Math 101—Midterm Exam #2, Practice Midterm B
Duration: 50 minutes

Surname (Last Name) ____________ Given Name ____________ Student Number ____________

Do not open this test until instructed to do so! This exam should have 8 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work. Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked. Continue on the back of the page if you run out of space.

Problem	**Out of**	**Score**	**Problem**	**Out of**	**Score**
1	6		5	8	
2	6		6	8	
3	6		7	8	
4	3		**Total**	45	

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

1a. **[3 pts]** Determine whether the following sequences converge or diverge. If they converge, find the limit

(a) $a_n = \frac{\sin n}{\sqrt{n+1}}$
(b) $b_n = \ln(n+1) - \ln n$
(c) $c_n = \cos(\frac{\pi n}{2})$

Solution

(a) We remember that $-1 \leq \sin n \leq 1$, for all n, hence $\frac{-1}{\sqrt{n+1}} \leq \frac{\sin n}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1}}$. However, both $d_n = \frac{-1}{\sqrt{n+1}}$ and $e_n = \frac{1}{\sqrt{n+1}}$ are easily seen to converge to zero, and by the Squeeze Theorem for sequences, $a_n \to 0$, as $n \to \infty$.

(b) Notice that if we were to take the limits of each component we would get $\infty - \infty$ which is a type of indeterminate form, so we have to algebraically manipulate the expression before we take the limit. $\ln(n+1) - \ln(n) = \ln \frac{n+1}{n}$ and given that the limit of $\frac{n+1}{n}$ is 1, and that $\ln x$ is a continuous function at 1, we get that

$$\lim_{n\to\infty} b_n = \ln 1 = 0$$

(c) This limit does not exist. Check that when n is an odd integer, $\cos(\frac{n\pi}{2}) = 0$ while when n is even $\cos(\frac{n\pi}{2})$ is either 1 or -1. The sequence never gets arbitrarily close to any value, so it diverges.

1b. **[3 pts]** Determine whether the improper integral

$$\int_1^\infty \frac{4 + \sin x}{\sqrt{x - 1/2}}, dx$$

converges or not. If it converges, find its value.

Solution

We will compare the given integral with $\int_1^\infty \frac{3}{\sqrt{x-\frac{1}{2}}} dx$.
Indeed, since $\sin x \geq -1$, $4 + \sin x \geq 3$. But

$$\int_1^\infty \frac{3}{\sqrt{x - \frac{1}{2}}} dx = \lim_{t\to\infty} \int_1^t \frac{3}{\sqrt{x - \frac{1}{2}}} dx$$

and it is straighforward to verify that $\int_1^t \frac{3}{\sqrt{x-\frac{1}{2}}} dx = 6[\sqrt{x - \frac{1}{2}}]_1^t = 6(\sqrt{t - \frac{1}{2}} - \frac{1}{2})$ and thus

$$\lim_{t\to\infty} \int_1^t \frac{3}{\sqrt{x - \frac{1}{2}}} dx = \infty,$$

so the integral $\int_1^\infty \frac{4+\sin x}{\sqrt{x-1/2}} dx$ diverges by the *comparison test*.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

2a. **[3 pts]** Use Simpson's Rule with $n = 4$ to approximate the value of the integral

$$\int_{-3}^{5} (2x^2 + 3x)\, dx.$$

Solution

$\Delta x = \frac{b-a}{n} = \frac{5-(-3)}{4} = 2$, so the x-coordinates for the relevant for Simpson's Rule approximation are: $-3, -1, 1, 3, 5$.

$S_4 = [f(-3) + 4f(-1) + 2f(1) + 4f(3) + f(5)]\frac{\Delta x}{3} = \frac{2}{3}[9 - 4 + 10 + 108 + 65] = \frac{376}{3}$

2b. **[3 pts]** Evaluate $\displaystyle\int_0^{\pi/4} \tan^3 x \sec^2 x\, dx$. Simplify your answer fully.

