

Math 101—Midterm Exam #2, Practice Midterm A
Duration: 50 minutes

Surname (Last Name)

Given Name

Student Number

Do not open this test until instructed to do so! This exam should have 8 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work. Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked. Continue on the back of the page if you run out of space.

UBC rules governing examinations:

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (c) purposely viewing the written papers of other examination candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem	Out of	Score	Problem	Out of	Score
1	6		5	8	
2	6		6	8	
3	6		7	8	
4	3		Total	45	

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

- 1a. [3 pts] Find the average value of the function $f(x) = \sqrt{x}$ on the interval $1 \leq x \leq 9$. Simplify your answer fully.

Use the formula: $\frac{\int_a^b f(x) dx}{b-a}$, we have:

$$\frac{\int_1^9 \sqrt{x} dx}{8} = \frac{1}{8} \cdot \frac{2}{3} x^{3/2} \Big|_1^9 = \frac{52}{24} = \frac{13}{6}$$

- 1b. [3 pts] Calculate $\int 9x^2 e^{-3x} dx$.

• Integration by parts: $u = 9x^2$ $du = 18x dx$
 $dv = e^{-3x} dx$ $v = \frac{e^{-3x}}{-3}$

$$uv - \int v du = -3x^2 e^{-3x} + \int 6x e^{-3x} dx$$

• For $\int 6x e^{-3x} dx$: integration by parts again:

$$u = 6x \quad du = 6 dx$$

$$dv = e^{-3x} dx \quad v = \frac{e^{-3x}}{-3}$$

$$uv - \int v du = -2x e^{-3x} + \int 2 e^{-3x} dx$$

$$= -2x e^{-3x} - \frac{2}{3} e^{-3x} + C$$

Final answer:

$$\boxed{-3x^2 e^{-3x} - 2x e^{-3x} - \frac{2}{3} e^{-3x} + C}$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

2a. [3 pts] Find $\int \sin^3 x \cos^4 x \, dx$.

$$= \int \sin^2 x \cos^4 x \sin x \, dx = \int (1 - \cos^2 x) \cos^4 x \sin x \, dx$$

$$u = \cos x$$

$$du = -\sin x \, dx$$

We have the integral:

$$\int (1 - u^2) u^4 (-du) = \int (-u^4 + u^6) \, du$$

$$= -\frac{u^5}{5} + \frac{u^7}{7} + C = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$$

2b. [3 pts] Find $\int \frac{1}{x^2 \sqrt{x^2 - 4}} \, dx$.

$$x = 2 \sec \theta \quad \left(0 \leq \theta < \frac{\pi}{2}, \pi \leq \theta < \frac{3\pi}{2} \right)$$

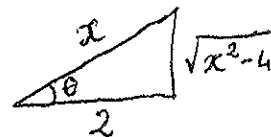
$$dx = 2 \sec \theta \tan \theta \, d\theta, \quad \text{and} \quad \sqrt{x^2 - 4} = 2 \tan \theta$$

The integral becomes

$$\int \frac{1}{4 \sec^2 \theta \cdot 2 \tan \theta} \cdot 2 \sec \theta \tan \theta \, d\theta = \frac{1}{4} \int \frac{1}{\sec \theta} \, d\theta$$

$$= \frac{1}{4} \int \cos \theta \, d\theta = \frac{1}{4} \sin \theta + C$$

From $x = 2 \sec \theta$, get $\cos \theta = \frac{2}{x}$. Triangle:



$$\text{answer: } \frac{1}{4} \frac{\sqrt{x^2 - 4}}{x} + C$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

3a. [3 pts] Approximate $\int_0^{4\pi} e^{\sin^2 x} dx$ by using Trapezoidal Rule with $n = 4$. Simplify your answer fully.

$$\Delta x = \frac{4\pi}{4} = \pi, \quad x_i = a + i\Delta x = i\pi \quad \text{for } i=0, 1, \dots, 4$$

$$f(x) = e^{\sin^2 x}. \quad \text{So } f(x_i) = e^{\sin^2(i\pi)} = e^0 = 1$$

Answer:

$$\begin{aligned} & \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)) \\ &= \frac{\pi}{2} (1 + 2 + 2 + 2 + 1) = 4\pi \end{aligned}$$

3b. [3 pts] Determine whether the integral

$$\int_e^{\infty} \frac{1}{x(\ln x)^{2015}} dx$$

is convergent or divergent. If it is convergent, evaluate it.

