Math 101—Practice Final Examination

Duration: 150 minutes

Surname (Last Name) ____________ Given Name ____________ Student Number ____________

Do not open this test until instructed to do so! This exam should have 13 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam. Phones cannot be visible at any point during the exam.

UBC rules governing examinations:

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (a) speaking or communicating with other examination candidates, unless otherwise authorized;
   (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (c) purposely viewing the written papers of other examination candidates;
   (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem	**Out of**	**Score**	**Problem**	**Out of**	**Score**
1	8		7	7	
2	8		8	7	
3	6		9	7	
4	6		10	7	
5	6		11	7	
6	6		**Total**	75	

- Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.
- Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.
- Problems 7–11 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.
- Continue on the backs of the pages if you run out of space.
- You have 150 minutes to complete this exam, and there are a total of 75 points to be earned. So, roughly speaking, you can allocate 2 minutes per point to each problem (for example, 14 minutes for a 7-point problem).

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

1a. **[2 pts]** For each series, write **C** if it converges, or write **D** if it diverges. (At least one of them converges, and at least one of them diverges.)

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$ Answer: ☐

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4+6}$ Answer: ☐

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{9+6^n}$ Answer: ☐

1b. **[2 pts]** For each integral, choose the substitution type that is most beneficial for evaluating the integral. (Write **E**, **F**, or **G** in each box; each answer will be used exactly once.)

E: $x = a\sec\theta$ **F:** $x = a\sin\theta$ **G:** $x = a\tan\theta$

(i) $\displaystyle\int \frac{x^4}{\sqrt{9-x^2}}\,dx$ Answer: ☐

(ii) $\displaystyle\int \frac{\tan^{-1} x}{\sqrt{4+x^2}}\,dx$ Answer: ☐

(iii) $\displaystyle\int \frac{3x^2+1}{\sqrt{4x^2-9}}\,dx$ Answer: ☐

1c. **[2 pts]** (It is known that the integral $\int_2^\infty \frac{1}{x\sqrt{\ln x}}\,dx$ diverges.) Choose which statement is true about the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln n}}$. Answer: ☐

H: The series converges to 0 by the Squeeze Theorem.
J: The series is conditionally convergent.
K: The series is absolutely convergent by the Ratio Test.
L: The series is divergent by the Comparison Test with the p-series with $p = 1$.
M: The series is divergent by the Integral Test.

1d. **[2 pts]** Which of the given integrals equals $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} \sin\frac{i}{n}$? Answer: ☐

N: $\displaystyle\int_0^1 (x-1)^2 \sin(x-1)\,dx$ **S**: $\displaystyle\int_1^2 x^2 \sin x\,dx$

P: $\displaystyle\int_1^2 (x+1)^2 \sin(x+1)\,dx$ **T**: $\displaystyle\int_1^2 (x-1)^2 \sin(x-1)\,dx$

Q: $\displaystyle\int_0^1 (x+1)^2 \sin(x+1)\,dx$

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

2a. **[4 pts]** Match each expression with its numerical value. (Write **A**, **B**, **C**, **D**, or **E** in each box; each answer will be used exactly once.)

A: $\ln 2$ **B:** 0 **C:** $\frac{1}{2}$ **D:** 1 **E:** e

(i) $\displaystyle\lim_{n\to\infty} \frac{e^{\sin n}}{n}$ Answer: ☐

(ii) $\displaystyle\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ Answer: ☐

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ Answer: ☐

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{2^{2n}3^n}$ Answer: ☐

(v) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1}-\frac{1}{n+2}\right)$ Answer: ☐

2b. **[4 pts]** For each of the following series, choose the appropriate statement. (Write **F**, **G**, **H**, **J**, or **K** in each box; each answer will be used at most once, and each series matches a single answer only.)

F: The series diverges.
G: The series converges by the Ratio Test.
H: The series converges by the Alternating Series Test.
J: The series converges absolutely by the Comparison Test with a p-series.
K: The series converges by the Integral Test.

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Answer: ☐

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{7^{n+3}}{3^{2n}-2}$ Answer: ☐

(iii) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \tan^{-1} n$ Answer: ☐

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n^3}$ Answer: ☐

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

3a. **[3 pts]** Compute $\int \cos^2(x)\sin(2x)\,dx$.

3b. **[3 pts]** Determine whether the series

$$\sum_{n=2}^{\infty} \frac{2^n + 3^n}{4^n}$$

converges, and if so, find its sum.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

4a. **[3 pts]** Given the function $f(x) = x^3 \sin(2x^2)$, use Maclaurin series to find $f^{(17)}(0)$, the seventeenth derivative of f at $x = 0$.

4b. **[3 pts]** A particle moves along a line having initial velocity $v(0) = -4$ m/s. The particle's *acceleration* function after t seconds is $a(t) = 2t$ m/s^2. Find the *total distance* the particle travels in the first 3 seconds.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

5a. **[3 pts]** Find $\displaystyle\int_0^{\pi/3} x^2 \cos x \, dx$.

5b. **[3 pts]** Evaluate the average value of the function $f(x) = \sqrt{2x+1}$ on the interval $[0, 4]$.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

6a. **[3 pts]** Suppose the power series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 6. What is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} c_n 3^n x^n$? What is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$? Justify your answers.

6b. **[3 pts]** Find the solution of the differential equation $y' = y^2 x$ that satisfies $y(2) = 1$. (Solve completely for y in terms of x.)

Problems 7–11 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

7. Both parts of this problem concern the power series $S = \sum_{n=2}^{\infty} \frac{(x-5)^n}{n \ln n}$.

(a) **[4 pts]** Using the Ratio Test, find the radius of convergence of the power series S.

(b) **[3 pts]** Find the interval of convergence of the power series S. (Hint: you may have to use the integral test.)

8. Let R be the region enclosed by the curves $y = x^2 + 1$ and $y = 3 - x^2$.

(a) **[2 pts]** Sketch the region R, labeling the points of intersection.

(b) **[5 pts]** Find the volume of the solid obtained by rotating the region R about the line $y = 1$.

9.

(a) **[3 pts]** Suppose that a function $f(x)$ satisfies $f(0) = 0$ and $f'(x) = \tan^{-1} x$. Find the complete Maclaurin series for $f(x)$.

(b) **[4 pts]** Using your answer to part (a), find the value of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)(2n+2)}.$$

You don't have to simplify your answer fully, but evaluate any trigonometric functions.

10. By Newton's Law of Gravitaton, the gravitational force acting on a body of mass M kg that is at a distance r metres from the centre of Earth is $\dfrac{4 \times 10^{17} \cdot M}{r^2}$ N.

(a) **[4 pts]** Write down an improper integral representing the amount of work required to move a 1000 kg spacecraft from the surface of Earth to infinitely far away from Earth. Use 6.4×10^6 m as the distance from the centre of Earth to its surface. (Assume there are no forces acting upon the spacecraft other than Earth's gravity.)

(b) **[3 pts]** Evaluate the improper integral you wrote down in part (a), including correct units.

11. Both parts of this problem refer to the function $f(x) = \dfrac{x^4 + x^3 - 6x^2 + 2x - 1}{x^3 - x^2}$.

(a) **[2 pts]** Use long division to write $f(x)$ as $Q(x) + \frac{R(x)}{D(x)}$ for polynomials $Q(x)$, $R(x)$, and $D(x)$ where the degree of $R(x)$ is less than the degree of $D(x)$.

(b) **[5 pts]** Evaluate $\displaystyle\int f(x)\,dx$.