Solution

Making the substitution $u = \tan x$: $du = \sec^2 x dx$, when $x = 0 : u = 0$, when $x = \frac{\pi}{4} : u = 1$ we see $\displaystyle\int_0^{\pi/4} \tan^3 x \sec^2 x\, dx = \int_0^1 u^3\, du = [\frac{u^4}{4}]_0^1 = \frac{1}{4}$.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

3a. **[3 pts]** Find the general antiderivative of $f(x) = x^2 \sin(\pi x)$.

Solution

We are asking for the indefinite integral $I = \displaystyle\int x^2 \sin(\pi x)\,dx$. Integrating by parts we get $I = -\frac{x^2}{\pi}\cos(\pi x) - \int \frac{(-\cos(\pi x)\cdot 2x}{\pi}\,dx$. Integrating by parts once more: $I = -\frac{x^2}{\pi}\cos(\pi x) + \frac{2x\sin(\pi x)}{\pi^2} - \frac{2}{\pi^2}\int \sin(\pi x)\,dx = \frac{(2-\pi^2x^2)\cos(\pi x)+2\pi x\sin(\pi x)}{\pi^3} + C$.

3b. **[3 pts]** Evaluate $\displaystyle\int_1^2 \frac{x^2}{\sqrt{x^2-1}}\,dx$.

Solution

This is an improper integral due the singularity at $x = 1$; $\displaystyle\int_1^2 \frac{x^2}{\sqrt{x^2-1}}\,dx = \lim_{t\to 1^+}\int_t^2 \frac{x^2}{\sqrt{x^2-1}}\,dx$. The last integral is rather complicated to compute, so, for simplicity, we will first evaluate the indefinite integral and plug in the endpoints at the end (before taking the limit).

We make the ("inverse") substitution $x = \sec s, 0 \le s < \pi/2$ and $\frac{dx}{ds} = \sec s \tan s$ and in differential form $dx = \sec s \tan s\,ds$ which results in the new integral $\displaystyle\int \frac{\sec^2 s \sec s \tan s}{\tan s}\,ds = \int \sec^3 s\,ds$. For $I = \displaystyle\int \sec^3 s\,ds$ we will perform integration by parts with $u = \sec s, dv = sec^2 sds$ (then $du = \sec s \tan s ds, v = \tan s$), so

$$I = \sec s \tan s - \int \tan^2 s \sec s\,ds = \sec s \tan s - \int (\sec^2 s - 1)\sec s\,ds$$

$$= \sec s \tan s + \int \sec s - \int \sec^3 s\,ds = \sec s \tan s + \ln|\sec s + \tan s| - I + C$$

because we observed that the integral we started with, appeared on the RHS. Rearranging:

$$2I = \sec s \tan s + \ln|\sec s + \tan s| + C \Rightarrow I = \frac{1}{2}(\sec s \tan s + \ln|\sec s + \tan s|) + C$$

Since we didn't change the endpoints after the substitution (we were only treating the indefinite integral) we need to go back to the original variable x, i.e., we need to express $\sec s, \tan s$ in terms of x. But $\sec s = x$ and to find $\tan s : \tan s = \sqrt{\sec^2 - 1} = \sqrt{x^2-1}$. Two remarks: (i) we picked the positive branch of the square root and that was because of the range of the original substitution ($0 \le s < \pi/2 : \tan s \ge 0$.) (ii) We could have arrived at the same conclusion by drawing a right triangle with hypotenuse x and one side with length 1 (as explained in section 7.3 in the book) .

