L_q(s, \chi) = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \chi(n) n^{-s} = \prod_{p \nmid q} (1 - \chi(p)p^{-s})^{-1} \qquad (\sigma > 1)$$

by the formula

(13) $$L_q(s, \chi) = L(s, \chi) \prod_{p|q} (1 - \chi(p)p^{-s}).$$

Using the abbreviations

$$\frac{m^{-s}}{s-1} \sum_{d|q} \mu(d) d^{-s} = Q, \qquad \sum_{k=1}^{m} \chi(k) G_k = \alpha(\chi),$$

we obtain, by (2), (3), (10),

$$L_q(s, \chi) = \sum_{k=1}^{m} \chi(k) \sum_{n=0}^{\infty} (k + nm)^{-s} = \sum_{k=1}^{m} \chi(k)(\epsilon_k k^{-s} + g_k)$$

$$L_q(s, \chi) = \sum_{k=1}^{m} \chi(k)(G_k + Q) = \alpha(\chi) + \beta(\chi), \tag{14}$$

say, where $\beta(\chi) = hQ$ in the case of the principal character $\chi = \chi_1$ and $\beta(\chi) = 0$ otherwise; the last formula holds in the half-plane $\sigma > 0$.

We now introduce all h characters modulo m and apply the inequality of the geometric and arithmetic means; then, by (14), in view of $\left|\frac{s-1}{s+1}\right| < 1$ for $\sigma > 0$,

$$\begin{aligned} h^{\frac{1}{2}} \left| \frac{s-1}{s+1} \prod_{\chi} L_q(s, \chi) \right|^{1/h} &\leqq \left(\left| \frac{s-1}{s+1} L_q(s, \chi_1) \right|^2 + \sum_{\chi \neq \chi_1} | L_q(s, \chi) |^2 \right)^{\frac{1}{2}} \\ &\leqq \left| \frac{s-1}{s+1} hQ \right| + \left(\sum_{\chi} | \alpha(\chi) |^2\right)^{\frac{1}{2}} \qquad (\sigma > 0); \end{aligned} \tag{15}$$

moreover

$$\sum_{\chi} | \alpha(\chi) |^2 = \sum_{\chi} \sum_{k_1, k_2 = 1}^{m} \chi(k_1)\bar{\chi}(k_2) G_{k_1} \bar{G}_{k_2} = h \sum_{\substack{k=1 \\ (k,m)=1}}^{m} | G_k |^2 \leqq h \sum_{k=1}^{m} | G_k |^2. \tag{16}$$

Since

$$\left| \frac{s-1}{s+1} hQ \right| \leqq | s + 1 |^{-1} hm^{-\sigma} \sum_{d|q} | \mu(d) | d^{-\sigma} \leqq h^{\frac{1}{2}} m^{\frac{1}{2}-\sigma} \sum_{d|q} | \mu(d) | \quad (\sigma > 0),$$

it follows from (8), (11), (15), (16) that

$$\begin{aligned} &\log \left| \frac{s-1}{s+1} \right| + \sum_{\chi} \log | L_q(s, \chi) | \\ &\qquad < h \log \left(\sum_{\substack{k=1 \\ (k,q)=1}}^{m} k^{-2\sigma} + (1 + \sigma^{-1} | t |) e^{\gamma_1 \log q / \log \log q} O(m^{\frac{1}{2}-\sigma}) \right), \end{aligned} \tag{17}$$

everywhere in the half-plane $\sigma > 0$.

In the particular case $q = 1$ we have $L_q(s, \chi) = L(s, \chi)$. Because of

$$\sum_{k=1}^{m} k^{-2\sigma} \leqq \begin{cases} \zeta(2\sigma) < \dfrac{2\sigma}{2\sigma - 1} & (\sigma > \frac{1}{2}) \\ m^{1-2\sigma} \displaystyle\sum_{k=1}^{m} k^{-1} \leqq m^{1-2\sigma}(1 + \log m) & (\sigma \leqq \frac{1}{2}), \end{cases}$$

the inequality (17) implies

$$\begin{aligned} &h^{-1}\left(\log \left| \frac{s-1}{s+1} \right| + \sum_{\chi} \log | L(s, \chi) |\right. \\ &\qquad < \begin{cases} \log \dfrac{1}{2\sigma - 1} + \log (1 + | t |) + O(1) & (\frac{1}{2} < \sigma < 3) \\ (1 - 2\sigma)m_1 + \log (1 + \sigma^{-1} | t |) + O(m_2) & (0 < \sigma \leqq \frac{1}{2}). \end{cases} \end{aligned} \tag{18}$$

