Define the “logarithmic integral” function \( \text{li}(x) = \int_2^x \frac{du}{\log u} \).

1. In this problem, we will explore various ways to write the error term in the prime number theorem for \( \pi(x) \).

(a) Using integration by parts, or otherwise, show that \( \text{li}(x) = \frac{x}{\log x} + \int_2^x \frac{du}{\log^2 u} - \frac{2}{\log 2} \).

(b) Show that \( \text{li}(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + O\left(\frac{x}{\log^4 x}\right) \).

(c) For any positive integer \( K \), prove that \( \pi(x) = \sum_{k=1}^{K} \frac{(k - 1)!x}{\log^k x} + O\left(\frac{x}{(\log x)^{K+1}}\right) \). You may assume equation (1) below to accomplish this task.

(d) For any fixed \( \alpha > 2 \), deduce that it is not the case that \( \pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^\alpha x}\right) \).

(a) Integration by parts (integrating 1 and differentiating \( 1/\log u \)) yields

\[
\text{li}(x) = \frac{u}{\log u}\bigg|_2^x - \int_2^x u \left( -\frac{1}{u \log^2 u} \right) du = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{du}{\log^2 u}.
\]

(b) We continue integrating by parts:

\[
\text{li}(x) = \frac{x}{\log x} + \int_2^x \frac{du}{\log^2 u} + O(1)
= \frac{x}{\log x} + \frac{u}{\log^2 u}\bigg|_2^x - \int_2^x u \left( -\frac{2}{u \log^3 u} \right) du + O(1)
= \frac{x}{\log x} + \frac{x}{\log^2 x} + \int_2^x \frac{2}{\log^3 u} du + O(1)
= \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2u}{\log^3 u}\bigg|_2^x - \int_2^x u \left( -\frac{6}{u \log^4 u} \right) du + O(1)
= \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \int_2^x \frac{6}{\log^4 u} du + O(1).
\]

As for the remaining integral, again we split at some \( 2 \leq y \leq x \) and estimate each integral trivially:

\[
\int_2^x \frac{6}{\log^4 u} du = \int_2^y \frac{6}{\log^4 u} du + \int_y^x \frac{6}{\log^4 u} du \ll y + x \cdot \frac{1}{\log^4 y},
\]

and many choices of \( y \) make the right-hand side \( \ll x/\log^4 x \) (for example, \( y = \sqrt{x} \)).

Another way of estimating this last integral: noting that

\[
\frac{d}{dx} \left( \frac{x}{\log^4 x} \right) = \frac{1}{\log^4 x} - \frac{4}{\log^5 x} \geq \frac{1/2}{\log^4 x} \quad \text{for } \log x \geq 8,
\]
we may write (when \( x \geq e^8 \))

\[
\int_2^x \frac{6}{\log^4 u} \, du = \int_2^{e^8} \frac{6}{\log^4 u} \, du + \int_{e^8}^x \frac{6}{\log^4 u} \, du \leq \int_2^{e^8} \frac{6}{\log^4 u} \, du + 12 \int_{e^8}^x \left( \frac{1}{\log^4 u} - \frac{4}{\log^5 u} \right) \, du,
\]

and therefore

\[
\int_2^x \frac{6}{\log^4 u} \, du \ll 1 + \int_{e^8}^x \left( \frac{1}{\log^4 u} - \frac{4}{\log^5 u} \right) \, du = 1 + \frac{u}{\log^4 u} \bigg|_{e^8}^x \ll \frac{x}{\log^4 x}.
\]

(c) Using repeated integration by parts as in part (b), it is easy to prove by induction on \( K \) that

\[
\text{li}(x) = \sum_{k=1}^{K} \frac{(k-1)!x}{\log^k x} + \int_2^x \frac{K!}{(\log u)^{K+1}} + O_K(1).
\]

(Notice a slight subtlety of the notation: adding \( K \) quantities that are each \( O(1) \) yields a quantity that is \( O_K(1) \), but not necessarily \( O(1) \) uniformly in \( K \).) As in part (b), splitting the remaining integral at \( y = \sqrt{x} \), say, shows that the integral is \( \ll_K x/(\log x)^{K+1} \).

Therefore by problem #1(b), there exists an absolute constant \( c > 0 \) such that

\[
\pi(x) = \text{li}(x) + O(x \exp(-c\sqrt{\log x}))
\]

\[
= \sum_{k=1}^{K} \frac{(k-1)!x}{\log^k x} + O_K \left( \frac{x}{(\log x)^{K+1}} + x \exp(-c\sqrt{\log x}) \right)
\]

\[
= \sum_{k=1}^{K} \frac{(k-1)!x}{\log^k x} + O_K \left( \frac{x}{(\log x)^{K+1}} \right),
\]

since \( (\log x)^{K+1} \ll_K \exp(c\sqrt{\log x}) \) for any \( K \). (No dependence on \( c \) is necessary since it is an absolute constant.)

[Note that it is tempting to extend this finite series to an infinite series, writing something like \( \text{li}(x) = \sum_{k=1}^{\infty} \frac{(k-1)!x}{\log^k x} \). However, the ratio test reveals that this series does not converge for any value of \( x \) ! This is an example of a divergent series, where any specific truncation provides a good approximation asymptotically even though the infinite series itself isn’t useful.]

