Math 101—Midterm Exam #1, Practice Midterm A
Duration: 50 minutes

Name: ______________________ Student Number: ______________________

Do not open this test until instructed to do so! This exam should have 8 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work. Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked. Continue on the back of the page if you run out of space.

UBC rules governing examinations:

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (a) speaking or communicating with other examination candidates, unless otherwise authorized;
   (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (c) purposely viewing the written papers of other examination candidates;
   (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem	**Out of**	**Score**	**Problem**	**Out of**	**Score**
1	6		5	8	
2	6		6	8	
3	6		7	8	
4	3		**Total**	45	

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

1a. **[3 pts]** Write $\displaystyle\int_2^7 \ln(x^2-1)\,dx$ as a limit of Riemann sums with right endpoints. Do not evaluate the Riemann sums or the limit.

Our function is $f(x) = \ln(x^2-1)$. We have $a = 2$ and $b = 7$, so $\Delta x = \frac{7-2}{n} = \frac{5}{n}$ and $x_i = a + i\Delta x = 2 + \frac{5i}{n}$. To use the right endpoints, we take the sample points x_i^* to just be x_i. Therefore

$$\begin{aligned}\int_2^7 \ln(x^2-1)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} \ln\left(\left(2+\frac{5i}{n}\right)^2 - 1\right)\frac{5}{n}.\end{aligned}$$

1b. **[3 pts]** Express $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i^2}{n^2}-\frac{3i}{n}\right)$ as a definite integral, and evaluate the integral.

We decide that Δx should equal $\frac{1}{n}$, and so we take $a = 0$ and $b = 1$ to make this happen. (Other choices are possible.) Then $x_i = a + i\Delta x = \frac{i}{n}$. We see that we can write the summand as $x_i^2 - 3x_i = f(x_i)$ where $f(x) = x^2 - 3x$. Therefore

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i^2}{n^2}-\frac{3i}{n}\right) &= \lim_{n\to\infty}\Delta x\sum_{i=1}^{n}(x_i^2-3x_i) \\ &= \int_0^1 (x^2-3x)\,dx.\end{aligned}$$

An antiderivative of $x^2 - 3x$ is $\frac{x^3}{3} - \frac{3x^2}{2}$, so by the Fundamental Theorem of Calculus, part 2,

$$\begin{aligned}\int_0^1 (x^2-3x)\,dx &= \left(\frac{x^3}{3}-\frac{3x^2}{2}\right)\Bigg]_0^1 \\ &= \left(\frac{1}{3}-\frac{3}{2}\right) - \left(\frac{0}{3}-\frac{0}{2}\right) = -\frac{7}{6}.\end{aligned}$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

2a. **[3 pts]** Evaluate the definite integral $\displaystyle\int_{-3}^{3} 3x^5\sqrt{9-x^2}\,dx$.

Notice that if we replace x by $-x$ in the integrand, it becomes its own negative: $3(-x^5)\sqrt{9-(-x)^2} = -3x^5\sqrt{9-x^2}$. In other words, the integrand is an odd function, and the interval of integration $[-3,3]$ is symmetric around 0. Therefore the value of the integral is automatically 0.

It is possible to evaluate this integral using the substitution $u = 9 - x^2$, if we don't see the easier way described above. We would have $du = -2x\,dx$; we would also have $x^2 = 9 - u$ and so $x^4 = (9-u)^2$. Therefore the substitution solution would begin

$$\int_{-3}^{3} 3x^5\sqrt{9-x^2}\,dx = -\frac{3}{2}\int_{-3}^{3} x^4\sqrt{9-x^2}(-2x)\,dx = -\frac{3}{2}\int_{9-(-3)^2}^{9-3^2} (9-u)^2\sqrt{u}\,du \ldots.$$

2b. **[3 pts]** Evaluate the indefinite integral $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$.

We use the substitution $u = \sqrt{x} = x^{1/2}$, so that $du = \frac{1}{2}x^{-1/2}\,dx$, or equivalently $2\,du = \frac{dx}{\sqrt{x}}$. Therefore

$$\begin{aligned}\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx &= \int (\sin u)\,2\,du \\ &= 2\int \sin u\,du \\ &= 2(-\cos u) + C \\ &= -2\cos(\sqrt{x}) + C.\end{aligned}$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

3a. **[3 pts]** Evaluate the indefinite integral $y = \displaystyle\int \frac{x^2}{1+x^6}\,dx$.