In the original variable x the integral is $= \frac{1}{2}(x\sqrt{x^2-1} + \ln|x + \sqrt{x^2-1}|) + C$ and plugging in the endpoints t and 2 we get:

$$\int_t^2 \frac{x^2}{\sqrt{x^2-1}}\,dx = \frac{1}{2}[2\sqrt{3} + \ln(2+\sqrt{3}) - t\sqrt{t^2-1} - \ln|t + \sqrt{t^2-1}|.$$

But

$$\int_1^2 \frac{x^2}{\sqrt{x^2-1}}\,dx = \lim_{t\to 1^+} \frac{1}{2}[2\sqrt{3}+\ln(2+\sqrt{3}) - t\sqrt{t^2-1} - \ln|t+\sqrt{t^2-1}| = \frac{2\sqrt{3}+\ln(2+\sqrt{3})}{2}$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

4. **[3 pts]** Evaluate $I = \int_0^4 \frac{x-1}{x^2-4x-5}\,dx$.

Solution

The numerator is of lower degree than the denominator so there is no need for long division. To factor the denominator we use the quadratic formula to get that the solutions of the quadratic equation $x^2-4x-5=0$ are $x=5$ and $x=-1$.

$$\frac{x-1}{x^2-4x-5} = \frac{A}{x-5} + \frac{B}{x+1}$$

Multiplying both sides with $(x-5)(x+1)$ and matching coefficients, we get $A-5B=-1$ and $A+B=1$. The solution to this system is $(A,B)=(\frac{2}{3},\frac{1}{3})$.

$$I = \int_0^4 \frac{x-1}{x^2-4x-5}\,dx = \frac{2}{3}\int_0^4 \frac{1}{x-5}\,dx + \frac{1}{3}\int_0^4 \frac{1}{x+1}\,dx = \frac{2}{3}[\ln|x-5|]_0^4 + \frac{1}{3}[\ln|x+1|]_0^4$$

$= \frac{2}{3}[\ln 1 - \ln 5] + \frac{1}{3}[\ln 5 + \ln 1] = \frac{-\ln 5}{3}$

Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

5.

(a) **[4 pts]** A lamina with density $\rho = 3$ is in the shape of the region under the graph of $y = e^{2x}$ between $x = -1$ and $x = 1$. Find M_y, the moment with respect to the y-axis, of this lamina.

Solution

$M_y = \rho \int_{-1}^{1} xe^{2x}\, dx = 3(x\frac{e^{2x}}{2} \mid_{-1}^{1} - \frac{1}{2}\int_{-1}^{1} e^{2x}\, dx) = \frac{3xe^{2x}}{2} \mid_{-1}^{1} - \frac{3e^{2x}}{4} \mid_{-1}^{1} = \frac{3e^2}{4} + \frac{9}{4e^2}.$

(b) **[4 pts]** Evaluate the integral

$$\int (x+1)\sqrt{x^2+2x+2}\, dx.$$

Solution

$I = \int (x+1)\sqrt{x^2+2x+2}\, dx = \int (x+1)\sqrt{(x+1)^2+1}\, dx.$

Making the ("inverse") substitution $x + 1 = \tan\theta$, $\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\frac{dx}{d\theta} = \sec^2\theta$:

$$\int (x+1)\sqrt{(x+1)^2+1}\, dx = \int \tan\theta\sqrt{\sec^2\theta}\sec^2\theta\, d\theta = \int \tan\theta\sec\theta\sec^2\theta d\theta$$

Now set $u = \sec\theta$, and thus $du = \sec\theta\tan\theta\, d\theta$ to get

$$I = \int u^2\, du = \frac{u^3}{3} + C = \frac{\sec^3\theta}{3} + C = \frac{(x^2+2x+2)^{3/2}}{3} + C.$$

6.

(a) **[4 pts]** Use integration to calculate the area of the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Hint: exploit the symmetry of the ellipse with respect to both axes.

Solution

Solving the equation of the ellipse for y we get $y = \pm\sqrt{9 - \frac{9x^2}{4}}$. Because the ellipse is symmetric to both axes, the total area A is four times the area in the first quadrant which the area of the region below the graph of $y = \sqrt{9 - \frac{9x^2}{4}}$ between $x = 0$ and $x = 2$.

