For all of these questions, we define the Bernoulli polynomials $B_k(x)$ as coefficients in the power series expansion

\[ f(z, x) = \frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} = 1 + \left( x - \frac{1}{2} \right) z + \left( x^2 - x + \frac{1}{6} \right) \frac{z^2}{2} + \cdots. \]

(1)

We also define the Bernoulli numbers $B_k = B_k(0)$, a few of which have been listed below.

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_k$</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{6}$</td>
<td>0</td>
<td>$-\frac{1}{30}$</td>
<td>0</td>
<td>$\frac{1}{42}$</td>
<td>0</td>
<td>$-\frac{1}{30}$</td>
<td>0</td>
<td>$\frac{5}{66}$</td>
<td>0</td>
<td>$-\frac{691}{2730}$</td>
</tr>
</tbody>
</table>

Finally, recall our notation for the fractional part $\{x\} = x - \lfloor x \rfloor$.

On this group work, you may differentiate or integrate infinite series term by term with impunity (that is, don’t worry about convergence issues on this group work).

1. Preliminaries:

   (a) Show that $\int_0^1 B_k(x) \, dx = 0$ for all $k \geq 1$.

   (b) Verify the identity $\frac{\partial f(z, x)}{\partial x} = zf(z, x)$, and conclude that $B_k'(x) = kB_k-1(x)$ for all $k \geq 1$.

   (c) Show that $f(z, 0) + z/2$ is an even function of $z$. Conclude that $B_1 = -1/2$ and that $B_{2j+1} = 0$ for all $j \geq 1$.

   (d) Prove that $B_k(1) = B_k$ for all $k \geq 2$, and conclude that $B_k(\{x\})$ is a continuous periodic function with period 1 for all $k \geq 2$. (Hint: part (b) has something to say about the difference $B_k(1) - B_k(0)$.)

   (e) Why do parts (a), (b), and (d) imply that $B_{k+1}(\{x\})/(k + 1)$ is an antiderivative for $B_k(\{x\})$ on the entire real line, for every $k \geq 1$?

(a) Integrating both sides of equation (1) with respect to $x$ yields

\[
\int_0^1 \frac{ze^{xz}}{e^z - 1} \, dx = \int_0^1 \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} \, dx
\]

\[
\left. \frac{ze^{xz}}{e^z - 1} \right|_0^1 = \sum_{k=0}^{\infty} \frac{z^k}{k!} \int_0^1 B_k(x) \, dx
\]

\[
1 = \frac{z}{e^z - 1} \int_0^1 B_k(x) \, dx
\]

By uniqueness of power series expansions, the coefficient $\frac{1}{k!} \int_0^1 B_k(x) \, dx$ of $x^k$ on the right-hand side equals the coefficient 0 of $z^k$ on the left-hand side for all $k \geq 1$, which gives the desired evaluation. (Integrating term by term is valid because, as one can check,
the series converges uniformly for $0 \leq x \leq 1$ for any fixed $z$ inside the disc of convergence \( \{|z| < 2\pi\} \) of the series.

(b) The verification is simple: \( \frac{\partial f(z, x)}{\partial x} = \frac{z}{e^z - 1} \frac{\partial e^z}{\partial x} = \frac{z}{e^z - 1} ze^x = zf(z, x) \). Writing this identity in terms of the series gives

\[
\frac{\partial}{\partial x} \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} = \sum_{k=0}^{\infty} B_k(x) \frac{\partial}{\partial x} \frac{z^k}{k!} = \sum_{k=0}^{\infty} B_k(x) \frac{z^{k+1}}{k!} = \sum_{k=1}^{\infty} B_{k-1}(x) \frac{z^k}{(k-1)!}.
\]

Again by uniqueness of power series expansions, we conclude that \( \frac{B_k'(x)}{k!} = \frac{B_{k-1}(x)}{(k-1)!} \) for all \( k \geq 1 \), which gives the desired identity.

(c) We have

\[
f(z, 0) + \frac{z}{2} = \frac{z}{e^z - 1} + \frac{z}{2} = \frac{2z + z(e^z - 1)}{2(e^z - 1)} = \frac{z(e^z + 1)}{e^z - 1} = \frac{z(e^{z/2} + e^{-z/2})}{e^{z/2} - e^{-z/2}},
\]

which we check is invariant under changing $z$ to $-z$. Therefore all of the odd power series coefficients of \( f(z, 0) + \frac{z}{2} = -\frac{1}{2} z + \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \) equal 0, which gives the required values.