The other important case is

$$q = \prod_{p<m_1} p = q_1,$$

say; then

(19) $$\log q = \sum_{p<m_1} \log p \sim m_1$$

and

(20) $$\sum_{\substack{k=1 \\ (k,q)=1}}^{m} k^{-2\sigma} < \sum_{\substack{k=1 \\ (k,q)=1}}^{\infty} k^{-2\sigma} = \prod_{p>m_1} (1 - p^{-2\sigma})^{-1} \qquad (\sigma > \tfrac{1}{2}).$$

Since, for $\sigma > \frac{1}{2}$,

(21) $$\sum_{p>m_1} \log (1 - p^{-2\sigma})^{-1} = O\Big(\sum_{p>m_1} p^{-2\sigma}\Big)$$

$$\sum_{p>m_1} p^{-2\sigma} = \sum_{n>m_1} \frac{\pi(n) - \pi(n-1)}{n^{2\sigma}} \leqq \sum_{n>m_1} \pi(n)\{n^{-2\sigma} - (n+1)^{-2\sigma}\}$$

$$= O(m_2^{-1}) \sum_{n>m_1} n\{n^{-2\sigma} - (n+1)^{-2\sigma}\}$$

(22) $$= O(m_2^{-1})(m_1^{1-2\sigma} + \sum_{n>m_1+1} n^{-2\sigma}) = \frac{\sigma}{2\sigma - 1} m_1^{1-2\sigma} O(m_2^{-1})$$

and

$$\log (e^a + b) = a + \log (1 + e^{-a}b) < a + e^{-a}b < a + b \qquad (a > 0; b > 0),$$

formulas (17), (19), (20), (21), (22) lead to the estimate

(23) $$\log \left|\frac{s-1}{s+1}\right| + \sum_{\chi} \log | L_{q_1}(s, \chi) | < \left\{\frac{\sigma}{2\sigma - 1} m_1^{1-2\sigma} m_2^{-1} + (1 + | t |)m^{\frac{1}{2}-\sigma} e^{\gamma_1(m_1/m_2)}\right\} O(h) \quad (\sigma > \tfrac{1}{2}).$$

4. Let $f(z)$ be regular analytic in the circle $| z | \leqq 1$ and $f(0) \neq 0$. Jensen's theorem states that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(e^{i\varphi}) | \, d\varphi = \log |f(0)| - \sum_{\alpha} \log | \alpha |,$$

where α runs over all zeros of $f(z)$ in the circle. If $B(\rho)$ denotes the number of zeros in the concentric circle $| z | \leqq \rho$, $0 < \rho < 1$, then

(24) $$B(\rho) \log \rho^{-1} \leqq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(e^{i\varphi}) | \, d\varphi - \log |f(0)| .$$

We observe, once more, that T_0 is a fixed positive constant > 1. Suppose $\epsilon > \gamma_2 m_2^{-1}$, and define

$$(25)\qquad r = \left\{T_0^2 + \left(\frac{1-\gamma_2}{m_2}\right)^2\right\} \Big/ \frac{2(1-\gamma_3)}{m_2} \sim \frac{T_0^2}{2(1-\gamma_3)}\, m_2\,,$$

$$\sigma_0 = \tfrac{1}{2} + \epsilon + r, \qquad z = r^{-1}(\sigma_0 - s),$$

$$(26)\qquad f(z) = \frac{s-1}{s+1} \prod_{\chi} L_{q_1}(s, \chi);$$

then $r > 1$, and the unit circle of the z-plane corresponds to the circle of radius r with the center σ_0 in the s-plane. This circle lies in the half-plane $\sigma \geqq \frac{1}{2} + \epsilon > \frac{1}{2} + \gamma_2 m_2^{-1}$; hence $f(z)$ is regular analytic for $|z| \leqq 1$.

