

Math 101—Midterm Exam #1, Practice Midterm B

Duration: 50 minutes

Name: _____ Student Number: _____

Do not open this test until instructed to do so! This exam should have 8 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work. Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked. Continue on the back of the page if you run out of space.

UBC rules governing examinations:

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (c) purposely viewing the written papers of other examination candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem	Out of	Score	Problem	Out of	Score
1	6		5	8	
2	6		6	8	
3	6		7	8	
4	3		Total	45	

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

1a. [3 pts] Find a function f and numbers a and b so that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \sqrt{1 - \left(\frac{i}{n}\right)^2}$$

represents the area under the graph of $y = f(x)$ between $x = a$ and $x = b$. (You don't have to calculate the area.)

It looks like we should arrange for $\Delta x = \frac{1}{n}$, so we choose $a = 0$ and $b = 1$ (other choices, though a little more complicated, are possible). Then $x_i = a + i\Delta x = \frac{i}{n}$, and so the given expression becomes

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x \sqrt{1 - x_i^2}.$$

If we define $f(x) = \sqrt{1 - x^2}$, then this becomes

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \int_0^1 f(x) dx$$

by the definition of the definite integral. The expression therefore represents the area under the graph of $y = f(x) = \sqrt{1 - x^2}$ between $x = 0$ and $x = 1$.

1b. [3 pts] Evaluate the derivative $F'(x)$, where

$$F(x) = \int_{\cos x}^{\sqrt{x}} \frac{z^2}{z^4 + 1} dz.$$

Define $g(u) = \int_0^u \frac{z^2}{z^4 + 1} dz$. Then by the properties of integrals,

$$F(x) = \int_0^{\sqrt{x}} \frac{z^2}{z^4 + 1} dz - \int_0^{\cos x} \frac{z^2}{z^4 + 1} dz = g(\sqrt{x}) - g(\cos x).$$

The Fundamental Theorem of Calculus part 1 tells us that $g'(u)$ is exactly $\frac{u^2}{u^4 + 1}$. Therefore by the Chain Rule,

$$\begin{aligned} F'(x) &= g'(\sqrt{x}) \cdot \frac{d}{dx} \sqrt{x} - g'(\cos x) \cdot \frac{d}{dx} \cos x \\ &= \frac{(\sqrt{x})^2}{(\sqrt{x})^4 + 1} \cdot \frac{1}{2\sqrt{x}} - \frac{(\cos x)^2}{(\cos x)^4 + 1} \cdot (-\sin x). \end{aligned}$$

If you wish, you could simplify this answer to

$$\frac{\sqrt{x}}{2x^2 + 2} + \frac{\cos^2 x \sin x}{\cos^4 x + 1}.$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

2a. [3 pts] Evaluate the indefinite integral

$$\int u(u+4)(2u+1) du.$$

We simply expand out the polynomial to get

$$\begin{aligned}\int u(u+4)(2u+1) du &= \int u(2u^2 + 9u + 4) du \\ &= \int (2u^3 + 9u^2 + 4u) du \\ &= \frac{1}{2}u^4 + 3u^3 + 2u^2 + C.\end{aligned}$$

2b. [3 pts] Evaluate the indefinite integral

$$\int e^{\sin \theta} \cos \theta d\theta.$$

We use the substitution $u = \sin \theta$, so that $du = \cos \theta d\theta$ and thus

$$\int e^{\sin \theta} \cos \theta d\theta = \int e^u du = e^u + C = e^{\sin \theta} + C.$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

3a. [3 pts] Evaluate the definite integral

$$\int_3^5 \frac{x}{(x^2 - 1) + \sqrt{x^2 - 1}} dx.$$

Your answer can remain in “calculator-ready” form.

We choose the substitution $u = \sqrt{x^2 - 1}$, so that

$$du = \frac{1}{2} \frac{1}{\sqrt{x^2 - 1}} 2x dx = \frac{x}{\sqrt{x^2 - 1}} dx.$$

We have $u = \sqrt{8}$ when $x = 3$, while $u = \sqrt{24}$ when $x = 5$; therefore the integral becomes

$$\begin{aligned} \int_3^5 \frac{x}{(x^2 - 1) + \sqrt{x^2 - 1}} dx &= \int_3^5 \frac{1}{\sqrt{x^2 - 1} + 1} \frac{x}{\sqrt{x^2 - 1}} dx \\ &= \int_{\sqrt{8}}^{\sqrt{24}} \frac{1}{u + 1} du \\ &= \ln |u + 1| \Big|_{\sqrt{8}}^{\sqrt{24}} = \ln(\sqrt{24} + 1) - \ln(\sqrt{8} + 1). \end{aligned}$$

3b. [3 pts] Consider a rod of length 10 cm whose mass density is given by the function

$$\rho(x) = \frac{\ln(x + 2)}{x + 2} \text{ g/cm}$$

for all $0 \leq x \leq 10$ (x is measured in cm). Compute the total mass of the rod. Your answer can remain in “calculator-ready” form.

