Math 101—Practice Final Examination
Duration: 150 minutes

Surname (Last Name)   Given Name   Student Number

Do not open this test until instructed to do so! This exam should have 13 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam. Phones cannot be visible at any point during the exam.

UBC rules governing examinations:

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (a) speaking or communicating with other examination candidates, unless otherwise authorized;
   (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (c) purposely viewing the written papers of other examination candidates;
   (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
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7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem	Out of	Score	Problem	Out of	Score
1	8		7	7	
2	8		8	7	
3	6		9	7	
4	6		10	7	
5	6		11	7	
6	6		**Total**	75	

- Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.
- Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.
- Problems 7–11 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.
- Continue on the backs of the pages if you run out of space.
- You have 150 minutes to complete this exam, and there are a total of 75 points to be earned. So, roughly speaking, you can allocate 2 minutes per point to each problem (for example, 14 minutes for a 7-point problem).

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

1a. **[2 pts]** For each series, write **C** if it converges, or write **D** if it diverges. (At least one of them converges, and at least one of them diverges.)

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$ Answer: C

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4+6}$ Answer: D

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{9+6^n}$ Answer: C

1b. **[2 pts]** For each integral, choose the substitution type that is most beneficial for evaluating the integral. (Write **E**, **F**, or **G** in each box; each answer will be used exactly once.)

E: $x = a\sec\theta$ **F:** $x = a\sin\theta$ **G:** $x = a\tan\theta$

(i) $\displaystyle\int \frac{x^4}{\sqrt{9-x^2}}\,dx$ Answer: F

(ii) $\displaystyle\int \frac{\tan^{-1} x}{\sqrt{4+x^2}}\,dx$ Answer: G

(iii) $\displaystyle\int \frac{3x^2+1}{\sqrt{4x^2-9}}\,dx$ Answer: E

1c. **[2 pts]** (It is known that the integral $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$ diverges.) Choose which statement is true about the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln n}}$. Answer: J

H: The series converges to 0 by the Squeeze Theorem.
J: The series is conditionally convergent.
K: The series is absolutely convergent by the Ratio Test.
L: The series is divergent by the Comparison Test with the p-series with $p = 1$.
M: The series is divergent by the Integral Test.

1d. **[2 pts]** Which of the given integrals equals $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} \sin\frac{i}{n}$? Answer: T

N: $\displaystyle\int_0^1 (x-1)^2 \sin(x-1)\,dx$

S: $\displaystyle\int_1^2 x^2 \sin x\,dx$

P: $\displaystyle\int_1^2 (x+1)^2 \sin(x+1)\,dx$

T: $\displaystyle\int_1^2 (x-1)^2 \sin(x-1)\,dx$

Q: $\displaystyle\int_0^1 (x+1)^2 \sin(x+1)\,dx$

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

2a. **[4 pts]** Match each expression with its numerical value. (Write **A**, **B**, **C**, **D**, or **E** in each box; each answer will be used exactly once.)

A: $\ln 2$ **B:** 0 **C:** $\frac{1}{2}$ **D:** 1 **E:** e

(i) $\lim_{n\to\infty} \frac{e^{\sin n}}{n}$ Answer: B

(ii) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ Answer: E

(iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ Answer: A

(iv) $\sum_{n=1}^{\infty} \frac{6^n}{2^{2n}3^n}$ Answer: D

(v) $\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$ Answer: C

2b. **[4 pts]** For each of the following series, choose the appropriate statement. (Write **F**, **G**, **H**, **J**, or **K** in each box; each answer will be used at most once, and each series matches a single answer only.)

F: The series diverges.
G: The series converges by the Ratio Test.
H: The series converges by the Alternating Series Test.
J: The series converges absolutely by the Comparison Test with a p-series.
K: The series converges by the Integral Test.

(i) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Answer: K

(ii) $\sum_{n=0}^{\infty} \frac{7^{n+3}}{3^{2n} - 2}$ Answer: G

(iii) $\sum_{n=0}^{\infty} (-1)^n \tan^{-1} n$ Answer: F

(iv) $\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n^3}$ Answer: J

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

3a. **[3 pts]** Compute $\int \cos^2(x)\sin(2x)\,dx$.