By (12),

$$\log \frac{\sigma_0 + 1}{\sigma_0 - 1} + \log f(0) = \sum_{\chi} \sum_{\substack{l=1 \\ p \nmid q_1}}^{\infty} l^{-1} \chi(p^l) p^{-l\sigma_0} = h \sum_{\substack{p > m_1 \\ p^l \equiv 1 \pmod{m}}} l^{-1} p^{-l\sigma_0} > 0$$

$$(27)\qquad -\log|f(0)| < \log \frac{\sigma_0 + 1}{\sigma_0 - 1} = O(r^{-1}) = O(m_2^{-1}).$$

The function $(2\sigma - 1)(mm_2^{-2})^{\frac{1}{2}-\sigma}$ is monotone decreasing for $\sigma - \frac{1}{2} > (m_1 - 2m_2)^{-1}$, and for $\sigma = \frac{1}{2} + \gamma_2 m_2^{-1}$ its value equals

$$2\gamma_2 m_2^{-1} e^{\gamma_2(2-(m_1/m_2))} = e^{-\gamma_1(m_1/m_2)} O(m_2^{-2}).$$

Suppose $|z| = 1$, then $|t| \leqq r = O(m_2)$, and it follows that the first term of the sum within the braces in (23) majorizes the second term; hence

$$\log|f(z)| < \frac{\sigma}{2\sigma - 1}\, m_1^{1-2\sigma} m_2^{-1} O(h) = \epsilon^{-1} m_1^{1-2\sigma} m_2^{-1} O(h) \qquad (|z| = 1).$$

Set $z = e^{i\varphi}$; then

$$\sigma - \tfrac{1}{2} = \epsilon + r(1 - \cos\varphi), \qquad d\varphi = \frac{d\sigma}{r \sin\varphi}.$$

Since $m_1^{1-2\sigma}$ is a monotone decreasing function of σ, we obtain the estimate

$$(28)\qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(e^{i\varphi})|\, d\varphi < O(h)\epsilon^{-1} m_2^{-1} \int_0^{\pi/2} m_1^{1-2\sigma}\, d\varphi.$$

Moreover,

$$\sin^2\varphi = 1 - \cos^2\varphi \geqq 1 - \cos\varphi = r^{-1}(\sigma - \tfrac{1}{2} - \epsilon) \qquad (0 \leqq \varphi \leqq \pi/2);$$

hence

$$(29)\qquad \begin{aligned} \int_0^{\pi/2} m_1^{1-2\sigma}\, d\varphi &< r^{-\frac{1}{2}} \int_{\frac{1}{2}+\epsilon}^{\infty} m_1^{1-2\sigma} (\sigma - \tfrac{1}{2} - \epsilon)^{-\frac{1}{2}}\, d\sigma \\ &= r^{-\frac{1}{2}} m_1^{-2\epsilon} m_2^{-\frac{1}{2}} \int_0^{\infty} u^{-\frac{1}{2}} e^{-2u}\, du = m_1^{-2\epsilon} O(m_2^{-1}), \end{aligned}$$

and (24), (27), (28), (29) imply the formula

$$B(\rho) \log \rho^{-1} = O(m_2^{-1}) + \epsilon^{-1} m_1^{-2\epsilon} m_2^{-2} O(h) \qquad (0 < \rho < 1).$$

In view of (1), (13), (26),

$$f(z) = (s+1)^{-1} P_m(s) \prod_{\substack{p \mid q_1 \\ \chi}} (1 - \chi(p) p^{-s});$$

consequently, the zeros of $f(z)$ and $P_m(s)$ coincide in the half-plane $\sigma > 0$. Choose $\rho = 1 - \vartheta r^{-1}$ with $\vartheta = (\gamma_3 - \gamma_2) m_2^{-1}$, and denote by B the number of zeros of $P(s)$ in the circle of radius $r - \vartheta$ with the center $\sigma_0 = \frac{1}{2} + \epsilon + r$; then