The mass of the small piece of the rod at position x , of length Δx cm, is $\rho(x)\Delta x = \frac{\ln(x+2)}{x+2} \Delta x$ g. Therefore the total mass is

$$\int_0^{10} \frac{\ln(x + 2)}{x + 2} dx.$$

We use the substitution $u = \ln(x + 2)$, so that $du = \frac{1}{x+2} dx$. The limits of integration $x = 0$ and $x = 10$ become $u = \ln 2$ and $u = \ln 12$ in the new variable. Therefore

$$\int_0^{10} \frac{\ln(x + 2)}{x + 2} dx = \int_{\ln 2}^{\ln 12} u du = \left. \frac{u^2}{2} \right|_{\ln 2}^{\ln 12} = \frac{(\ln 12)^2 - (\ln 2)^2}{2}.$$

Problems 1–4 are short-answer questions.

4. **[3 pts]** If $w'(t)$ is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t) dt$ represent? (For this single problem, you don't have to show any work—just write your answer.)

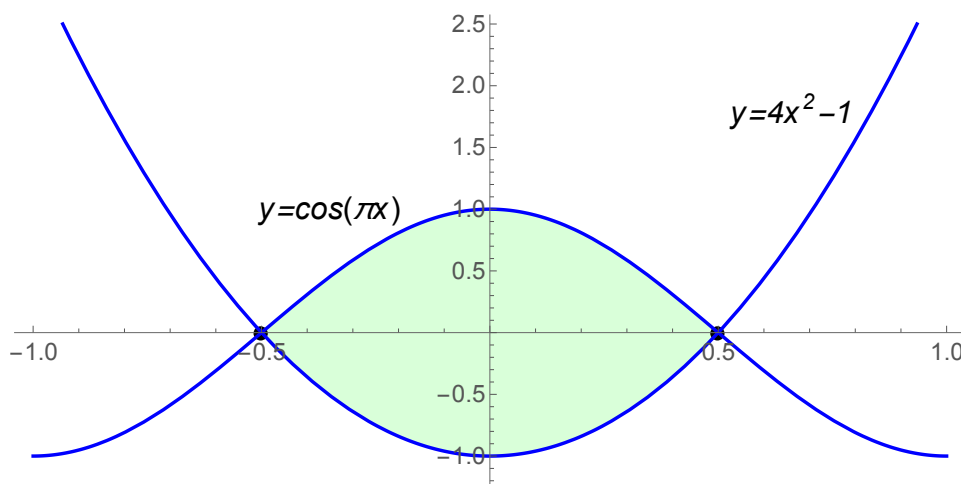
By the Net Change theorem, the integral of the rate of change of a quantity is the net change in the quantity. Since the integrand is the rate at which the child's weight is changing, the integral is the net change in the child's weight on the interval $5 \leq t \leq 10$. In other words (since $w'(t)$ is given in pounds per year), the integral is the number of pounds the child gained between age 5 and age 10.

Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

5. This problem concerns the two curves $y = \cos \pi x$ and $y = 4x^2 - 1$ and the region enclosed by the two curves.

- (a) [4 pts] Find the x -intercepts of the two curves and make a rough sketch of the region. Label the curves and all intersection points.

Note that the x -intercepts are in fact points of intersection of the two curves. Outside of the interval $[-1, 1]$, the function $4x^2 - 1$ is greater than 1; since $\cos(\pi x)$ is never larger than 1, there can't be any intersection points outside $[-1, 1]$. Finally, on the interval $[0, 1]$, one function is increasing while the other is decreasing, so there's only one intersection point; and similarly for the interval $[-1, 0]$.



