First Name: _____ Last Name: _____

Student-No: _____ Section: _____

Grade:

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Short answer question

1. $\boxed{\text{2 marks}}$ Each part is worth 1 marks. Please write your answers in the boxes. Only your answers in the boxes will be marked (no partial credit).

   (a) Evaluate the indefinite integral $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$.

   Answer: $-2\cos(\sqrt{x}) + C$

   Solution: We use the substitution $u = \sqrt{x} = x^{1/2}$, so that $du = \frac{1}{2}x^{-1/2}\,dx$, or equivalently $2\,du = \frac{dx}{\sqrt{x}}$. Therefore

   $$\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx = \int (\sin u)\,2\,du = 2(-\cos u) + C = -2\cos(\sqrt{x}) + C.$$

   (b) Find the area between the curves $y = 2x$ and $y = 6x - 4x^2$, by first identifying the points of intersection and then integrating.

   Answer: $\left(2 - \dfrac{4}{3}\right) = \dfrac{2}{3}$

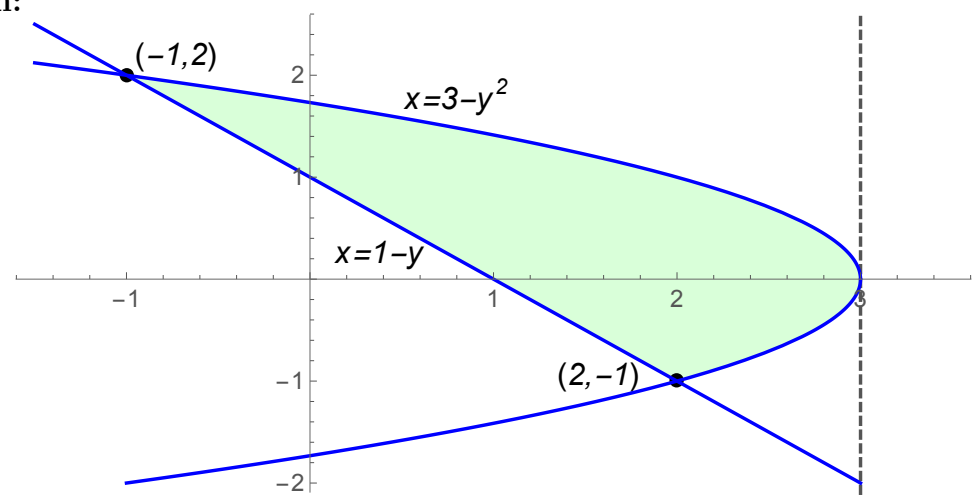
   Solution: Equating the two curves, we find that the intersections are when $2x = 6x - 4x^2$, or equivalently $x^2 = x$ which has solutions $x = 0$ and $x = 1$. Moreover, between these values, $6x - 4x^2$ is larger than $2x$ (which we see either because $y = 6x - 4x^2$ is concave down, or by comparing their values at an interior point such as $x = \frac{1}{2}$). We therefore evaluate the integral

   $$\int_0^1 \left[(6x - 4x^2) - 2x\right] dx = \int_0^1 [4x - 4x^2]\,dx = \left(2x^2 - \frac{4x^3}{3}\right)\Bigg|_0^1 = \left(2 - \frac{4}{3}\right) - 0 = \frac{2}{3}.$$

Long answer question—you must show your work

2. $\boxed{\text{4 marks}}$ The region enclosed by the line $x+y=1$ and the curve $x=3-y^2$ is rotated about the line $x=3$. Find the volume of the resulting solid. A calculator-ready answer is acceptable.

Solution:



Since we are rotating horizontally, we need both equations to give x as a function of y; in particular, we write the line as $x=1-y$. To find the points of intersection, we set $1-y=3-y^2$, which gives us $y^2-y-2=0$ or $(y+1)(y-2)=0$; therefore the two graphs intersect when $y=-1$ and $y=2$. On this interval, the outer radius is the distance from $x=1-y$ to $x=3$, which is $3-(1-y)=2+y$; similarly, the inner radius is $3-(3-y^2)=y^2$. The volume of the solid of revolution is therefore

$$\begin{aligned}\int_{-1}^{2}\pi\big((2+y)^2-(y^2)^2\big)\,dy &= \pi\int_{-1}^{2}\left(4+4y+y^2-y^4\right)dy\\ &= \pi\left(4y+2y^2+\frac{y^3}{3}-\frac{y^5}{5}\right)\Bigg]_{-1}^{2}\\ &= \pi\left(8+8+\frac{8}{3}-\frac{32}{5}\right)-\pi\left(-4+2-\frac{1}{3}+\frac{1}{5}\right)=\frac{72}{5}\pi.\end{aligned}$$

Marking scheme:

- 2 marks for the correct integrand
- 1 mark for the correct limits of integration
- 1 mark for the correct answer to their integral

Long answer question—you must show your work

3. $\boxed{\text{4 marks}}$ Evaluate $\displaystyle\int \big(3x\cos(\pi x) + 4x\log x\big)\,dx$.

Solution: We split the integral into two parts:

$$\int \big(3x\cos(\pi x) + 4x\log x\big)\,dx = \int 3x\cos(\pi x)\,dx + \int 4x\log x\,dx.$$

For the first part, we use integration by parts with $u = 3x$ and $dv = \cos(\pi x)\,dx$, so that $du = 3\,dx$ and $v = \frac{1}{\pi}\sin(\pi x)$:

$$\begin{aligned}\int 3x\cos(\pi x)\,dx &= 3x\cdot\frac{1}{\pi}\sin(\pi x) - \int 3\cdot\frac{1}{\pi}\sin(\pi x)\,dx \\ &= \frac{3x}{\pi}\sin(\pi x) + \frac{3}{\pi^2}\cos(\pi x) + C.\end{aligned}$$

For the second part, we again use integration by parts, but this time with $u = \log x$ and $dv = 4x\,dx$, so that $du = \frac{1}{x}\,dx$ and $v = 2x^2$:

$$\begin{aligned}\int 4x\log x\,dx &= \log x\cdot 2x^2 - \int \frac{1}{x}\cdot 2x^2\,dx \\ &= 2x^2\log x - \int 2x\,dx = 2x^2\log x - x^2 + C.\end{aligned}$$

Therefore

$$\int \big(3x\cos(\pi x) + 4x\log x\big)\,dx = \frac{3x}{\pi}\sin(\pi x) + \frac{3}{\pi^2}\cos(\pi x) + 2x^2\log x - x^2 + C.$$

(Note that we only need a single $+C$ at the end, since expressions like $+2C$ or $+C_1 + C_2$ still just represent arbitrary real numbers.)

Marking scheme:

- 1 mark for the choice $u = 3x$ and $dv = \cos(\pi x)\,dx$
- 1 mark for correctly evaluating $\int 3x\cos(\pi x)\,dx$
- 1 mark for the choice $u = \log x$ and $dv = 4x\,dx$
- 1 mark for correctly evaluating $\int 4x\log x\,dx$