(d) By the fundamental theorem of calculus and part (b),

\[
B_k(1) - B_k(0) = \int_0^1 B_k'(x) \, dx = \int_0^1 k B_{k-1}(x) \, dx,
\]

which equals 0 for \( k - 1 \geq 1 \) by part (a). Since \( \{x\} \) is a periodic function with period 1, so is \( B_k(\{x\}) \); and whenever \( x \) is not an integer, \( B_k(\{x\}) \) is a composition of two continuous functions and hence is itself continuous. On the other hand, when \( x \) is an integer, then (by continuity of \( B_k \)) we have \( \lim_{x \to x^+} B_k(\{x\}) = B_k(\lim_{x \to x^+} \{x\}) = B_k(1) = B_k(1) \) and \( \lim_{u \to x^+} B_k(\{u\}) = B_k(\lim_{u \to x^+} \{u\}) = B_k(0+), \) both of which equal \( B_k(\{x\}) = B_k(0) \). Therefore \( B_k(\{x\}) \) is continuous for all real numbers \( x \).

(e) Certainly part (b) implies that \( B_{k+1}(\{x\})/(k + 1) \) is an antiderivative for \( B_k(\{x\}) \) for \( 0 < x < 1 \), since \( \{x\} = x \) there. On the other hand, part (a) implies that \( \int_0^x B_k(\{u\}) \, du = \int_{[x]} B_k(\{u\}) \, du = \int_0^x B_k(u) \, du \), so that this antiderivative must be periodic with period 1. Part (d) then implies that \( B_{k+1}(\{x\})/(k + 1) \) is the appropriate antiderivative for all \( x \in \mathbb{R} \).

Remark: Although not relevant to this group work, it is a wonderful fact that the Bernoulli polynomials also provide the formulas for the sum of the first \( N \) powers of integers, generalizing the well-known formulas for the sum of the first \( N \) integers, squares, or cubes:

\[
\sum_{0 \leq n < N} n^k = \frac{B_{k+1}(N) - B_{k+1}}{k + 1} = \int_0^N B_k(x) \, dx.
\]

Thus the Bernoulli polynomial \( B_{k+1}(x) \) also functions as a “discrete antiderivative” of the simple power function \( x^k \).
For the next problem, you may use the known Fourier expansion of the sawtooth function

\[ -\frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin(2\pi mx) = \begin{cases} \{x\} - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{if } x \text{ is an integer.} \end{cases} \]  

(2)

2. A formula for the values of \( \zeta(s) \) at positive even integers:

(a) When \( x \notin \mathbb{Z} \), show that \( B_{2j}(\{x\}) = (-1)^{j-1}(2j)! \sum_{m=1}^{\infty} \frac{2\cos(2\pi mx)}{(2\pi m)^{2j}} \) for all \( j \geq 1 \).

(b) Deduce that for all \( j \geq 1 \),

\[ \zeta(2j) = \frac{(-1)^{j-1}2^{j-1}\pi^{2j}B_{2j}}{(2j)!}. \]  

(3)

Conclude that in particular, \( \zeta(2j) \) equals a rational number times \( \pi^{2j} \) for all \( j \geq 1 \).

(a) We prove, by induction on \( k \geq 1 \), that when \( x \) is not an integer,

\[ B_k(\{x\}) = (-1)^{\lfloor k/2 \rfloor - 1}k! \sum_{m=1}^{\infty} \begin{cases} 2\sin(2\pi mx)/(2\pi m)^k, & \text{if } k \text{ is odd,} \\ 2\cos(2\pi mx)/(2\pi m)^k, & \text{if } k \text{ is even}; \end{cases} \]  

(4)

the assertion follows upon taking \( k = 2j \). Since both expressions are periodic functions with period 1, we may assume \( 0 < x < 1 \). Note that \( B_1(x) = x - \frac{1}{2} \), as we see in equation (1), and therefore the case \( k = 1 \) of equation (4) is exactly the known identity (2).