$$1/\log \rho^{-1} = O(rm_2) = O(m_2^2)$$

$$B = B(\rho) = O(m_2) + \epsilon^{-1} m_1^{-2\epsilon} O(h).$$

Finally, let $m_2^{-1} < \delta < \frac{1}{2}$, $\epsilon = \delta + (\gamma_2 - 1) m_2^{-1}$. By (25),

$$2\left(\frac{1-\gamma_2}{m_2} - \vartheta\right) r + \vartheta^2 > T_0^2 + \left(\frac{1-\gamma_2}{m_2}\right)^2$$

$$(r - \vartheta)^2 > \left(r - \frac{1-\gamma_2}{m_2}\right)^2 + T_0^2 = (\delta - \epsilon - r)^2 + T_0^2;$$

this proves that the whole rectangle $\frac{1}{2} + \delta < \sigma < 1$, $-T_0 < t < T_0$ lies in the circle $|s - \sigma_0| < r - \vartheta$. We denote by A the number of zeros of $P(s)$ in this rectangle. Since

$$\epsilon^{-1} m_1^{-2\epsilon} = m_1^{-2\delta} O(\delta^{-1})$$

and, by (6),

$$\delta m_1^{2\delta} m_2 h^{-1} < m_1 m_2 h^{-1} = m_1 m_2 m^{-1} \prod_{p \mid m} (1 - p^{-1})^{-1}$$

$$< \zeta(2) m_1 m_2 m^{-1} \prod_{p \mid m} (1 + p^{-1}) = m_1 m_2 m^{-1} O(m_2) = O(1),$$

it follows that

$$A \leq B = \delta^{-1} m_1^{-2\delta} O(h).$$

This is the assertion of Theorem I.

Let $m_3 > 2$ and $m_3 > 2c_1$; then the number $\delta = \frac{1}{2} m_2^{-1} m_3$ satisfies the condition $m_2^{-1} < \delta < \frac{1}{2}$ of Theorem I. Consequently, the number of zeros of $P(s)$ in the rectangle $\frac{1}{2}(1 + m_2^{-1} m_3) < \sigma < 1$, $-T_0 < t < T_0$ is less than

$$c_1 \delta^{-1} m_1^{-2\delta} h = 2 c_1 m_3^{-1} h < h,$$

so at least one of the h functions $L(s, \chi)$ has no zeros in this rectangle. This proves Theorem II.

5. Each character χ modulo m determines in a unique way a divisor d of m and a proper character $\hat{\chi}$ modulo d such that

$$L(s, \chi) = L(s, \hat{\chi}) \prod_{p \mid m} (1 - \hat{\chi}(p) p^{-s}).$$

It is well known that

$$\prod_{\chi} L(s, \hat{\chi}) = \zeta_m(s)$$

is the zeta-function of the field K of the m^{th} roots of unity. The degree of K is $h = \varphi(m)$, the absolute value of the discriminant of K is

$$D = \left(m \prod_{p \mid m} p^{-1/(p-1)}\right)^h, \tag{30}$$

and K is totally imaginary in case $m > 2$. In virtue of Hecke's theorem, the function

$$\psi(s) = (2\pi)^{-(h/2)s} D^{s/2} \Gamma^{h/2}(s) \zeta_m(s) \qquad (m > 2) \tag{31}$$

fulfills the equation

$$\psi(s) = \psi(1 - s). \tag{32}$$

Let $0 < T < T_0$ and denote by $A_0(T)$ the number of zeros of $\psi(s)$ in the rectangle $-\frac{1}{2} < \sigma < \frac{3}{2}$, $-T < t < T$. Since $\psi(s)$ is regular in the whole s-plane except for the two poles of first order at $s = 0$ and $s = 1$, we infer that $2\pi A_0(T) - 4\pi$ equals the variation of arg $\psi(s)$, if s runs through the boundary of the rectangle in positive direction and all zeros of $\psi(s)$ on the contour itself are avoided by sufficiently small half-circles in the interior. Starting from the right lower vertex $s = \frac{3}{2} - Ti$, we denote the successive variations on the four sides by Δ_1, Δ_2, Δ_3, Δ_4; moreover, let Δ be the variation on the right half of the second side, i.e., in the interval $\frac{3}{2} \geqq \sigma \geqq \frac{1}{2}$, $t = T$.

