Math 101—Midterm Exam #2, Practice Midterm A

Duration: 50 minutes

Surname (Last Name) Given Name Student Number

Do not open this test until instructed to do so! This exam should have 8 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work. Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked. Continue on the back of the page if you run out of space.

Problem	Out of	Score	Problem	Out of	Score
1	6		5	8	
2	6		6	8	
3	6		7	8	
4	3		**Total**	45	

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

1a. **[3 pts]** Find the average value of the function $f(x) = \sqrt{x}$ on the interval $1 \le x \le 9$. Simplify your answer fully.

Use the formula: $\dfrac{\int_a^b f(x)\,dx}{b-a}$, we have:

$$\frac{\int_1^9 \sqrt{x}\,dx}{8} = \frac{1}{8}\cdot\frac{2}{3}x^{3/2}\Big]_1^9 = \frac{52}{24} = \frac{13}{6}$$

1b. **[3 pts]** Calculate $\int 9x^2e^{-3x}\,dx$.

- Integration by parts: $u = 9x^2$ $du = 18x\,dx$
  $dv = e^{-3x}dx$ $v = \frac{e^{-3x}}{-3}$

$$uv - \int v\,du = -3x^2e^{-3x} + \int 6xe^{-3x}\,dx$$

- For $\int 6xe^{-3x}dx$: integration by parts again:

$u = 6x$ $du = 6dx$

$dv = e^{-3x}dx$ $v = \frac{e^{-3x}}{-3}$

$$uv - \int v\,du = -2xe^{-3x} + \int 2e^{-3x}dx$$
$$= -2xe^{-3x} - \frac{2}{3}e^{-3x} + C$$

Final answer: $\boxed{-3x^2e^{-3x} - 2xe^{-3x} - \frac{2}{3}e^{-3x} + C}$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

2a. **[3 pts]** Find $\int \sin^3 x \cos^4 x \, dx$.

$$= \int \sin^2 x \cos^4 x \sin x \, dx = \int (1-\cos^2 x) \cos^4 x \sin x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

We have the integral:

$$\int (1-u^2) u^4 (-du) = \int (-u^4 + u^6) \, du$$

$$= -\frac{u^5}{5} + \frac{u^7}{7} + C = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$$

2b. **[3 pts]** Find $\int \frac{1}{x^2\sqrt{x^2-4}} \, dx$.

$$x = 2\sec\theta \quad \left(0 \le \theta < \frac{\pi}{2}, \ \pi \le \theta < \frac{3\pi}{2}\right)$$

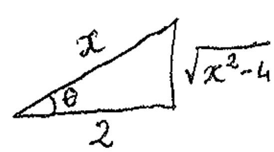
$dx = 2\sec\theta \tan\theta \, d\theta$, and $\sqrt{x^2-4} = 2\tan\theta$

The integral becomes

$$\int \frac{1}{4\sec^2\theta \cdot 2\tan\theta} 2\sec\theta \tan\theta \, d\theta = \frac{1}{4}\int \frac{1}{\sec\theta} \, d\theta$$

$$= \frac{1}{4}\int \cos\theta \, d\theta = \frac{1}{4}\sin\theta + C$$

From $x = 2\sec\theta$, get $\cos\theta = \frac{2}{x}$. Triangle:



answer: $\frac{1}{4} \frac{\sqrt{x^2-4}}{x} + C$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

3a. **[3 pts]** Approximate $\int_0^{4\pi} e^{\sin^2 x}\, dx$ by using Trapezoidal Rule with $n = 4$. Simplify your answer fully.

$\Delta x = \frac{4\pi}{4} = \pi$, $x_i = a + i\Delta x = i\pi$ for $i = 0, 1, \dots, 4$

$f(x) = e^{\sin^2 x}$. So $f(x_i) = e^{\sin^2(i\pi)} = e^0 = 1$

Answer:

$$\frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)$$

$$= \frac{\pi}{2}(1 + 2 + 2 + 2 + 1) = 4\pi$$

3b. **[3 pts]** Determine whether the integral

$$\int_e^{\infty} \frac{1}{x(\ln x)^{2015}}\, dx$$

is convergent or divergent. If it is convergent, evaluate it.

