First Name: _____ Last Name: _____

Student-No: _____ Section: _____

Grade:

This page will be overwritten with the fancy auto-multiple-choice front page.

Short answer question

1. $\boxed{\text{2 marks}}$ Each part is worth 1 mark. Please write your answers in the boxes. Only answers in the boxes will be marked. "Calculator-ready" answers are acceptable.

   (a) The function $f(x)$ satisfies $\int_0^2 f(x)\,dx = 4$ and $\int_1^2 f(x)\,dx = 2$.

   Find $\int_0^1 \big[3f(x)\big]\ dx$.

   Answer: 6

   Solution: We have

$$\begin{aligned}\int_0^1 \big[3f(x)\big]\ dx &= 3\int_0^1 f(x)\ dx = 3\left[\int_0^2 f(x)\ dx + \int_2^1 f(x)\ dx\right]\\ &= 3\left[\int_0^2 f(x)\ dx - \int_1^2 f(x)\ dx\right]\\ &= 3\big[4-2\big] = 6.\end{aligned}$$

   (b) Calculate $\int_{-2}^{-1} x^3\ dx$.

   Answer: $\dfrac{1-2^4}{4} = \dfrac{-15}{4}$

   Solution: One function with derivative x^3 is $\frac{x^4}{4}$. So, by the Fundamental Theorem of Calculus,

$$\int_{-2}^{-1} x^3\ dx = \frac{x^4}{4}\bigg|_{-2}^{-1} = \frac{(-1)^4}{4} - \frac{(-2)^4}{4} = \frac{-15}{4}.$$

Long answer question—you must show your work

2. $\boxed{\text{4 marks}}$ Part (a) is worth 1 mark and part (b) is worth 3 marks. Define $f(x) = \int_{x^2}^{x} e^{t^2}\,dt$.

(a) Find $f(1)$. Simplify your answer.

Answer: 0

Solution: Evaluating the definition of $f(x)$ at $x = 1$ gives

$$f(1) = \int_{1^2}^{1} e^{t^2}\,dt = \int_{1}^{1} e^{t^2}\,dt = 0$$

since the upper and lower endpoints of integration are equal.

Marking scheme:

- 1 mark for the answer 0

(b) Find $f'(1)$. Simplify your answer.

Answer: $-e$

Solution: Write

$$\begin{aligned} f(x) &= \int_{x^2}^{x} e^{t^2}\,dt \\ &= \int_{0}^{x} e^{t^2}\,dt - \int_{0}^{x^2} e^{t^2}\,dt \\ &= G(x) - G(x^2) \end{aligned}$$

where

$$G(y) = \int_{0}^{y} e^{t^2}\,dt$$

By the Fundamental Theorem of Calculus,

$$G'(y) = e^{y^2}$$

Hence, by the chain rule,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[G(x) - G(x^2)\right] \\ &= G'(x) - 2xG'(x^2) \\ &= e^{x^2} - 2xe^{x^4} \end{aligned}$$

When $x = 1$,

$$f'(1) = e^1 - 2e^1 = -e$$

Marking scheme:

- 1 mark for splitting up the integral
- 1 mark for a correct application of the chain rule
- 1 mark for the correct evaluation at $x = 1$

Long answer question—you must show your work

3. $\boxed{\text{4 marks}}$ Each part is worth 2 marks.

(a) For a certain function $f(x)$ and a certain number b, the following equation holds:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\left|\left(i-\frac{1}{2}\right)\frac{4}{n}-2\right| = \int_2^b f(x)\,dx.$$

Find $f(x)$ and b.

Answer: $f(x) = |x-4|$, $b = 6$

Solution: The given sum is of the form

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\left|\left(i-\frac{1}{2}\right)\frac{4}{n}-2\right| &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\left|2+\left(i-\frac{1}{2}\right)\frac{4}{n}-4\right| \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x\left|x_i^*-4\right| = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x\end{aligned}$$

with $\Delta x = \frac{4}{n}$, $x_i^* = 2+\left(i-\frac{1}{2}\right)\frac{4}{n} = a+\left(i-\frac{1}{2}\right)\Delta x$, $a = 2$, and $f(x) = |x-4|$. Since $x_0 = \left[2+i\frac{3}{n}\right]_{i=0} = 2$ and $x_n = \left[2+i\frac{4}{n}\right]_{i=n} = 6$, the left hand side is the definition (using the midpoint Riemann sum) of $\int_2^6 f(x)\,dx$.

Marking scheme:

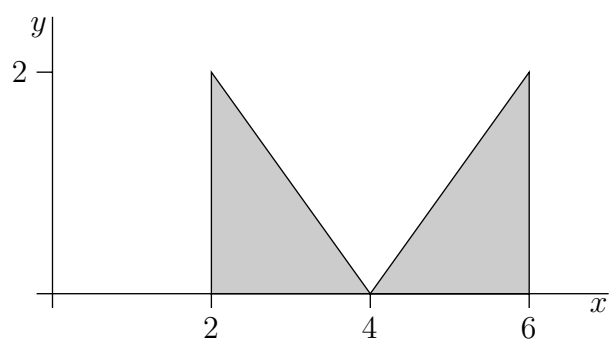
- 1 mark for $f(x) = |x-4|$
- 1 mark for $b = 6$

(b) Evaluate $\displaystyle\int_2^b f(x)\,dx$, using the $f(x)$ and b you found in part (a). Simplify your answer.

Answer: 4

Solution: The integral $\int_2^6 |x-4|\,dx$ represents the shaded area in the figure below, which is

$$\frac{1}{2}\times 2\times 2+\frac{1}{2}\times 2\times 2 = 4$$



Marking scheme:

- 1 mark for identifying their integral from part (a) as the area of a correct figure
- 1 mark for the getting the area of their figure correct.