(d) Suppose that the estimate did hold; then from part (c) with \( K = 2 \),

\[
\frac{x}{\log x} + O \left( \frac{x}{\log^2 x} \right) = \pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + O \left( \frac{x}{\log^3 x} \right);
\]

after rearranging this becomes

\[
\frac{x}{\log^2 x} = O \left( \frac{x}{\log^3 x} + \frac{x}{\log^2 x} \right)
\]

which is certainly false when \( \alpha > 2 \).

2. In this problem, we will give an asymptotic formula for \( \pi(x) \) with a better error term than what we saw in class.

(a) Show that

\[
\pi(x) - \text{li}(x) = \frac{\theta(x) - x}{\log x} + 2 + \int_2^x \frac{\theta(u) - u}{u \log^2 u} \, du.
\]
(b) Suppose that $c > 0$ is a constant such that $\theta(x) = x + O(x \exp(-c \sqrt{\log x}))$. Prove that

$$\pi(x) = \text{li}(x) + O(x \exp(-c \sqrt{\log x})).$$  \hfill (1)

(a) We can write $\pi(x) = \sum_{p \leq x} 1$ in terms of $\theta(x) = \sum_{p \leq x} \log p$ using Riemann–Stieltjes integrals:

$$\pi(x) = \int_{2-}^{x} \frac{1}{\log u} \, d\theta(u) = \int_{2-}^{x} \frac{1}{\log u} \, d(\theta(u) - u) + \int_{2-}^{x} \frac{1}{\log u} \, du$$

$$= \int_{2-}^{x} \frac{1}{\log u} \, d(\theta(u) - u) + \text{li}(x) - \text{li}(2-).$$

Rearranging terms, replacing $\text{li}(2-)$ by $\text{li}(2) = 0$ (due to the implicit limit in that lower endpoint that will soon be taken), and integrating by parts, we obtain

$$\pi(x) - \text{li}(x) = \frac{\theta(u) - u}{\log u} \bigg|_{2-}^{x} - \int_{2-}^{x} (\theta(u) - u) \frac{1}{\log u} \, du$$

$$= \frac{\theta(x) - x}{\log x} - 0 - 2 + \int_{2}^{x} (\theta(u) - u) \frac{1}{u \log^2 u} \, du.$$

(b) From part (a),

$$\pi(x) - \text{li}(x) \ll \frac{x \exp(-c \sqrt{\log x})}{\log x} + 1 + \int_{2}^{x} \frac{u \exp(-c \sqrt{\log u})}{u \log^2 u} \, du$$

$$\ll \frac{x \exp(-c \sqrt{\log x})}{\log x} + \int_{2}^{y} \frac{\exp(-c \sqrt{\log u})}{\log^2 u} \, du + \int_{y}^{x} \frac{\exp(-c \sqrt{\log u})}{\log^2 u} \, du$$

for any $2 \leq y \leq x$. Since the integrand is positive and decreasing for $u \geq 2$, it is also bounded, and so

$$\pi(x) - \text{li}(x) \ll x \exp(-c \sqrt{\log x}) + y + (x - y) \frac{\exp(-c \sqrt{\log y})}{\log^2 y}$$

$$\ll x \exp(-c \sqrt{\log x}) + y + x \exp(-c \sqrt{\log y}).$$

A reasonable choice for $y$ seems to be $y = x \exp(-c \sqrt{\log x})$. With this choice,

$$\log y = \log x - c \sqrt{\log x} = (\log x) \left(1 + O\left(\frac{\sqrt{\log y}}{\log x}\right)\right);$$

since $\sqrt{1 + O(\varepsilon)} = 1 + O(\varepsilon)$ by the tangent line for $\sqrt{1+t}$ at $t = 0$,

$$\sqrt{\log y} = \sqrt{\log x} \left(1 + O\left(\frac{\sqrt{\log y}}{\log x}\right)\right) = \sqrt{\log x} + O(1).$$

We conclude that

$$\pi(x) - \text{li}(x) \ll x \exp(-c \sqrt{\log x}) + x \exp(-c(\sqrt{\log x} + O(1))) \ll x \exp(-c \sqrt{\log x}),$$

since $\exp(O(1)) \ll 1$.

Alternatively, we can use the “wishful thinking derivative” method we saw in #1(b): since

$$\frac{d}{dx}(x \exp(-c \sqrt{\log x})) = \exp(-c \sqrt{\log x}) - \frac{c \exp(-c \sqrt{\log x})}{2 \sqrt{\log x}} \gg x \exp(-c \sqrt{\log x}),$$
we have
\[
\int_2^x \frac{\exp(-c\sqrt{\log u})}{\log^2 u} \, du \ll \int_2^x \exp(-c\sqrt{\log u}) \, du
\]
\[
\ll \int_2^x \frac{d}{du}(u \exp(-c\sqrt{\log u})) \, du
\]
\[
= x \exp(-c\sqrt{\log x}) - 2 \exp(-c\sqrt{\log 2}) \ll x \exp(-c\sqrt{\log x}),
\]
with which the required estimate follows from equation (2).