Solution 1:

$$\int \cos^2(x)\sin(2x)\,dx = \int \cos^2 x(2\sin x\cos x)\,dx$$
$$= \int u^3(-2\,du) = -\frac{u^4}{2} + C = -\frac{\cos^4 x}{2} + C.$$

Solution 2:

$$\int \cos^2(x)\sin(2x)\,dx = \int \frac{1+\cos 2x}{2}\sin 2x\,dx$$
$$= \int \left(\frac{\sin 2x}{2} + \frac{\sin 4x}{4}\right)dx$$
$$= -\frac{\cos 2x}{4} - \frac{\cos 4x}{16} + C.$$

[One can show using double-angle formulas that these are actually the same answer! Other variants are also possible.]

3b. **[3 pts]** Determine whether the series

$$\sum_{n=2}^{\infty} \frac{2^n+3^n}{4^n}$$

converges, and if so, find its sum.

We split the series into the sum of two series:

$$\sum_{n=2}^{\infty} \frac{2^n+3^n}{4^n} = \sum_{n=2}^{\infty} \frac{2^n}{4^n} + \sum_{n=2}^{\infty} \frac{3^n}{4^n} = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n.$$

Each series is a convergent geometric series, for which we have the formula

$$\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n = \frac{(1/2)^2}{1-1/2} + \frac{(3/4)^2}{1-3/4} = \frac{11}{4}.$$

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

4a. **[3 pts]** Given the function $f(x) = x^3 \sin(2x^2)$, use Maclaurin series to find $f^{(17)}(0)$, the seventeenth derivative of f at $x = 0$.

We start with the known Maclaurin series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

From this we get

$$\begin{aligned} \sin(2x^2) &= (2x^2) - \frac{(2x^2)^3}{3!} + \frac{(2x^2)^5}{5!} - \frac{(2x^2)^7}{7!} + \cdots \\ &= 2x^2 - \frac{2^3x^6}{3!} + \frac{2^5x^{10}}{5!} - \frac{2^7x^{14}}{7!} + \cdots ; \\ x^3 \sin(2x^2) &= 2x^5 - \frac{2^3x^9}{3!} + \frac{2^5x^{13}}{5!} - \frac{2^7x^{17}}{7!} + \cdots . \end{aligned}$$

For any Maclaurin series, the coefficient of x^{17} always equals $f^{(17)}(0)/17!$, and so

$$f^{(17)}(0) = -\frac{2^7}{7!} \cdot 17!.$$

4b. **[3 pts]** A particle moves along a line having initial velocity $v(0) = -4$ m/s. The particle's *acceleration* function after t seconds is $a(t) = 2t$ m/s^2. Find the *total distance* the particle travels in the first 3 seconds.

Since acceleration is the derivative of velocity, we must have $v(t) = \int 2t\,dt = t^2 + C$ for some constant C; since $v(0) = -4$, we deduce that $v(t) = t^2 - 4$. The total distance traveled in the first three seconds is therefore

$$\begin{aligned} \int_0^3 |v(t)|\,dt &= \int_0^3 |t^2 - 4|\,dt \\ &= \int_0^2 (4 - t^2)\,dt + \int_2^3 (t^2 - 4)\,dt \\ &= \left(4t - \frac{t^3}{3}\right)\Big|_0^2 + \left(\frac{t^3}{3} - 4t\right)\Big|_2^3 \\ &= \left(8 - \frac{8}{3}\right) - 0 + (9 - 12) - \left(\frac{8}{3} - 8\right) = \frac{23}{3} \text{ m.} \end{aligned}$$

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

5a. **[3 pts]** Find $\displaystyle\int_0^{\pi/3} x^2 \cos x \, dx$.

We first find the indefinite integral, using integration by parts twice:

$$\begin{aligned}\int x^2 \cos x \, dx &= x^2 \sin x - \int 2x \sin x \, dx \\ &= x^2 \sin x - \left(2x(-\cos x) - \int 2(-\cos x) \, dx\right) \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C.\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^{\pi/3} x^2 \cos x \, dx &= \left(x^2 \sin x + 2x \cos x - 2 \sin x\right)\Big|_0^{\pi/3} \\ &= \left(\frac{\pi^2}{9} \cdot \frac{\sqrt{3}}{2} + \frac{2\pi}{3} \cdot \frac{1}{2} - 2 \cdot \frac{\sqrt{3}}{2}\right) - 0.\end{aligned}$$

5b. **[3 pts]** Evaluate the average value of the function $f(x) = \sqrt{2x+1}$ on the interval $[0, 4]$.