Suppose equation (4) holds for some positive integer \( k \). Integrating both sides of the equation and using problem #1(b), we see that

\[ B_{k+1}(x) = \int (k + 1)B_k(x) \, dx \]

\[ = \int (k + 1) \left( (-1)^{\lfloor k/2 \rfloor - 1}k! \sum_{m=1}^{\infty} \begin{cases} 2\sin(2\pi mx)/(2\pi m)^k, & \text{if } k \text{ is odd,} \\ 2\cos(2\pi mx)/(2\pi m)^k, & \text{if } k \text{ is even}; \end{cases} \right) \, dx \]

\[ = C + \left( (-1)^{\lfloor k/2 \rfloor - 1}(k + 1)! \sum_{m=1}^{\infty} \begin{cases} -2\cos(2\pi mx)/(2\pi m)^{k+1}, & \text{if } k \text{ is odd,} \\ 2\sin(2\pi mx)/(2\pi m)^{k+1}, & \text{if } k \text{ is even}; \end{cases} \right) dx, \]

which we can see is exactly the desired formula (4) in the case \( k + 1 \), except possibly for the constant of integration \( C \). On the other hand, integrating both sides of equation (4) from 0 to 1 results in 0 on both sides (where we have used problem #1(a) for the left-hand side), and therefore the constant of integration must be \( C = 0 \).

Remark: The term-by-term integration can be justified by the uniform convergence of the series (4), which is easy to establish when \( k \geq 2 \) but not so obvious when \( k = 1 \). However, general theorems from the subject of Fourier series exist that justify the term-by-term integration for nice enough functions, including \( B_1(\{x\}) \).

(b) Taking the limit as \( x \to 0^+ \) of both sides of the identity proved in part (a) yields

\[ B_{2j} = B_{2j}(0) = (-1)^{j-1}(2j)! \sum_{m=1}^{\infty} \frac{2}{(2\pi m)^{2j}} = (-1)^{j-1}(2j)! \sum_{m=1}^{\infty} \frac{2}{(2\pi)^{2j}} \zeta(2j) \]

for all \( j \geq 1 \), which is the desired identity.
If we can prove that $B_k$ is a rational number for every $k \geq 0$, then the given formula for $\zeta(2j)$ is indeed a rational number times $\pi^{2j}$. Perhaps the easiest way to prove this is via the following logic: if $g(z, y)$ is a rational function of $z$ and $y$ with integer coefficients, then every derivative of $g$ is also a rational function of $z$ and $y$ with integer coefficients. Therefore every derivative of $g(z, e^z)$ is a rational function of $z$ and $e^z$ with integer coefficients (since the chain rule only produces additional factors of $e^z$); in particular, every derivative of $g(z, e^z)$ evaluated at $z = 0$ is a rational function of $0$ and $1$ with integer coefficients, that is, a rational number.

Remark: If we try to find the value of $\zeta(2j + 1)$ in this way, the corresponding series in part (a) is a sine series instead of a cosine series, and when we take the limit as $x \to 0^+$ we simply reprove $B_{2j+1} = 0$ for $j \geq 1$ without gaining any knowledge about $\zeta(2j + 1)$.

For the final problem, we will need the notation

$$\binom{z}{k} = \frac{z(z - 1) \cdots (z - (k - 1))}{k!}$$

for all $z \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$. (This is exactly the usual binomial coefficient $\binom{z}{k} = \frac{z!}{k!(z-k)!}$ when $z \geq k$ is an integer; the given formula shows that we can think of $\binom{z}{k}$ as a polynomial of degree $k$ in the complex variable $z$.) We also recall from equation (1.24) the formula, valid for $\sigma > 0$:

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \{x\} x^{-s-1} \, dx. \quad (5)$$

3. An alternate way to meromorphically continue $\zeta(s)$ to the entire complex plane:

(a) Show that for all integers $K \geq 1$ and all $\sigma > 0$,

$$\int_1^\infty B_K(\{x\}) x^{-s-K} \, dx = -\frac{B_{K+1}}{K+1} + \frac{\sigma - K}{K+1} \int_1^\infty B_{K+1}(\{x\}) x^{-(s-K+1)} \, dx.$$

(b) By induction on $K$ (or otherwise), show that for all integers $K \geq 1$ and all $\sigma > 0$,

$$\zeta(s) = 1 + \frac{1}{s-1} - \sum_{k=1}^K (-1)^k \binom{s-1}{k-1} \frac{B_k}{k} + (-1)^K \int_1^\infty B_K(\{x\}) x^{-s-K} \, dx. \quad (6)$$

(c) Show that the integral on the right-hand side of the above equation converges for $\sigma > 1 - K$. Conclude that $\zeta(s)$ can be analytically continued to the entire complex plane except for a simple pole at $s = 1$.

(d) Prove that $\zeta(-n) \in \mathbb{Q}$ for all $n \in \mathbb{Z}_{\geq 0}$.