If $\mathfrak{p}$ runs over all prime ideals in K, then

$$\log \zeta_m(s) = \sum_{\mathfrak{p}} \log (1 - N\mathfrak{p}^{-s})^{-1} = \sum_{\substack{l=1 \\ \mathfrak{p}}}^{\infty} l^{-1} N\mathfrak{p}^{-ls} \qquad (\sigma > 1)$$

$$|\log \zeta_m(s)| \leqq \sum_{l,\mathfrak{p}} l^{-1} N\mathfrak{p}^{-l\sigma} \leqq h \sum_{l,p} l^{-1} p^{-l\sigma} = h \log \zeta(\sigma).$$

In particular, this holds on $\sigma = \frac{3}{2}$, and it follows, because of (31), (32), that

$$\Delta_3 = \Delta_1 = T \log D + O(h). \tag{33}$$

Furthermore, $\psi(\bar{s}) = \overline{\psi(s)}$; hence

$$\Delta_4 = \Delta_2 = 2\Delta. \tag{34}$$

It remains to deduce an upper estimate of $|\Delta|$.

Set

$$\frac{P_m(s)}{\zeta_m(s)} = (s - 1) \prod_{\substack{p \mid m \\ \chi}} (1 - \hat{\chi}(p)p^{-s}) = R(s), \qquad \tfrac{3}{2} + Ti = s_0,$$

$$\frac{1}{2}\left\{\frac{P(s)}{P(s_0)} + \frac{P(s - 2Ti)}{P(s_0 - 2Ti)}\right\} = g(s). \tag{35}$$

Plainly, $g(s_0) = 1$, and $g(s)$ is real on $t = T$, viz., equal to the real part of $P(s)/P(s_0)$. If C denotes the number of zeros of $g(s)$ on the segment $\frac{3}{2} \geqq \sigma \geqq \frac{1}{2}$, $t = T$, then the variation of arg $P(s)$ on this segment satisfies the inequality

$$| \Delta\{P(s)\} | < \pi(C + 1),$$

whence

$$(36) \quad \Delta = \Delta\{\psi(s)\} = \Delta\left\{(2\pi)^{-(h/2)s} D^{s/2} \Gamma^{(h/2)s} \frac{P(s)}{R(s)}\right\} = -\Delta\{R(s)\} + O(h) + O(C).$$

Applying (7) and the inequality

$$| \log (1 - z)^{-1} | \leqq \log (1 - | z |)^{-1} \leqq (1 - | z |)^{-1} \log (1 + | z |) \qquad (| z | < 1),$$

we obtain

$$-\log \frac{R(s)}{s - 1} = \sum_{\substack{p | m \\ \chi}} \log(1 - \hat{\chi}(p)p^{-s})^{-1} = m_1^{\frac{1}{2}} m_2^{-1} O(h)$$

$$(37) \qquad -\Delta\{R(s)\} = m_1^{\frac{1}{2}} m_2^{-1} O(h) + O(1) = m_1^{\frac{1}{2}} m_2^{-1} O(h).$$

6. In order to estimate the number C, we apply Jensen's inequality (24) with

$$(38) \qquad f(z) = g(s), \qquad z = \frac{s_0 - s}{1 + \epsilon}, \qquad s_0 = \tfrac{3}{2} + Ti, \qquad \epsilon = m_1^{-\frac{1}{4}},$$

