Math 331–Homework #6
due at the beginning of class Monday, April 2, 2007

Reality-check problems. Not to write up; just ensure that you know how to do them.

I. Verify the following binomial coefficient identites hold for all complex numbers x and all integers k:

(UP) $(x-k)\binom{x}{k} = x\binom{x-1}{k}$

(DOWN) $k\binom{x}{k} = (x-k+1)\binom{x}{k-1}$

(CHARM) $k\binom{x}{k} = x\binom{x-1}{k-1}$

II. Wilf, p. 188, #3

III. Wilf, p. 190, #7

Homework problems. To write up and hand in.

I. Prove using the WZ-method that

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!(k+x)} = \frac{1}{x(x+1)(x+2)\cdots(x+n)}$$

for all nonnegative integers n and all $x \in \mathbb{C} \setminus \{0, -1, \dots, -n\}$. It might help make the calculations easier to write things in terms of binomial coefficients, but it's up to you. (A spooky voice in your mind says: "$R(n,k) = (x+k-1)/(n+1)\dots$.")

II. Let $u(x) = \dfrac{1-\sqrt{1-4x}}{2x} - 1$.

(a) Show that $u(x)$ is the unique function satisfying $u(0) = 0$ that is implicitly defined by $u = x(1+u)^2$.

(b) Use the Lagrange Inversion Formula to prove equation (2.5.16) on page 54 of Wilf. (Remark: equation (2.5.10) on the previous page is a special case of this.)

III. (a) Suppose that $u(x)$ is a function defined implicitly by $u = x\phi(u)$, where ϕ is a power series with $\phi(0) \neq 0$ (but not necessarily $\phi(0) = 1$). By making a suitable change of variable and appealing to the Lagrange Inversion Formula as stated in class (or otherwise), prove that we still have

$$[x^n]u(x) = \frac{1}{n}[t^{n-1}]\phi(t)^n.$$

(b) Define A_m to be the coefficient of $[t^{m-1}]$ in the power series expansion of

$$\frac{(t+1)^{2m}}{(t+2)^m}.$$

By working the Lagrange Inversion Formula backwards, show that

$$A_m = m[x^m]u(x)$$

for a function u implicitly defined by $u = x\phi(u)$ for a suitable ϕ. Solve that equation to get an explicit formula for $u(x)$, and by recognizing it as a function with a known power series (or otherwise), prove that

$$A_m = \frac{m}{4^m}\binom{2m}{m}.$$

IV. For this problem define $A(x) = 6/(2x^2 - 5x + 2)$ and $B(x) = A(x)^3$.

(a) Show that $PP(B; \frac{1}{2})$, the principal part of $B(x)$ around the pole at $x = \frac{1}{2}$, equals

$$-\frac{64/3}{(x-1/2)} - \frac{16}{(x-1/2)^2} - \frac{8}{(x-1/2)^3}\ .$$

(b) Define a_n to be the coefficients of the power series for $A(x) = \sum_{n=0}^{\infty} a_n x^n$ (with the usual convention that $a_n = 0$ for $n < 0$). Calculate an asymptotic formula for

$$b_n = \sum_j \sum_k a_j a_k a_{n-j-k}\ ,$$

with an error term that tends to zero as n tends to infinity.

V. Wilf, p. 188, #2

VI. Wilf, p. 188, #4. Note that γ_n is the number of ways of getting a total of n when you roll n "dice", each of which has an equal chance of yielding 0, 1, or 2.

Another interesting interpretation is that if two people repeatedly play rock-paper-scissors against each other, each person getting a point for every win and nobody getting a point for a draw, then $\gamma_n/3^n$ is the probability that the two players have the same number of points after n games.

VII. Wilf, p. 191, #10. Also, use the answer to part (d) to deduce that the average number of fixed points of an involution of n letters is $(1 + o(1))\sqrt{n}$.

VIII. (a) Let $T(x) = \sum_{n\geq 1} T_n x^n/n!$ be the exponential generating function of the sequence $\{T_n\}$ which counts the number of rooted labeled trees with n vertices. What is the radius of convergence of $T(x)$?

(b) Find an asymptotic formula for the quantity A_m defined in problem III(b). Your answer should turn out to be $A_m \sim \alpha m^\beta$ for some real numbers α and β.

Bonus problem.

Click on the link to Gosper's Algorithm on the course web page. Figure out and explain to me how step 3 is accomplished.