Need to study: $\lim_{t \to \infty} \int_e^t \frac{1}{x(\ln x)^{2015}}\, dx$

Substitution

$u = \ln x$

$du = \frac{1}{x}\, dx$

The integral becomes $\int_1^{\ln t} \frac{1}{u^{2015}}\, du = \left.\frac{u^{-2014}}{-2014}\right]_1^{\ln t}$

$$= -\frac{1}{2014}\frac{1}{(\ln t)^{2014}} + \frac{1}{2014}$$

As $t \to \infty$: $\frac{1}{\ln t} \to 0$. So the improper integral converges to $\frac{1}{2014}$.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

4. **[3 pts]** Determine if the sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{\sqrt{n^4 + 2014}}{\sqrt[3]{n^6 + 2016}}$$

Divide the numerator and denominator by

$$n^2 = \sqrt{n^4} = \sqrt[3]{n^6}$$

We have

$$a_n = \frac{\sqrt{1 + \frac{2014}{n^4}}}{\sqrt[3]{1 + \frac{2016}{n^6}}}$$

Since $\lim_{n \to \infty} \frac{2014}{n^{4}} = 0$ and $\lim_{n \to \infty} \frac{2016}{n^{6}} = 0$, we have

$$\lim_{n \to \infty} a_n = \frac{\sqrt{1+0}}{\sqrt[3]{1+0}} = 1$$

Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

5.

(a) **[3 pts]** Find constants A, B, and C such that:

$$\frac{2x^2 - x + 5}{x^3 - 2x^2 + 5x} = \frac{A}{x} + \frac{Bx + C}{x^2 - 2x + 5}.$$

Multiply common denominator: $2x^2 - x + 5 = A(x^2 - 2x + 5) + (Bx + C)x$
$= (A + B)x^2 + (-2A + C)x + 5A$

Hence we have $A + B = 2$, $-2A + C = -1$, $5A = 5$

This gives: $A = 1$, $B = 1$, $C = 1$.

(b) **[3 pts]** Evaluate $\displaystyle\int \frac{2x^2 - x + 5}{x^3 - 2x^2 + 5x}\,dx$.

By part (a): $\displaystyle\int \frac{1}{x}\,dx + \int \frac{x + 1}{x^2 - 2x + 5}\,dx$

For the 2nd integral: $x^2 - 2x + 5 = (x-1)^2 + 4$. Put $u = \frac{x-1}{2}$ so $dx = 2du$ and $x + 1 = 2u + 2$. We have

$$\int \frac{2u + 2}{4u^2 + 4} 2du = \frac{2}{4}\left(\int \frac{2u}{u^2 + 1}\,du + \int \frac{2}{u^2 + 1}\,du\right) = \frac{1}{2}\left(\ln(u^2 + 1) + 2\arctan(u)\right) + C$$

Answer: $\ln|x| + \frac{1}{2}\left(\ln\left(\left(\frac{x-1}{2}\right)^2 + 1\right) + 2\arctan\left(\frac{x-1}{2}\right)\right) + C$

(c) **[2 pts]** If the integrand had been the more complicated function

$$\frac{x + 7}{(x + 2)(x^3 + x)(x^3 - x^2 + x)},$$

what would the general partial fraction form have been? Just write down the form of the partial fractions, with constants A, B, $\ldots$; do not find numerical values for the constants.