The average value in question is $\displaystyle\frac{1}{4-0}\int_0^4 \sqrt{2x+1} \, dx$.

Solution 1: by finding the antiderivative directly,

$$\begin{aligned}\frac{1}{4}\int_0^4 \sqrt{2x+1} \, dx &= \frac{1}{4} \cdot \frac{1}{2} \frac{(2x+1)^{3/2}}{3/2}\Bigg|_0^4 \\ &= \frac{9^{3/2}}{12} - \frac{1^{3/2}}{12} = \frac{13}{6}.\end{aligned}$$

Solution 2: using the substitution $u = 2x + 1$,

$$\frac{1}{4}\int_0^4 \sqrt{2x+1} \, dx = \frac{1}{4}\int_1^9 u^{1/2} \frac{du}{2} = \frac{1}{8} \cdot \frac{2}{3} u^{3/2}\Big|_1^9 = \frac{1}{12}(27 - 1) = \frac{26}{12} = \frac{13}{6}.$$

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

6a. **[3 pts]** Suppose the power series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 6. What is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} c_n 3^n x^n$? What is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$? Justify your answers.

We know that $\sum_{n=0}^{\infty} c_n x^n$ converges when $|x| < 6$ and diverges when $|x| > 6$. Plugging in $3x$ for x, we see that $\sum_{n=0}^{\infty} c_n (3x)^n = \sum_{n=0}^{\infty} c_n 3^n x^n$ converges when $|3x| < 6$ and diverges when $|3x| > 6$, that is, converges when $|x| < 2$ and diverges when $|x| > 2$. Therefore the radius of convergence of $\sum_{n=0}^{\infty} c_n 3^n x^n$ equals 2.

As for $\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$, it is a term-by-term antiderivative of $\sum_{n=0}^{\infty} c_n x^n$, and we know that two power series related in this way always have the same radius of convergence. Therefore the radius of convergence of $\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$ is also 6.

6b. **[3 pts]** Find the solution of the differential equation $y' = y^2 x$ that satisfies $y(2) = 1$. (Solve completely for y in terms of x.)

Separating the variables, we get

$$\frac{dy}{y^2} = x\,dx$$

$$\int \frac{dy}{y^2} = \int x\,dx$$

$$-\frac{1}{y} = \frac{x^2}{2} + C.$$

Plugging in $x = 2$ and $y = 1$, we see that $-\frac{1}{1} = \frac{2^2}{2} + C$ which gives $C = -3$; therefore

$$-\frac{1}{y} = \frac{x^2}{2} - 3$$

$$y = \frac{1}{3 - x^2/2}.$$

Problems 7–11 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

7. Both parts of this problem concern the power series $S = \sum_{n=2}^{\infty} \frac{(x-5)^n}{n \ln n}$.

(a) **[4 pts]** Using the Ratio Test, find the radius of convergence of the power series S.

For the Ratio Test, we consider

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-5)^{n+1}/((n+1)\ln(n+1))}{(x-5)^n/(n \ln n)} \right|$$
$$= |x-5| \cdot \lim_{n\to\infty} \frac{n}{n+1} \cdot \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = |x-5| \cdot 1 \cdot 1.$$

(The last limit was evaluated by l'Hôpital's Rule: $\lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} = \lim_{x\to\infty} \frac{1/x}{1/(x+1)} = 1$.) Therefore the series converges when $|x-5| < 1$, giving a radius of convergence of $R = 1$.

(b) **[3 pts]** Find the interval of convergence of the power series S. (Hint: you may have to use the integral test.)

Since the radius of convergence is $R = 1$ and the centre of the power series is $a = 5$, we need to check the endpoints $a \pm R = 5 \pm 1$. At $x = 6$, the series is

$$\sum_{n=2}^{\infty} \frac{(6-5)^n}{n \ln n} = \sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

Notice that by the substitution $u = \ln x$,

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1}{u}\, du = \ln|u| = \ln|\ln x| + C.$$

Therefore the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the integral test, since

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} \ln|\ln x| \Big|_2^t = \lim_{t\to\infty} \big(\ln(\ln t) - \ln(\ln 2) \big) = \infty.$$