We notice that $x^6 = (x^3)^2$, and that the derivative of x^3 appears (other than the multiplicative constant 3) in the numerator. Therefore we decide to set $u = x^3$, so that $du = 3x^2\,dx$. We obtain

$$\begin{aligned}\int \frac{x^2}{1+x^6}\,dx &= \frac{1}{3}\int \frac{3x^2\,dx}{1+(x^3)^2}\\ &= \frac{1}{3}\int \frac{du}{1+u^2}\\ &= \frac{1}{3}\tan^{-1} u + C\\ &= \frac{1}{3}\tan^{-1}(x^3) + C.\end{aligned}$$

3b. **[3 pts]** Find the slope of the tangent line to the curve $y = \displaystyle\int_{-2}^{x^3} e^{-t^2}\,dt$ at $x = -1$.

Let $f(x) = \int_{-2}^{x^3} e^{-t^2}\,dt$. The slope of the tangent line to the graph of $y = f(x)$ at $x = -1$ is exactly $f'(-1)$, so we start by finding the derivative $f'(x)$. We write $f(x) = g(h(x))$ as a composition of the two functions $g(x) = \int_{-2}^{x} e^{-t^2}\,dt$ and $h(x) = x^3$. By the Fundamental Theorem of Calculus, part 1, $g'(x) = e^{-x^2}$. Therefore, by the Chain Rule,

$$\begin{aligned}f'(x) &= g'(h(x))h'(x)\\ &= e^{-h(x)^2}h'(x)\\ &= e^{-(x^3)^2}3x^2 = 3e^{-x^6}x^2.\end{aligned}$$

We conclude that the slope of the tangent line at $x = -1$ is

$$f'(-1) = 3e^{-(-1)^6}(-1)^2 = 3e^{-1} = \frac{3}{e}.$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

4. **[3 pts]** A particle moves along a line so that its velocity at time t is given $v(t) = \cos(\pi t)$ (measured in metres per second). Find the total distance traveled during the first 3 seconds.

We know that the total distance traveled is the integral of the absolute value of the velocity, so we need to evaluate $\int_0^3 |\cos(\pi t)|\,dt$. The function $\cos(\pi t)$ is positive for $0 \le t \le \frac{1}{2}$ and $\frac{3}{2} \le t \le \frac{5}{2}$, and negative for $\frac{1}{2} \le t \le \frac{3}{2}$ and $\frac{5}{2} \le t \le 3$. Therefore we need to evaluate

$$\int_0^{1/2} \cos(\pi t)\,dt + \int_{1/2}^{3/2} -\cos(\pi t)\,dt + \int_{3/2}^{5/2} \cos(\pi t)\,dt + \int_{5/2}^{3} -\cos(\pi t)\,dt.$$

One can either notice directly, or else use the substition $u = \pi t$ to work out, that an antiderivative of $\cos(\pi t)$ is $\frac{\sin(\pi t)}{\pi}$. So by the Fundamental Theorem of Calculus, part 2, the above expression equals

$$\frac{\sin(\pi t)}{\pi}\bigg]_0^{1/2} + \frac{-\sin(\pi t)}{\pi}\bigg]_{1/2}^{3/2} + \frac{\sin(\pi t)}{\pi}\bigg]_{3/2}^{5/2} + \frac{-\sin(\pi t)}{\pi}\bigg]_{5/2}^{3}$$

$$= \frac{1}{\pi}\Big(\big(\sin\tfrac{\pi}{2} - \sin 0\big) - \big(\sin\tfrac{3\pi}{2} - \sin\tfrac{\pi}{2}\big) + \big(\sin\tfrac{5\pi}{2} - \sin\tfrac{3\pi}{2}\big) - \big(\sin 3\pi - \sin\tfrac{5\pi}{2}\big)\Big)$$

