Last Name: _____ First Name: _____ Student-No: _____ Section: _____

Very short answer question

1. [1 mark] If $\int_0^2 f(x)\,dx = 5$ and $\int_0^2 g(x)\,dx = 3$, calculate $\int_0^2 \big(2f(x) - 4g(x)\big)\,dx$. Please write your answer in the box. Simplify your answer completely.

Answer: -2

Marking scheme: 1 for a correct answer in the box

Solution: The operation of integration is linear, so that:

$$\begin{aligned}\int_0^2 [2f(x) - 4g(x)]\,dx &= \int_0^2 2f(x)\,dx - \int_0^2 4g(x)\,dx \\ &= 2\int_0^2 f(x)\,dx - 4\int_0^2 g(x)\,dx = (2\times 5) - (4\times 3) = -2\end{aligned}$$

Short answer questions—you must show your work

2. [2 marks] Use elementary geometry to calculate $\int_0^3 f(x)\,dx$, where

$$f(x) = \begin{cases} 1, & \text{if } x \le -1, \\ -x, & \text{if } x > -1. \end{cases}$$

Solution: On the interval of integration $[0,3]$, the function $f(x)$ is simply equal to $-x$, whose graph on that interval is a straight line segment below the x-axis. [The intention was for both parts of the definition of f to be relevant, but a typo in the statement of the problem made that not happen.] Using the interpretation of $\int_{-3}^0 f(x)\,dx$ as the signed area bounded by the function $f(x)$ between 0 and -3, we see from its graph that the value of the integral is the negative of the area of the triangle with vertices $(0,0)$, $(3,0)$, and $(3,-3)$. This triangle has base 3 and height 3, and so the value of the integral is $\int_{-3}^0 f(x)\,dx = -\frac{1}{2}\cdot 3\cdot 3 = -\frac{9}{2}$.

Marking scheme:

- 1 mark for calculating the correct answer.
- 1 mark for recognizing that the integral involves the area of a triangle, or recognizing that the result must be negative due to the graph lying below the x-axis.

3. [2 marks] The value of the following limit is equal to the area below a graph of $y = f(x)$, integrated over the interval $[0, b]$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \left[\sin\left(2 + \frac{4i}{n}\right)\right]^2.$$

Find $f(x)$ and b. (Do not evaluate the limit.) Please write your answers in the boxes.

Answer: $b = 4$

Answer: $f(x) = \sin^2(2 + x)$

Solution: We identify this as the right Riemann sum, with interval $\Delta x = 4/n$ and $x_i = 4i/n$. Therefore, $b = 4$ and $f(x) = \sin^2(2 + x)$.

Marking scheme: 1 mark for each answer

Long answer question—you must show your work

4. [5 marks] Consider the integral:

$$\int_0^4 (6 + x^2)\, dx. \qquad (*)$$

(a) (1 mark) Approximate this integral using the left Riemann sum with $n = 4$ intervals.

Solution: The left Riemann sum is defined as

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x.$$

We subdivide into 4 intervals, (i.e. $\Delta x = 1$), defining for $i = 0, \ldots, 3$, the values $x_i = 0,\ 1,\ 2,\ 3$. The function $f(x) = 6 + x^2$ has the values: $f(x_i) = 6,\ 7,\ 10,\ 15$, from which we evaluate $L_4 = 38$. **Marking scheme:** 1 mark for a correct answer

(b) (4 marks) Write down the expression for the right Riemann sum with n intervals and calculate the sum. Now take the limit $n \to \infty$ in your expression for the Riemann sum, to evaluate the integral $(*)$ exactly. You can leave your answer in "calculator-ready" form.

Hint: you may use the identity

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6}.$$

Solution: We divide into n intervals $\Delta x = 4/n$ and set $x_i = 4i/n$. The right Riemann sum is therefore:

$$R_n = \sum_{i=1}^{n} \frac{4}{n}\left[6 + \frac{(4i)^2}{n^2}\right].$$

To calculate the sum:

$$\begin{aligned} R_n &= \left(\frac{24}{n}\sum_{i=1}^{n} 1\right) + \left(\frac{64}{n^3}\sum_{i=1}^{n} i^2\right) \\ &= 24 + \frac{64}{n^3} \times \frac{2n^3 + 3n^2 + n}{6} = 24 + \frac{64}{6}(2 + 3/n + 1/n^2) \end{aligned}$$

To evaluate the limit exactly, we take $n \to \infty$. The expressions involving $1/n$ vanish leaving:

$$\int_0^4 (6 + x^2)\, dx = \lim_{n\to\infty} R_n = 24 + \frac{64}{3} = 45\frac{1}{3}.$$

Marking scheme:

- 1 mark for writing R_n.
- 2 marks for calculating the sum using the formula in the hint, or otherwise.
- 1 mark for taking the limit and getting the answer.