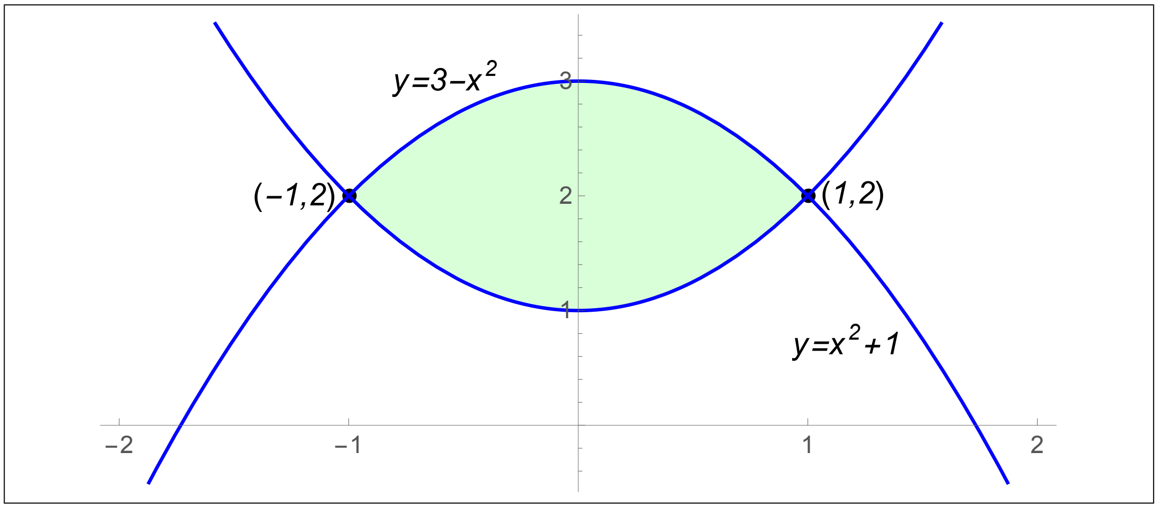
At $x = 4$, the series is

$$\sum_{n=2}^{\infty} \frac{(4-5)^n}{n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n},$$

which converges by the Alternating Series Test, since $b_n = \frac{1}{n \ln n}$ is decreasing and has limit 0. Therefore the interval of convergence is $[4, 6)$.

8. Let R be the region enclosed by the curves $y = x^2 + 1$ and $y = 3 - x^2$.

(a) **[2 pts]** Sketch the region R, labeling the points of intersection.



(b) **[5 pts]** Find the volume of the solid obtained by rotating the region R about the line $y = 1$.

The volume is

$$\int_{-1}^{1} \pi\Big(\big((3-x^2)-1\big)^2 - \big((x^2+1)-1\big)^2\Big)\,dx$$
$$= \pi\int_{-1}^{1} \big((x^4 - 4x^2 + 4) - x^4\big)\,dx$$
$$= \pi\int_{-1}^{1} \big(4 - 4x^2\big)\,dx$$
$$= \pi\left(4x - \frac{4x^3}{3}\right)\Bigg|_{-1}^{1} = \frac{16\pi}{3}.$$

9.

(a) **[3 pts]** Suppose that a function $f(x)$ satisfies $f(0) = 0$ and $f'(x) = \tan^{-1} x$. Find the complete Maclaurin series for $f(x)$.

The known Maclaurin series for $\tan^{-1} x = \arctan x$ gives

$$f'(x) = \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

Integrating the power series term by term yields

$$f(x) = \int \arctan x \, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)}.$$

Plugging in $x = 0$ gives $0 = f(0) = C + \sum_{n=0}^{\infty} 0$, so that $C = 0$ and therefore

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)}.$$

(b) **[4 pts]** Using your answer to part (a), find the value of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)(2n+2)}.$$

You don't have to simplify your answer fully, but evaluate any trigonometric functions.

Everything in this series matches the answer to part (a), except that it has $x^{2n+2} = (x^2)^n x^2$ and we want $\frac{1}{3^n}$. A little experimentation shows that plugging in $x = \frac{1}{\sqrt{3}}$ is helpful:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)(2n+2)} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/\sqrt{3})^{2n+2}(\sqrt{3})^2}{(2n+1)(2n+2)} = 3f\left(\frac{1}{\sqrt{3}}\right).$$