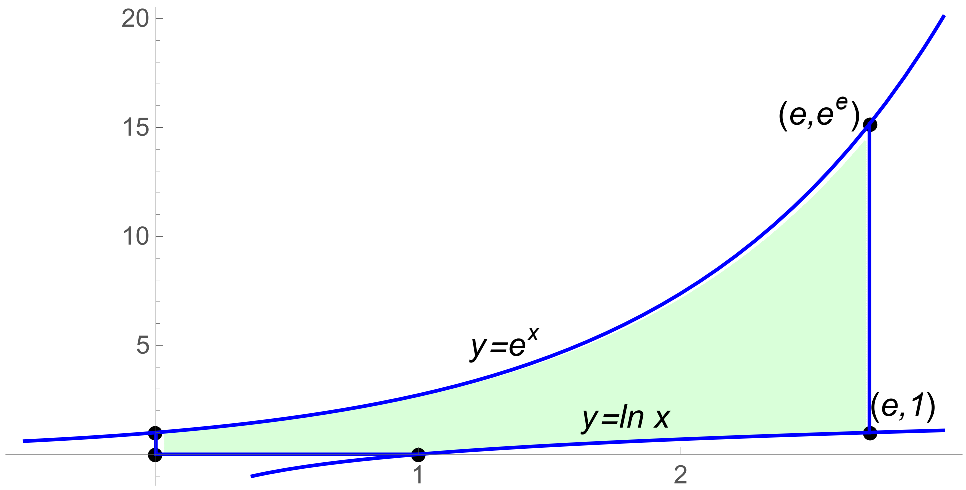
$$= \frac{1}{\pi}\big((1-0) - (-1-1) + (1-(-1)) - (0-1)\big) = \frac{6}{\pi}.$$

(Alternatively, one could see from the graph of $y = |\cos(\pi t)|$ that the area under the graph between $0 \le x \le 3$ is composed of six congruent pieces of equal area, and so the total area is 6 times the area from $0 \le x \le \frac{1}{2}$; then only the integral $\int_0^{1/2} \cos(\pi t)\,dt = \frac{1}{\pi}$ would need to be calculated.)

Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

5a. **[2 pts]** Sketch the region enclosed by the graphs of $y = e^x$ and $y = \ln x$, the x- and y-axes, and the line $x = e$. Label the curves and the points of intersection.

We always have $e^x > x > \ln x$, so the two curves don't intersect. Other than x- and y-intercepts, the only relevant points are the intersections with $x = e$, which are found simply by plugging in e to both functions.



5b. **[6 pts]** Evaluate the area of the region from part a. (Hint: Cut the region in two using the line $y = 1$.)

We divide the area in two with the line $y = 1$, which connects the point $(0, 1)$ with the point $(e, 1)$. For the lower section, we integrate with respect to y: the curve $y = \ln x$ becomes $x = e^y$, and so the area below the line $y = 1$ is

$$\int_0^1 e^y\, dy = e^y]_0^1 = e^1 - e^0 = e - 1.$$

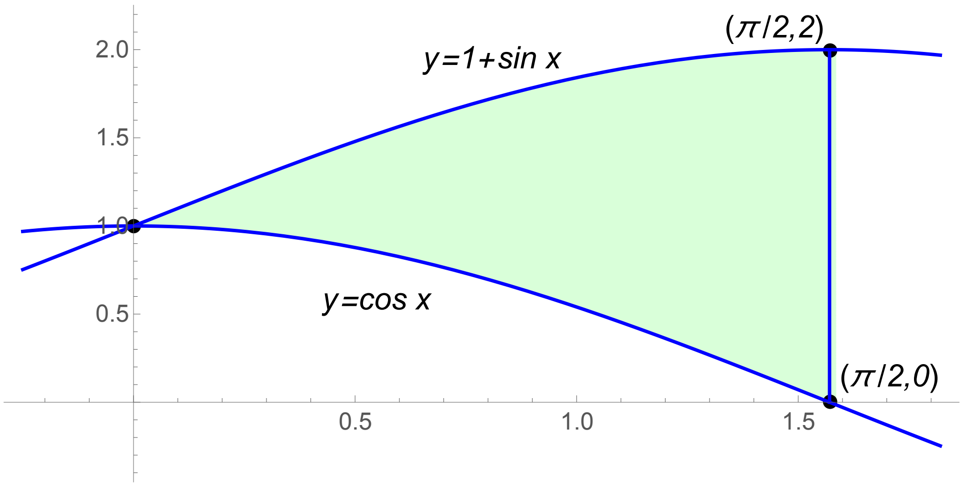
The area above the line $y = 1$ is just the area between the graphs of $y = e^x$ and $y = 1$ from $0 \le x \le e$, which is

$$\int_0^e (e^x - 1)\, dx = (e^x - x)]_0^e = (e^e - e) - (e^0 - 0) = e^e - e - 1.$$

The total area is therefore $(e - 1) + (e^e - e - 1) = e^e - 2$.

6a. **[2 pts]** Sketch the region bounded by the curves $y = 1 + \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \frac{\pi}{2}$. Label the curves and the points of intersection.