Need to study: $\lim_{t \rightarrow \infty} \int_e^t \frac{1}{x(\ln x)^{2015}} dx$

Substitution

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

The integral becomes $\int_1^{\ln t} \frac{1}{u^{2015}} du = \left. \frac{u^{-2014}}{-2014} \right|_1^{\ln t}$

$$= -\frac{1}{2014} \frac{1}{(\ln t)^{2014}} + \frac{1}{2014}$$

As $t \rightarrow \infty$: $\frac{1}{\ln t} \rightarrow 0$. So the improper integral converges to $\frac{1}{2014}$.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

4. [3 pts] Determine if the sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{\sqrt{n^4 + 2014}}{\sqrt[3]{n^6 + 2016}}$$

Divide the numerator and denominator by

$$n^2 = \sqrt{n^4} = \sqrt[3]{n^6}$$

We have

$$a_n = \frac{\sqrt{1 + \frac{2014}{n^4}}}{\sqrt[3]{1 + \frac{2016}{n^6}}}$$

Since $\lim_{n \rightarrow \infty} \frac{2014}{n^4} = 0$ and $\lim_{n \rightarrow \infty} \frac{2016}{n^6} = 0$, we

have

$$\lim_{n \rightarrow \infty} a_n = \frac{\sqrt{1+0}}{\sqrt[3]{1+0}} = 1$$

Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

5.

(a) [3 pts] Find constants A , B , and C such that:

$$\frac{2x^2 - x + 5}{x^3 - 2x^2 + 5x} = \frac{A}{x} + \frac{Bx + C}{x^2 - 2x + 5}$$

Multiply common denominator: $2x^2 - x + 5 = A(x^2 - 2x + 5) + (Bx + C)x$
 $= (A+B)x^2 + (-2A+C)x + 5A$

Hence we have $A + B = 2$, $-2A + C = -1$, $5A = 5$

This gives: $A = 1$, $B = 1$, $C = 1$.

(b) [3 pts] Evaluate $\int \frac{2x^2 - x + 5}{x^3 - 2x^2 + 5x} dx$.

By part (a): $\int \frac{1}{x} dx + \int \frac{x+1}{x^2 - 2x + 5} dx$

For the 2nd integral: $x^2 - 2x + 5 = (x-1)^2 + 4$. Put $u = \frac{x-1}{2}$ so
 $dx = 2du$ and $x+1 = 2u+2$. We have

$$\int \frac{2u+2}{4u^2+4} 2du = \frac{2}{4} \left(\int \frac{2u}{u^2+1} du + \int \frac{2}{u^2+1} du \right) = \frac{1}{2} \left(\ln(u^2+1) + 2\arctan(u) \right) + C$$

Answer: $\ln|x| + \frac{1}{2} \left(\ln\left(\left(\frac{x-1}{2}\right)^2 + 1\right) + 2\arctan\left(\frac{x-1}{2}\right) \right) + C$

(c) [2 pts] If the integrand had been the more complicated function

$$\frac{x+7}{(x+2)(x^3+x)(x^3-x^2+x)},$$

what would the general partial fraction form have been? Just write down the form of the partial fractions, with constants A, B, \dots ; do not find numerical values for the constants.