But we can find a formula for $f(x)$ using integration by parts:

$$f(x) = \int \arctan x \, dx = x \arctan - \int \frac{x}{x^2+1} \, dx = x \arctan x - \frac{1}{2} \ln |x^2 + 1| + C,$$

and plugging in $x = 0$ again shows that $C = 0$. Therefore

$$3f\left(\frac{1}{\sqrt{3}}\right) = 3\left(\frac{1}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}} - \frac{1}{2} \ln \left| \left(\frac{1}{\sqrt{3}}\right)^2 + 1 \right| \right)$$
$$= 3\left(\frac{1}{\sqrt{3}} \cdot \frac{\pi}{6} - \frac{1}{2} \ln \frac{4}{3}\right).$$

10. By Newton's Law of Gravitaton, the gravitational force acting on a body of mass M kg that is at a distance r metres from the centre of Earth is $\dfrac{4 \times 10^{17} \cdot M}{r^2}$ N.

(a) **[4 pts]** Write down an improper integral representing the amount of work required to move a 1000 kg spacecraft from the surface of Earth to infinitely far away from Earth. Use 6.4×10^6 m as the distance from the centre of Earth to its surface. (Assume there are no forces acting upon the spacecraft other than Earth's gravity.)

The work required to move the spacecraft from r m away from Earth's centre to $r + \Delta r$ m away is approximately

$$\frac{4 \times 10^{17} \cdot 1000}{r^2} \text{ N} \cdot \Delta r \text{ m} = 4 \times 10^{20} \frac{\Delta r}{r^2} \text{ J}.$$

Therefore the total amount of work required, in joules, is

$$\int_{6.4\times 10^6}^{\infty} 4 \times 10^{20} \frac{dr}{r^2}.$$

(b) **[3 pts]** Evaluate the improper integral you wrote down in part (a), including correct units.

By the definition of the improper integral,

$$\begin{aligned}
\int_{6.4\times 10^6}^{\infty} 4 \times 10^{20} \frac{dr}{r^2} &= \lim_{t\to\infty} \int_{6.4\times 10^6}^{t} 4 \times 10^{20} \frac{dr}{r^2} \\
&= \lim_{t\to\infty} 4 \times 10^{20} \left(-\frac{1}{r} \right) \Bigg|_{6.4\times 10^6}^{t} \\
&= \lim_{t\to\infty} 4 \times 10^{20} \left(-\frac{1}{t} + \frac{1}{6.4 \times 10^6} \right) \\
&= \frac{4 \times 10^{20}}{6.4 \times 10^6} \text{ J}.
\end{aligned}$$

11. Both parts of this problem refer to the function $f(x) = \dfrac{x^4 + x^3 - 6x^2 + 2x - 1}{x^3 - x^2}$.

(a) **[2 pts]** Use long division to write $f(x)$ as $Q(x) + \frac{R(x)}{D(x)}$ for polynomials $Q(x)$, $R(x)$, and $D(x)$ where the degree of $R(x)$ is less than the degree of $D(x)$.

Of course $D(x) = x^3 - x^2$, and the following calculation shows that $Q(x) = x + 2$ and $R(x) = -4x^2 + 2x - 1$:

$$
\begin{array}{r|l}
 & \phantom{x^4 + x^3 - 6x^2 +{}} x + 2 \\
\hline
x^3 - x^2 & x^4 + x^3 - 6x^2 + 2x - 1 \\
 & -x^4 + x^3 \\
\hline
 & \phantom{-x^4 +{}} 2x^3 - 6x^2 \\
 & -2x^3 + 2x^2 \\
\hline
 & -4x^2
\end{array}
$$

(b) **[5 pts]** Evaluate $\displaystyle\int f(x)\,dx$.

Since $x^3 - x^2 = x^2(x-1)$, we use partial fractions to write

$$\frac{-4x^2 + 2x - 1}{x^3 - x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1},$$

or

$$-4x^2 + 2x - 1 = Ax(x-1) + B(x-1) + Cx^2.$$

Plugging in $x = 0$ and $x = 1$ gives $B = 1$ and $C = -3$, respectively, at which point finding $A = -1$ is easy. Therefore

$$\begin{aligned}
\int \frac{x^4 + x^3 - 6x^2 + 2x - 1}{x^3 - x^2}\,dx &= \int \left(x + 2 - \frac{1}{x} + \frac{1}{x^2} - \frac{3}{x-1} \right) dx \\
&= \frac{x^2}{2} + 2x - \ln|x| - \frac{1}{x} - 3\ln|x-1| + C.
\end{aligned}$$