Obviously, $f(z)$ is regular for $| z | \leqq 1$, and $f(0) = g(s_0) = 1$. Since

$$\log \frac{P(s_0)}{s_0 - 1} = \sum_{\chi} \log L(s_0 , \chi) = \sum_{\chi, p, l} l^{-1} \chi(p^l) p^{-ls_0} = h \sum_{p^l \equiv 1 (\mathrm{mod}\, m)} l^{-1} p^{-ls_0}$$

$$\left| \log \frac{P(s_0)}{s_0 - 1} \right| \leqq h \sum_{p^l \equiv 1 (\mathrm{mod}\, m)} p^{-\frac{3}{2}l} < hm^{-\frac{1}{2}} \zeta(\tfrac{3}{2}) = O(1),$$

we conclude from (18), (35), (38) that

$$h^{-1} \log | f(z) | < \begin{cases} \log \dfrac{1}{2\sigma - 1} + O(1) & (\frac{1}{2} < \sigma < 3) \\ 2\epsilon m_1 + O(m_2) & (\frac{1}{2} - \epsilon \leqq \sigma \leqq \frac{1}{2}), \end{cases}$$

on the circle $| z | = 1$.

Set $z = e^{i\varphi}$, then

$$8\pi^{-2} \varphi^2 \leqq 1 - \cos \varphi = 1 + \frac{\sigma + \frac{3}{2}}{1 + \epsilon} = \frac{\sigma - \frac{1}{2} - \epsilon}{1 + \epsilon} < \epsilon$$

for $\frac{1}{2} - \epsilon \leqq \sigma \leqq \frac{1}{2}$, whence

$$\int_{\sigma \leqq \frac{1}{2}} \log | f(e^{i\varphi}) | \, d\varphi < O(h)(\epsilon^{\frac{3}{2}} m_1 + \epsilon^{\frac{1}{2}} m_2) = O(h).$$

On the other hand, since

$$d\varphi = \frac{d\sigma}{(1+\epsilon)\sin\varphi},$$

$$\int_{\sigma>\frac{1}{2}} \log|f(e^{i\varphi})|\,d\varphi$$

$$< O(h)\left\{\int_{\frac{1}{2}}^{\frac{3}{2}}\left(\log\frac{1}{2\sigma-1}+O(1)\right)(\sigma-\tfrac{1}{2})^{-\frac{1}{2}}d\sigma+\int_{\pi/2}^{\pi}d\varphi\right\} = O(h).$$

Therefore

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(e^{i\varphi})|\,d\varphi < O(h).$$

If $\rho = \dfrac{1}{1+\epsilon}$, then the circle $|z| \leqq \rho$ contains the segment corresponding to $\frac{1}{2} \leqq \sigma \leqq \frac{3}{2}$, $t = T$; hence

$$C\log(1+\epsilon) = O(h), \qquad C = m_1^{2/3}O(h).$$

Because of (33), (34), (36), (37), we obtain

$$\Delta = m_1^{2/3}O(h)$$

$$2\pi A_0(T) - 4\pi = \Delta_1 + \Delta_3 + 4\Delta = 2T\log D + m_1^{2/3}O(h).$$

Furthermore, by (5), (30),

$$hm_1 - \log D = h\sum_{p\mid m}\frac{\log p}{p-1} = O(h)\sum_{p<m_1}p^{-1}\log p = m_2\,O(h);$$

consequently,

$$(39) \qquad A_0(T) = \pi^{-1}m_1hT + m_1^{2/3}O(h).$$

On the other hand, all zeros of $\psi(s)$ lie in the critical strip $0 < \sigma < 1$; therefore $A_0(T)$ equals the number of zeros of $P(s)$ in the rectangle $0 < \sigma < 1$, $-T < t < T$. We apply (39) for $T \to 0$ and use the formula $P(\bar{s}) = \overline{P(s)}$; it follows that the number of zeros of $P(s)$ in the rectangle $0 < \sigma < 1$, $0 \leq t < T$ has the value

$$\tfrac{1}{2}A_0(T) + m_1^{2/3}O(h) = \frac{1}{2\pi}m_1hT + m_1^{2/3}O(h).$$

This is the assertion of Theorem III.

7. For the proof of Theorem IV we use the following lemma; it is somewhat simpler than a related proposition involving elliptic functions which was introduced by Littlewood and Hoheisel in their above-mentioned publications.