The denominator is factorized as

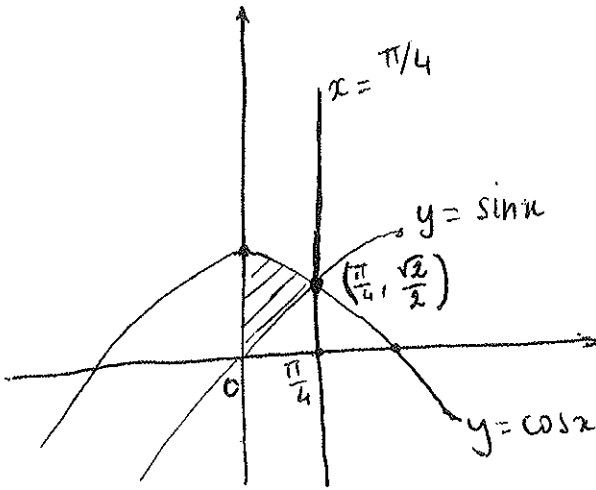
$$(x+2)x^2(x^2+1)(x^2-x+1)$$

Hence we have the form

$$\frac{A}{x+2} + \frac{B}{x} + \frac{C}{x^2} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{x^2-x+1}$$

6. Let R be the region bounded by $0 \leq x \leq \frac{\pi}{4}$ and the graphs of $y = \cos x$ and $y = \sin x$.

(a) [2 pts] Sketch R and find its area.



$$\begin{aligned} A = \text{area} &= \int_0^{\pi/4} (\cos x - \sin x) dx \\ &= (\sin x + \cos x) \Big|_0^{\pi/4} \\ &= \sqrt{2} - 1 \end{aligned}$$

(b) [6 pts] Find the centroid of R .

$$\bar{x} = \frac{1}{A} \int_0^{\pi/4} x (\cos x - \sin x) dx$$

$$\begin{aligned} \left(\begin{array}{l} u = x \\ dv = (\cos x - \sin x) dx \end{array} \right. & \quad \begin{array}{l} du = dx \\ v = \sin x + \cos x \end{array} \end{aligned}$$

$$= \frac{1}{\sqrt{2} - 1} \left(x(\sin x + \cos x) \Big|_0^{\pi/4} - \int_0^{\pi/4} (\sin x + \cos x) dx \right)$$

$$= \frac{1}{\sqrt{2} - 1} \left(\frac{\pi}{4} \cdot \sqrt{2} - (-\cos x + \sin x) \Big|_0^{\pi/4} \right)$$

$$= \frac{1}{\sqrt{2} - 1} \left(\frac{\sqrt{2} \pi}{4} - 1 \right)$$

$$\bar{y} = \frac{1}{A} \int_0^{\pi/4} \frac{1}{2} (\cos^2 x - \sin^2 x) dx = \frac{1}{2(\sqrt{2} - 1)} \int_0^{\pi/4} \cos 2x dx$$

$$= \frac{1}{2(\sqrt{2} - 1)} \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} = \frac{1}{4(\sqrt{2} - 1)} \cdot \frac{\sin \pi/2}{2} = \frac{1}{4(\sqrt{2} - 1)}$$

Answer: $\left(\frac{1}{\sqrt{2} - 1} \left(\frac{\sqrt{2} \pi}{4} - 1 \right), \frac{1}{4(\sqrt{2} - 1)} \right)$

7. [8 pts] Find the solution of the differential equation

$$y' = \sqrt{1-y^2} \arctan(x)$$

satisfying the initial condition $y(0) = \frac{\sqrt{2}}{2}$.

Write the equation into the form

$$\frac{1}{\sqrt{1-y^2}} dy = \int \arctan(x) dx$$

We have $\int \frac{1}{\sqrt{1-y^2}} dy = \arcsin(y) + \text{constant}$

For $\int \arctan(x) dx$: integration by parts
 $u = \arctan(x)$ $du = \frac{1}{x^2+1} dx$
 $dv = dx$ $v = x$

$$= x \arctan(x) - \int \frac{x}{x^2+1} dx$$

$$= x \arctan(x) - \frac{1}{2} \ln(x^2+1) + \text{constant}$$

So we have:

$$\arcsin(y) = x \arctan(x) - \frac{1}{2} \ln(x^2+1) + C$$

where C is a constant

$$y = \sin \left(x \arctan(x) - \frac{1}{2} \ln(x^2+1) + C \right)$$

Plug-in $x = 0$: $\frac{\sqrt{2}}{2} = y(0) = \sin(C)$

Take $C = \frac{\pi}{4}$

$$y = \sin \left(x \arctan(x) - \frac{1}{2} \ln(x^2+1) + \frac{\pi}{4} \right)$$