$A = 4\int_0^2 \sqrt{9 - \frac{9x^2}{4}}\, dx = 12\int_0^2 \sqrt{1 - (\frac{x}{2})^2}\, dx.$

Set $\frac{x}{2} = \sin\theta, \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$, so $x = 2\sin\theta$ and $\frac{dx}{d\theta} = 2\cos\theta$. When $x = 0, \theta = 0$ and when $x = 2, \theta = \pi/2$ (because of the range of the substitution).

Then $A = 12\int_0^{\pi/2} 2\cos\theta\sqrt{1-\sin^2\theta}\, d\theta = 24\int_0^{\pi/2} \cos^2\theta\, d\theta$, which by the half-angle formula becomes $A = 12\int_0^{\pi/2}(1 + \cos(2\theta))\, d\theta = 12[\theta + \frac{1}{2}\sin(2\theta)]_0^{\pi/2} = 6\pi$. ("sanity" check: $A = a \cdot b \cdot \pi = 3 \cdot 2 \cdot \pi$ (where a and b are the lengths of the axes of the ellipse))

6b. **[4 pts]** Determine whether the integral

$$\int_0^{e^3} \ln x\, dx$$

converges or diverges. If it converges, find its value.

Solution

$$\int_0^{e^3} \ln x\, dx = \lim_{t\to 0^+}\int_t^{e^3} \ln x\, dx$$

Integrating by parts with $u = \ln x, dv = dx$ to get

$$= \lim_{t\to 0^+}[x\ln x\,|_t^{e^3} - \int_t^{e^3} x\frac{1}{x}\, dx] = e^3\ln(e^3) - \lim_{t\to 0^+}(t\ln t + e^3 - t)$$
$$= 3e^3 - e^3 + \lim_{t\to 0^+}(t - t\ln t)$$

But

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+}\frac{\ln t}{1/t}$$

which by L'Hospital's Rule is the same as $\lim_{t\to 0^+}\frac{1/t}{-1/t^2} = -\lim_{t\to 0^+} t = 0$ thus the limit (and consequently the integral) equals $2e^3$.

7. **[8 pts]** Find the solution to the differential equation

$$\frac{dy}{dx} = \frac{y}{x^4 + x^2}$$

which satisfies the initial condition $y(1) = 1/e$.

Solution

We first consider the case $y = 0$. Then $\frac{dy}{dx} = 0 \Rightarrow y = \text{constant} = 0$. However this cannot happen since $y(1) = \frac{1}{e}$.

So we can divide, and writing in differential form: $\frac{dy}{y} = \frac{dx}{x^4+x^2}$. Integrating:
$\ln|y| = \int \frac{1}{x^4+x^2}\,dx = \int \frac{1}{x^2(x^2+1)}\,dx$.

To calculate the last integral, you can either use the trigonometric substitution $x = \tan\theta$ (but in the process you will have to remember the anti-derivative of $\csc^2 x$) or you resort to a partial fraction decomposition, which is what we will do here:
$\frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$ since there is a repeated and an irreducible quadratic term. Multiplying both sides with $x^2(x^2+1)$ and matching coefficients, it is easy to see that $A = C = 0, B = 1, D = -1$

Consequently, $\int \frac{1}{x^2(x^2+1)}\,dx = \int (\frac{1}{x^2} - \frac{1}{x^2+1})\,dx = \frac{-1}{x} - \arctan x + C$.
Putting everything together and "exponentiating": $|y| = e^{-(\frac{1}{x}+\arctan x)} \cdot C$. To find the constant C we use the information that $y(1) = 1/e$ to get $C = e^{\pi/4}$

$y = \pm e^{\frac{\pi}{4} - \frac{1}{x} - \arctan x}$ and we can exclude the negative case by another use of $y(1) = 1/e$.

In conclusion, the solution of the differential equation with the extra condition $y(1) = 1/e$ is

$$y = e^{\frac{\pi}{4} - \frac{1}{x} - \arctan x}$$