LEMMA: *Let* $\lambda > 0$, $0 < \xi < 1$, $0 < M_0 < M$, *and assume that* $f = f(z)$ *is a regular analytic function of* $z = x + iy$ *in the rectangle* $0 \leqq x \leqq 1$, $-\dfrac{1}{2\lambda} \leqq y \leqq \dfrac{1}{2\lambda}$, *that the real part* $\Re\{f(z)\} \leqq M$ *in the whole rectangle and that the absolute value* $|f(z)| \leqq M_0$ *on the right side* $x = 1$; *then*

$$\log\left|\frac{2M}{f(\xi)} - 1\right| \Big/ \log\left(\frac{2M}{M_0} - 1\right) \geqq \sinh(\pi\lambda\xi) \Big/ \sinh(\pi\lambda).$$

PROOF: We consider the entire function

$$g(z) = \alpha^{\sinh(\pi\lambda z)} = \exp\{\log\alpha \sinh(\pi\lambda z)\}, \qquad \log\alpha = \frac{\log\left(\frac{2M}{M_0} - 1\right)}{\sinh(\pi\lambda)} > 0.$$

Since $\Re\{\sinh z\} = \sinh x \cos y$, we have $|g(z)| = 1$ on the three straight lines $x = 0$ and $y = \pm\frac{1}{2\lambda}$. Moreover,

$$\Re\{\sinh(\pi\lambda z)\} \leqq \sinh(\pi\lambda), \qquad |g(z)| \leqq \frac{2M}{M_0} - 1 \qquad (x = 1).$$

On the other hand, set

$$f_0(z) = (1 - 2Mf^{-1})^{-1} = \frac{(f - M) + M}{(f - M) - M};$$

then $|f_0(z)| \leqq 1$ in the whole rectangle $0 \leqq x \leqq 1$, $-\frac{1}{2\lambda} \leqq y \leqq \frac{1}{2\lambda}$, and

$$|f_0(z)|^{-1} \geqq 2M|f|^{-1} - 1 \geqq \frac{2M}{M_0} - 1 \geqq |g(z)| \quad . \qquad (x = 1).$$

It follows that the product $f_0(z)g(z) = h(z)$ satisfies the inequality $|h(z)| \leqq 1$ on the contour of the rectangle, and also in the interior, because of the regularity of $h(z)$. Hence

$$|h(\xi)| \leqq 1, \qquad |g(\xi)| \leqq |f_0(\xi)|^{-1}$$

$$\log\left|\frac{2M}{f(\xi)} - 1\right| = -\log|f_0(\xi)| \geqq \log|g(\xi)|$$

$$= \log\alpha \sinh(\pi\lambda\xi) = \frac{\sinh(\pi\lambda\xi)}{\sinh(\pi\lambda)} \log\left(\frac{2M}{M_0} - 1\right);$$

q.e.d.

It suffices to prove Theorem IV in the case of a proper character χ. Set $\frac{1}{2}\{\chi(1) - \chi(-1)\} = a$, then

$$\varphi(s, \chi) = \left(\frac{m}{\pi}\right)^{s/2} \Gamma\left(\frac{s + a}{2}\right) L(s, \chi) \tag{40}$$

satisfies the functional equation

$$\varphi(s, \chi) = \omega(\chi)\varphi(1 - s, \bar{\chi}), \quad . \quad |\omega(\chi)| = 1; \tag{41}$$

furthermore, $\overline{L(s, \chi)} = L(\bar{s}, \bar{\chi})$. Let $-T_0 < T_1 < T_2 < T_0$, $T_2 - T_1 > 4m_3^{-1}$, and assume that $L(s) = L(s, \chi)$ has no zero in the rectangle $\frac{1}{2} \leqq \sigma < 1$, $T_1 < t < T_2$. On the other hand, no zero of $L(s)$ lies in the half-plane $\sigma \geqq 1$. We define

$$\sigma_0 = \frac{15}{14}, \qquad \tfrac{1}{2}(T_1 + T_2) = T, \qquad s_0 = \sigma_0 + Ti,$$

$$\epsilon = 4m_3^{-1}, \qquad \lambda = \epsilon^{-1}(\sigma_0 + 1), \qquad z = \frac{s + 1 - Ti}{\sigma_0 + 1};$$

then it follows from (40), (41) that the function

$$f(z) = \log \frac{L(s)}{(s + 1)L(s_0)}$$

is regular in the rectangle $0 \leqq x \leqq 1$, $-\frac{1}{2\lambda} \leqq y \leqq \frac{1}{2\lambda}$.