(Note that the line $x = 0$ isn't visible, since the curves intersect at the point $(0, 1)$.)



6b. **[6 pts]** Find the volume of the solid obtained by rotating the region from part a about the x-axis. (Hint: $\sin^2 x - \cos^2 x = -\cos(2x)$.)

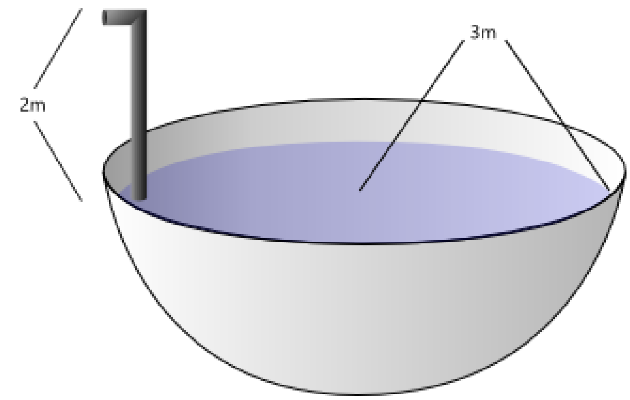
The outer radius is $1 + \sin x$ and the inner radius is $\cos x$, so the volume in question is

$$\int_0^{\pi/2} \pi\big((1+\sin x)^2 - (\cos x)^2\big)\,dx = \pi \int_0^{\pi/2} \big(1 + 2\sin x + \sin^2 x - \cos^2 x\big)\,dx.$$

Using the double angle formula $\cos(2x) = \cos^2 x - \sin^2 x$, this becomes

$$\begin{aligned}\pi \int_0^{\pi/2} \big(1 + 2\sin x - \cos(2x)\big)\,dx &= \pi\big(x - 2\cos x - \tfrac{1}{2}\sin(2x)\big)\Big]_0^{\pi/2} \\ &= \pi\big(\tfrac{\pi}{2} - 2\cos\tfrac{\pi}{2} - \tfrac{1}{2}\sin\pi\big) - \pi\big(0 - 2\cos 0 - \tfrac{1}{2}\sin 0\big) \\ &= \pi\big(\tfrac{\pi}{2} + 2\big).\end{aligned}$$

7. **[8 pts]** A tank in the shape of a hemispherical bowl of radius 3 m, with an outlet that rises 2 m above its top (see the diagram below), is full of water. Using the fact that the density of water is $1000\ \mathrm{kg/m^3}$, find the work (in joules) required to pump all the water out of the outlet. You may use the value $g = 9.8\ \mathrm{m/s^2}$ for the acceleration due to gravity. *You do not need to simplify your answer, but you must completely evaluate any integral that arises.*



We have to choose coordinates and stick to our choice throughout. We will let $y = 0$ correspond to the top of the tank, with negative y below and positive y above. Then the tank runs from $y = -3$ to $y = 0$. At a particular height y in that range, the cross section of the tank is a circle with radius $\sqrt{9-y^2}$. Therefore the mass of the water at coordinate y is the volume times the density of water, which is

$$\pi\left(\sqrt{9-y^2}\right)^2 \Delta y \cdot 1000\ \mathrm{kg/m^3} = 1000\pi(9-y^2)\Delta y\ \mathrm{kg}.$$

The water at coordinate y needs to be lifted a total of $2-y$ metres (note that y is negative in these coordinates, so this is more than 2 m), so the work required to lift that water is the mass times the force of gravity times the distance, or

$$1000\pi(9-y^2)\Delta y\ \mathrm{kg} \cdot 9.8\ \mathrm{m/s^2} \cdot (2-y)\ \mathrm{m} = 9800\pi(9-y^2)(2-y)\Delta y\ \mathrm{J}.$$

The total work is therefore

$$\begin{aligned}\int_{-3}^{0} 9800\pi(9-y^2)(2-y)\,dy &= 9800\pi\int_{-3}^{0}(18-9y-2y^2+y^3)\,dy \\ &= 9800\pi\left(18y - \frac{9}{2}y^2 - \frac{2}{3}y^3 + \frac{1}{4}y^4\right)\Bigg]_{-3}^{0} \\ &= 0 - 9800\pi\left(-54 - \frac{81}{2} + 18 + \frac{81}{4}\right)\ \mathrm{J}.\end{aligned}$$

(Although it isn't necessary to simplify completely, we can check that the answer equals 551250π J.)