Math 331–Homework #3

due at the beginning of class Monday, February 12, 2007

Reality-check problems. Not to write up; just ensure that you know how to do them.

I. Wilf, p. 66, #11

II. Show that the various parts of Wilf's "definition" of lim sup on page 46 are all consequences of the definition of lim sup given in class.

III. If $\lim a_n$ exists, show that $\limsup a_n = \lim a_n$.

IV. Let $A(x)$ be the ordinary power series generating function for a sequence $\{a_n\}$, and let $E(x)$ be the exponential power series generating function for the same sequence.

(a) If $A(x)$ has a nonzero radius of convergence, show that the radius of convergence of $E(x)$ is $+\infty$.
(b) If $E(x)$ has a finite radius of convergence, show that the radius of convergence of $A(x)$ is 0.

V. For any positive integer n, show that $\sum_{k_1} \cdots \sum_{k_n} x^{|k_1|+\cdots+|k_n|} = \left(\frac{1+x}{1-x}\right)^n$.

VI. Wilf, p. 104, #3 and 6

Homework problems. To write up and hand in.

I. Using generating functions, compute the following numbers, where F_n denotes the Fibonacci sequence:

(a) $\sum_{n=0}^{\infty} \frac{F_n}{2^n}$

(b) $\sum_{n=0}^{\infty} \frac{F_n^2}{3^n}$

(c) $\sum_{n=0}^{\infty} \frac{F_n^2}{2^n}$ (be suspicious!)

(d) $\sum_{n=0}^{\infty} \frac{F_n}{n!}$

II. Recall that $\cos x$ is the exponential power series generating function for the periodic sequence

$$\{1, 0, -1, 0, 1, 0, -1, 0, \dots\} = \begin{cases} (-1)^{n/2}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

while $\sin x$ is the exponential power series generating function for the periodic sequence

$$\{0, 1, 0, -1, 0, 1, 0, -1, \dots\} = \begin{cases} 0, & \text{if } n \text{ is even,} \\ (-1)^{(n-1)/2}, & \text{if } n \text{ is odd.} \end{cases}$$

Using exponential power series generating functions, prove $\cos^2 x + \sin^2 x = 1$.

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III. Use the method of Wilf, p. 37, Example 4 to find a formula for the sum $\sum_{j=0}^{n} j^3$.

IV. The definition (2.4.1) of the radius of convergence of a series is reminiscent of the "root test" for convergence of a series. This problem discusses the "ratio test" version.

(a) Show that if $\lim |a_{n+1}/a_n|$ exists, then $\lim |a_n|^{1/n}$ also exists and $\lim |a_n|^{1/n} = \lim |a_{n+1}/a_n|$. Conclude that in this case, the radius of convergence R of the power series $\sum_{n=1}^{\infty} a_n x^n$ satisfies

$$R = \frac{1}{\lim |a_{n+1}/a_n|}.$$

(b) Give an example of a sequence $\{a_n\}$ of positive real numbers such that the quantity $\limsup(a_{n+1}/a_n)$ exists, but $\limsup a_n^{1/n} \neq \limsup(a_{n+1}/a_n)$. Conclude that we *cannot* in general say

$$R = \frac{1}{\limsup |a_{n+1}/a_n|}.$$

V. For positive integers m and n, let $f(m,n)$ denote the number of m-tuples of integers $(x_1, x_2, \ldots, x_m)$ such that $|x_1| + |x_2| + \cdots + |x_m| \leq n$.

(a) Explain why it's sensible to define $f(0,n) = 1$ and $f(m,0) = 1$ for all nonnegative integers m and n.

(b) Show that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m,n) x^m y^n = \frac{1}{1 - x - y - xy}.$$

(c) Using part (b), show that $f(m,n) = f(n,m)$.

VI. For a positive integer n, define $s(n)$ to be the number of subsets of $\{1, 2, \ldots, n\}$ whose size is a multiple of 3. For example, $s(4) = 5$ due to the five subsets $\{\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, and $\{2,3,4\}$ of $\{1,2,3,4\}$. We might expect that about a third of the 2^n subsets have size that is a multiple of three. Prove that in fact,

$$s(n) = \frac{2^n}{3} + \begin{cases} 2/3, & \text{if } n \text{ is of the form } 6k, \\ 1/3, & \text{if } n \text{ is of the form } 6k \pm 1, \\ -1/3, & \text{if } n \text{ is of the form } 6k \pm 2, \\ -2/3, & \text{if } n \text{ is of the form } 6k + 3. \end{cases}$$

VII. Wilf, p. 104, #7

VIII. Wilf, p. 105, #15(a). Hint: prove it directly in the case where there's only one nonempty deck; then merge and flood.

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IX. BONUS PROBLEM. A group of n children is being signed up to form k song-and-dance teams. Each child will be signed up as either a singer or a dancer on that child's team. Each team must have at least one singer and at least one dancer. We're interested in the number $h(n,k)$ of different ways these song-and-dance teams can be formed.

(a) Describe this problem as one of cards, decks, and hands, with the number of children in each song-and-dance team being the weight of that card. Show that the nth deck has $d_n = 2^n - 2$ cards for $n \geq 1$ (with $d_0 = 0$).

(b) Show that the hand enumerator generating function $\mathcal{H}(x,y)$ is given by

$$\mathcal{H}(x,y) = e^{y(e^{2x}-2e^x+1)}.$$

Conclude that

$$h(n,k) = n![x^n]\left(\frac{(e^{2x}-2e^x+1)^k}{k!}\right). \qquad (*)$$

(c) Count $h(n,k)$ another way, as follows: first imagine the n children all deciding whether they will be singers or dancers. Then for each set of decisions, split the singers into k groups and the dancers into k groups; then pair off the k singer groups with the k dancer groups. Using this approach, justify why

$$h(n,k) = k! \sum_j \binom{n}{j} \left\{ {j \atop k} \right\} \left\{ {n-j \atop k} \right\}.$$

(d) Using problem VIII(b) on Homework #2 and part (c) above, conclude that

$$\{h(n,k)\}_{n=0}^{\infty} \overset{egf}{\longleftrightarrow} \frac{(e^x-1)^{2k}}{k!}.$$

How is this related to equation $(*)$ above?