(a) From problem #1(e), we know that $B_{K+1}(\{x\})/(K+1)$ is an antiderivative for $B_K(\{x\})$. Therefore integration by parts gives

$$\int_1^\infty B_K(\{x\}) x^{-s-K} \, dx = \frac{B_{K+1}(\{x\})}{K+1} x^{-s-K} \bigg|_1^\infty - \int_1^\infty B_{K+1}(\{x\}) (-s - K) x^{-s-K-1} \, dx$$

$$= 0 - \frac{B_{K+1}}{K+1} - \frac{\sigma - K}{K+1} \int_1^\infty B_{K+1}(\{x\}) x^{-s-(K+1)} \, dx,$$

since $B_K(1) = B_K$; the upper endpoint yields 0 because the continuous, periodic function $B_K(\{x\})$ is bounded and $\sigma > 0$ (indeed, even $\sigma > -K$ would suffice here).
(b) When we note from problem #1 that $B_1(x) = x - \frac{1}{2}$ and $B_1 = -\frac{1}{2}$, and that $\binom{-s}{1} = -s$ from its definition, we see that the base case $K = 1$ to be proved is the identity

$$\zeta(s) = 1 + \frac{1}{s - 1} - \frac{1}{2} - s \int_{1}^{\infty} \left( \{n\} - \frac{1}{2} \right) x^{-s - 1} \, dx,$$

which is the same as equation (5) once we observe that $s \int_{1}^{\infty} \frac{1}{2} x^{-s - 1} \, dx = \frac{1}{2}$ since $\sigma > 0$. As for the induction step, part (a) implies that

$$-(-1)^K \binom{-s}{K} \int_{1}^{\infty} B_K(\{x\}) x^{-s-K} \, dx = (-1)^K \binom{-s}{K} \frac{B_{K+1}}{K+1} + (-1)^K \binom{-s}{K} \frac{-s - K}{K+1} \int_{1}^{\infty} B_{K+1}(\{x\}) x^{-s-(K+1)} \, dx$$

$$= -(-1)^{K+1} \binom{-s}{K} \frac{B_{K+1}}{K} \frac{-s}{K+1} \int_{1}^{\infty} B_{K+1}(\{x\}) x^{-s-(K+1)} \, dx$$

from the definition of $\binom{-s}{K}$; therefore equation (6) for the case $K$ implies equation (6) for the case $K + 1$.

Remark: This identity for $\zeta(s)$ is an example of a much more general technique called Euler–Maclaurin summation (see Appendix B), in which the Bernoulli polynomials figure prominently. For example, one can get extremely good versions of Stirling’s formula (approximations to $n!$) by applying Euler–Maclaurin summation to $\log(n!) = \sum_{k=1}^{n} \log k$.

(c) Because the continuous periodic function $B_K(\{x\})$ is bounded (by some constant depending on $K$), we have the estimate

$$\int_{1}^{\infty} B_K(\{x\}) x^{-s-K} \, dx \ll_K \int_{1}^{\infty} x^{-\sigma-K} \, dx = \frac{1}{\sigma + K - 1}$$

as long as $\sigma > 1 - K$. Therefore equation (6) provides a meromorphic continuation of $\zeta(s)$ to the half-plane $\sigma > 1 - K$, with the only singularity caused by the term $1/(s - 1)$ (the binomial coefficients are simply polynomials in $s$). Since $K$ can be taken as large as we wish, we obtain a (necessarily unique) analytic continuation of $\zeta(s)$ to the entire complex plane other than the pole at $s = 1$.

Remark: Parts (b) and (c) illustrate a general phenomenon about integrals whose integrand contains an oscillatory term, namely that one can often get better convergence by integrating the integral by parts, integrating the oscillating function (here $B_K(\{x\})$) and differentiating the other part.

(d) If we choose $K = n + 1$ (or larger), then the definition of $\binom{-s}{K}$ includes a factor $-s - n$ on top, which vanishes when we set $s = -n$. Therefore

$$\zeta(-n) = 1 + \frac{1}{-n - 1} - \sum_{k=1}^{n+1} (-1)^k \binom{n}{k-1} \frac{B_k}{k}.$$

We observed in problem #2(b) that the Bernoulli numbers $B_k$ are all rational, and therefore this right-hand side is indeed a rational number.

Remark: With another half-hour or so, we could establish the exact formula

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

(7)
with similar elementary methods that use other combinatorial properties of the Bernoulli numbers and polynomials (see Appendix B for the details). On the other hand, it is a simple exercise to verify that the formula (7) follows directly from the formula (3) if we use the functional equation for $\zeta(s)$ that we saw in class (although the case $n = 0$ of equation (7) is slightly harder than the other cases).