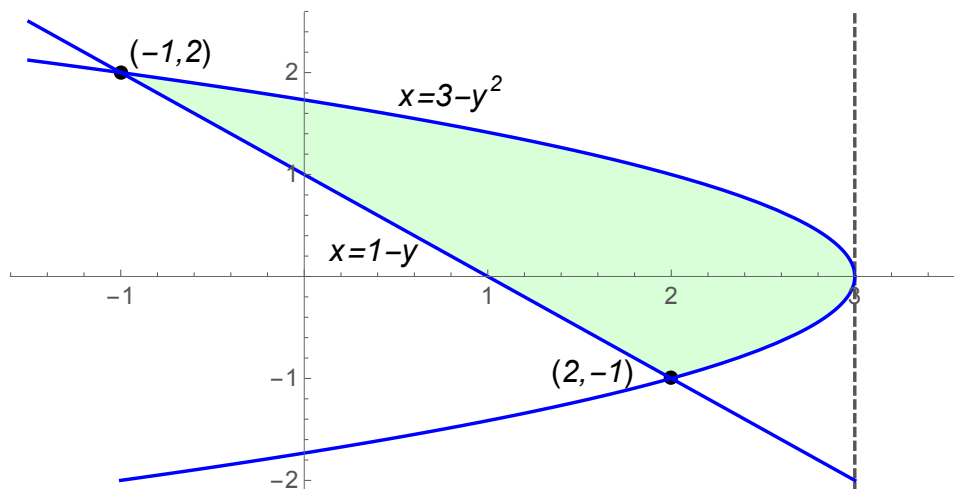
- (b) [4 pts] Calculate the area of the region enclosed by the two curves.

From $-\frac{1}{2} \leq x \leq \frac{1}{2}$, the larger function is $\cos(\pi x)$ and the smaller function is $4x^2 - 1$ (one is positive and the other negative, for example). Therefore the area between the two curves is

$$\begin{aligned} \int_{-1/2}^{1/2} (\cos(\pi x) - (4x^2 - 1)) dx &= \left(\frac{1}{\pi} \sin(\pi x) - \frac{4}{3}x^3 + x \right) \Big|_{-1/2}^{1/2} \\ &= \left(\frac{1}{\pi} \sin\left(\frac{\pi}{2}\right) - \frac{4}{3} \cdot \frac{1}{8} + \frac{1}{2} \right) - \left(\frac{1}{\pi} \sin\left(-\frac{\pi}{2}\right) + \frac{4}{3} \cdot \frac{1}{8} - \frac{1}{2} \right) \\ &= \frac{2}{\pi} + \frac{2}{3}. \end{aligned}$$

(One could have also integrated just from $0 \leq x \leq \frac{\pi}{2}$ and doubled the answer, since the region is symmetric with respect to the y -axis, that is, since both functions are even.)

6. [8 pts] The region enclosed by the line $x + y = 1$ and the curve $x = 3 - y^2$ is rotated about the line $x = 3$. Find the volume of the resulting solid.



Since we are rotating horizontally, we need both equations to give x as a function of y ; in particular, we write the line as $x = 1 - y$. To find the points of intersection, we set $1 - y = 3 - y^2$, which gives us $y^2 - y - 2 = 0$ or $(y + 1)(y - 2) = 0$; therefore the two graphs intersect when $y = -1$ and $y = 2$. On this interval, the outer radius is the distance from $x = 1 - y$ to $x = 3$, which is $3 - (1 - y) = 2 + y$; similarly, the inner radius is $3 - (3 - y^2) = y^2$. The volume of the solid of revolution is therefore

$$\begin{aligned} \int_{-1}^2 \pi((2+y)^2 - (y^2)^2) dy &= \pi \int_{-1}^2 (4 + 4y + y^2 - y^4) dy \\ &= \pi \left(4y + 2y^2 + \frac{y^3}{3} - \frac{y^5}{5} \right) \Big|_{-1}^2 \\ &= \pi \left(8 + 8 + \frac{8}{3} - \frac{32}{5} \right) - \pi \left(-4 + 2 - \frac{1}{3} + \frac{1}{5} \right) \\ &= \frac{72}{5} \pi \end{aligned}$$

(which you could write as 14.4π if you wished).

7. **[8 pts]** A 10-foot long chain weighing 25 pounds hangs from a high ceiling. Calculate the amount of work done when the bottom end of the chain is lifted straight up and attached to the top of the chain (so that the end result is the chain doubled back on itself).

We choose the coordinates so that $y = 0$ is the original bottom of the chain and $y = 10$ is the ceiling. (Other choices of where to put $y = 0$ are fine, as long as one remains consistent throughout the solution.) The part of the chain starting from $5 \leq y \leq 10$ doesn't move at all. If $0 \leq y \leq 5$, then the small part of the chain (of length Δy) that started at height y ends up at height $10 - y$, and therefore is lifted a total of $(10 - y) - y = 10 - 2y$ feet. The chain has linear density $\frac{25 \text{ lb}}{10 \text{ ft}} = 2.5 \text{ lb/ft}$, and so this small piece of chain weighs $2.5\Delta y$ lb; thus the work done while raising this piece is $(10 - 2y)2.5\Delta y = (25 - 5y)\Delta y$ ft-lb. The total work is therefore

$$\int_0^5 (25 - 5y) dy = (25y - 2.5y^2) \Big|_0^5 = (25 \cdot 5 - 2.5 \cdot 25) - (0 - 0) = 62.5 \text{ ft-lb.}$$