Since

$$|\log L(s)| = |\sum_{p,l} l^{-1}\chi(p^l)p^{-ls}| \leqq \sum_{p,l} l^{-1}p^{-l\sigma_0} = c_6 \qquad (\sigma = \sigma_0),$$

we may choose $M_0 = c_7$. Moreover, because of (9), (10), (14),

$$L(s) = \sum_{k=1}^{m} \chi(k)(k^{-s} + O(m^{-\sigma})) = O(m^{1/2}) \qquad (\sigma \geqq \tfrac{1}{2};\ T_1 < t < T_2)$$

$$\log |L(s)| < O(m_1).$$

Gonsequently, in virtue of (41), for $-1 \leqq \sigma \leqq \frac{1}{2}$ and $T_1 < t < T_2$,

$$\log \left| \frac{L(s, \chi)}{s + a} \right| = \log |L(1 - \bar{s}, \chi)| + \log \left| \frac{\Gamma(\frac{1}{2}(1 - \bar{s} + a))}{(s + a)\Gamma(\frac{1}{2}(s + a))} \right|$$
$$+ (\tfrac{1}{2} - \sigma) \log \frac{m}{\pi} < O(m_1).$$

We conclude that the latter estimate holds good in the whole rectangle $-1 \leqq \sigma \leqq \sigma_0$, $T_1 < t < T_2$, and we may choose $M = c_8 m_1$. Plainly,

$$\log\left(\frac{2M}{M_0} - 1\right) = \log\left(\frac{2c_8}{c_7} m_1 + 1\right) = m_2 + O(1).$$

Set

$$s_1 = 1 - \bar{s}_0 = 1 - \sigma_0 + Ti, \qquad \xi = \frac{s_1 + 1 - Ti}{\sigma_0 + 1} = \frac{2 - \sigma_0}{\sigma_0 + 1} = \frac{13}{29};$$

then

$$\Re\{f(\xi)\} = \log \left| \frac{L(s_1, \chi)}{(s_1 + a)L(s_0, \chi)} \right| = \log \left| \frac{\Gamma(\frac{1}{2}(s_0 + a))}{(s_1 + a)\Gamma(\frac{1}{2}(s_1 + a))} \right| + (\sigma_0 - \tfrac{1}{2}) \log \frac{m}{\pi}$$
$$= (\sigma_0 - \tfrac{1}{2})m_1 + O(1) = \frac{4}{7} m_1 + O(1)$$

$$\log\left|\frac{2M}{f(\xi)}-1\right| \leq \log\left|\frac{2M}{\Re\{f(\xi)\}}-1\right| = \log\left|\frac{2c_8 m_1}{4/7(m_1+O(1))}-1\right| = O(1)$$

$$\log\left|\frac{2M}{f(\xi)}-1\right| \Big/ \log\left(\frac{2M}{M_0}-1\right) < O(m_2^{-1}) \tag{42}$$

and

$$\sinh(\pi\lambda\xi)/\sinh(\pi\lambda) \sim \exp(\pi\lambda\xi - \pi\lambda) = \exp\{\pi\epsilon^{-1}(1-2\sigma_0)\} = m_2^{-2\pi/7}. \tag{43}$$

Since $2\pi/7 < 1$, formulas (42), (43) contradict the lemma, provided m is sufficiently large. This proves Theorem IV.

It is easily seen that the essential properties of the quantities σ_0 and ϵ, in the last proof, are the inequalities $\sigma_0 > 1$ and $\epsilon m_3 > (2\sigma_0 - 1)\pi$. This leads to the following slight refinement of Theorem IV: Suppose $-T_0 < T_1 < T_2 < T_0$, $\alpha > \pi$ and $T_2 - T_1 > \alpha m_3^{-1}$; if m is larger than a certain number depending only on T_0 and α, then $L(s, \chi)$ possesses a zero in the rectangle $\frac{1}{2} \leqq \sigma < 1$, $T_1 < t < T_2$.

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