The denominator is factorized as

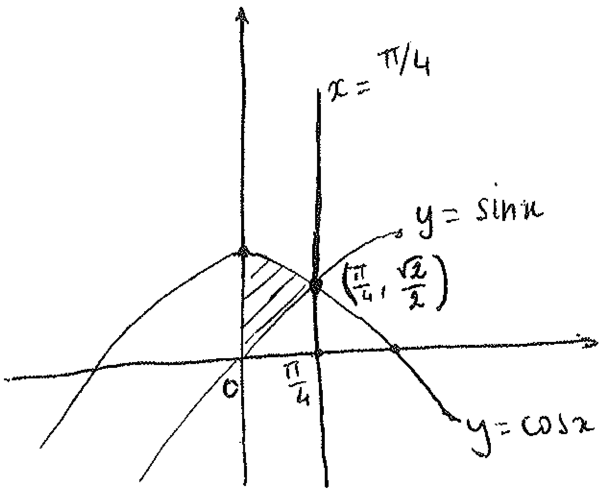
$$(x + 2)\, x^2 (x^2 + 1)(x^2 - x + 1)$$

Hence we have the form

$$\frac{A}{x + 2} + \frac{B}{x} + \frac{C}{x^2} + \frac{Dx + E}{x^2 + 1} + \frac{Fx + G}{x^2 - x + 1}$$

6. Let R be the region bounded by $0 \le x \le \frac{\pi}{4}$ and the graphs of $y = \cos x$ and $y = \sin x$.

(a) **[2 pts]** Sketch R and find its area.



$$A = \text{area} = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx$$

$$= (\sin x + \cos x)\Big]_0^{\pi/4}$$

$$= \sqrt{2} - 1$$

(b) **[6 pts]** Find the centroid of R.

$$\bar{x} = \frac{1}{A}\int_0^{\pi/4} x(\cos x - \sin x)\,dx$$

$$\left(\begin{array}{ll} u = x & du = dx \\ dv = (\cos x - \sin x)\,dx & v = \sin x + \cos x \end{array} \right)$$

$$= \frac{1}{\sqrt{2}-1}\left(x(\sin x + \cos x)\Big]_0^{\pi/4} - \int_0^{\pi/4} (\sin x + \cos x)\,dx \right)$$

$$= \frac{1}{\sqrt{2}-1}\left(\frac{\pi}{4}\cdot\sqrt{2} - (-\cos x + \sin x)\Big]_0^{\pi/4} \right)$$

$$= \frac{1}{\sqrt{2}-1}\left(\frac{\sqrt{2}\,\pi}{4} - 1 \right)$$

$$\bar{y} = \frac{1}{A}\int_0^{\pi/4} \frac{1}{2}(\cos^2 x - \sin^2 x)\,dx = \frac{1}{2(\sqrt{2}-1)}\int_0^{\pi/4} \cos 2x\,dx$$

$$= \frac{1}{2(\sqrt{2}-1)} \quad \frac{\sin 2x}{2}\Big]_0^{\pi/4} = \frac{1}{4(\sqrt{2}-1)}$$

Answer: $\left(\frac{1}{\sqrt{2}-1}\left(\frac{\sqrt{2}\,\pi}{4} - 1\right), \frac{1}{4(\sqrt{2}-1)} \right)$

7. **[8 pts]** Find the solution of the differential equation

$$y' = \sqrt{1-y^2}\arctan(x)$$

satisfying the initial condition $y(0) = \frac{\sqrt{2}}{2}$.

Write the equation into the form

$$\frac{1}{\sqrt{1-y^2}}\,dy = \cancel{\int} \arctan(x)\,dx$$

We have $\int \frac{1}{\sqrt{1-y^2}}\,dy = \arcsin(y) + \text{constant}$

For $\int \arctan(x)\,dx$: integration by parts

$u = \arctan(x)$ $du = \frac{1}{x^2+1}dx$

$dv = dx$ $v = x$

$= x\arctan(x) - \int \frac{x}{x^2+1}\,dx$

$= x\arctan(x) - \frac{1}{2}\ln(x^2+1) + \text{constant}$

So we have:

$\arcsin(y) = x\arctan(x) - \frac{1}{2}\ln(x^2+1) + C$

where C is a constant

$$y = \sin\left(x\arctan(x) - \frac{1}{2}\ln(x^2+1) + C\right)$$

Plug-in $x=0$: $\frac{\sqrt{2}}{2} = y(0) = \sin(C)$

Take $C = \frac{\pi}{4}$

$$y = \sin\left(x\arctan(x) - \frac{1}{2}\ln(x^2+1) + \frac{\pi}{4}\right)$$