Absolutely abnormal numbers

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Outline

1. Introduction
2. Constructing our number
3. Proving irrationality and abnormality
4. Generalizing the construction
Simply normal numbers

A real number is simply normal to the base $b$ if each digit occurs in its $b$-ary expansion with the expected asymptotic frequency.

$$N(\alpha; b, a, x) = \#\{1 \leq n \leq x : \text{the } n\text{th digit in the base-}b\text{ expansion of } \alpha \text{ is } a\}$$

**Definition**

$\alpha$ is simply normal to the base $b$ if for each $0 \leq a < b$,

$$\lim_{x \to \infty} \frac{N(\alpha; b, a, x)}{x} = \frac{1}{b}.$$  

$b$-adic rational numbers $\alpha$ (those for which $b^j \alpha$ is an integer for some $j$) have two $b$-ary expansions; but $\alpha$ is not simply normal for either one.
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Normal numbers

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A number is **normal to the base** $b$ if it is simply normal to each of the bases $b, b^2, b^3, \ldots$.

Equivalently:

For any finite string $a_1a_2\ldots a_\ell$ of base-$b$ digits, the limiting frequency of occurrences of this string in the $b$-ary expansion of $\alpha$ exists and equals $1/b^\ell$.

- The set of numbers normal to any base $b$ has full Lebesgue measure, and thus the same is true of the set of absolutely normal numbers.
- Proving specific numbers normal is notoriously hard.
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How hard?

Champernowne’s number

0.1234567891011121314151617181920212223242526... 

is normal to the base 10 (and hence to bases 100, 1000, etc.).

Theorem (Stoneham, 1973; Bailey–Crandall 2002)

\[
\sum_{n=1}^{\infty} \frac{1}{c^n b^n} \text{ is normal to the base } b \text{ if } \gcd(b, c) = 1.
\]

The bad news

No real number has ever been proved normal to two multiplicatively independent bases.

But they all are... (pretty pictures)
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Absolute abnormality

Definition

A number is **absolutely abnormal** if it is not normal to any base $b \geq 2$.

- Every rational number is absolutely abnormal.
- The set of absolutely abnormal numbers (while Lebesgue measure 0) is uncountable and dense.

Well, then:

Can we write down an irrational, absolutely abnormal number?
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A sequence of integers

**Definition (recursive)**

\[ d_2 = 2^2, \quad d_3 = 3^2, \quad d_4 = 4^3, \quad d_5 = 5^{16}, \]

\[ d_6 = 6^{30,517,578,125}, \quad \ldots \]

\[ d_j = j^{d_{j-1}/(j-1)} \quad (j \geq 3). \]

**Explicitly:**

\[ d_4 = 4^{3^2-1}, \quad d_5 = 5^{4(3^2-1)-1}, \quad d_6 = 6^{5(4(3^2-1)-1)-1}, \quad \ldots \]

and in general,

\[ d_j = j^{(j-2)(j-3)(\ldots(4(3^2-1)-1)\ldots)-1)-1}. \]
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- $d_j = j^{d_{j-1} / (j-1)} \quad (j \geq 3)$.

**Explicitly:**

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d_4 = 4^{32-1}, \quad d_5 = 5^{4(32-1)-1}, \quad d_6 = 6^{5(4(32-1)-1)-1}, \ldots
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and in general,

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d_j = j^{(j-2)(j-3)(\ldots(4(32-1)-1)\ldots)-1}-1).
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constructing our number

Proving irrationality and abnormality

Generalizing the construction

(a typesetting nightmare)

If you think it’s easy to get:

$$d_j = j^{(j-1)} \left( (j-2) \left( (j-3) \left( \ldots \left( 4^{(3^j-1)-1} \right) \ldots \right) -1 \right) -1 \right)$$

try not to get:

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A sequence of rational numbers

**Definition**

\[ \alpha_k = \prod_{j=2}^{k} \left( 1 - \frac{1}{d_j} \right) \]

- \( \alpha_2 = \frac{3}{4}, \alpha_3 = \frac{2}{3}, \alpha_4 = \frac{21}{32}, \alpha_5 = \frac{100,135,803,222}{152,587,890,625}, \ldots \)

**Some nice cancellation**

It seems that the denominator of \( \alpha_k \) should contain powers of 2, 3, \ldots, \( k \). But in the listed terms, the denominator contains only powers of \( k \); in other words, \( \alpha_k \) is a \( k \)-adic fraction.

- \( 4 = 2^2, \ 3 = 3^1, \ 32 \mid 64 = 4^3, \ 152,587,890,625 = 5^{16} \)
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A little elementary number theory

Fun fact (for you to prove if you get bored)

\[(k + 1)^{km} - 1 \text{ is divisible by } k^{m+1} \text{ for any integers } k, m \geq 1.\]

Lemma

\[(k + 1)^{d_k/k} - 1 \text{ is divisible by } d_k \text{ for any integer } k \geq 2.\]

Proof.

Since \( d_k = k^{d_{k-1}/(k-1)} \),

\[(k + 1)^{d_k/k} - 1 = (k + 1)^{k^{d_{k-1}/(k-1)-1}} - 1;\]

apply the fun fact with \( m = d_{k-1}/(k - 1) - 1.\)
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The cause of the cancellation

**Lemma**

$d_k \alpha_k$ is an integer for each $k \geq 2$. In particular, $\alpha_k$ is a $k$-adic fraction (since $d_k$ is a power of $k$).

**Proof by induction on $k$.**

\[
d_{k+1} \alpha_{k+1} = d_{k+1} \prod_{j=2}^{k+1} \left( 1 - \frac{1}{d_j} \right) = (d_{k+1} - 1) \alpha_k
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= \left( (k + 1)^{d_k/k} - 1 \right) \alpha_k = \left( \frac{(k + 1)^{d_k/k} - 1}{d_k} \right) d_k \alpha_k
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The fraction is an integer by the lemma on the last slide; so if $d_k \alpha_k$ is an integer, then $d_{k+1} \alpha_{k+1}$ is also an integer.
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Our candidate

### Definition

\[ \alpha = \lim_{k \to \infty} \alpha_k = \prod_{j=2}^{\infty} \left( 1 - \frac{1}{d_j} \right) \]

- In Peter’s honour, I’ve memorized the first twenty-three billion decimal places of \( \alpha \):

\[ \ldots 0.6562499999956992 \]

### Compare \( \alpha \) to its partial products

\[ \alpha_4 = \frac{21}{32} = 0.65625 \]
\[ \alpha_5 = \frac{100,135,803,222}{152,587,890,625} = 0.6562499999956992 \]
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\[ \alpha = 0.656249999995699199999 \ldots 999998528404201690728 \ldots \]

23,747,291,559 nines

---

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**easy exercise**

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**harder exercise**

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**Compare \( \alpha \) to its partial products**

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- \( \alpha_5 = \frac{100,135,803,222}{152,587,890,625} = 0.6562499999569992 \)
α is irrational

**Definition (reminder)**

- \( d_j = j^{d_{j-1}/(j-1)} \)
- \( \alpha = \prod_{j=2}^{\infty} \left( 1 - 1/d_j \right) \)

\{d_j\} grows ridiculously quickly

One can show that \( d_{j+1} > d_j^{d_j-1} \) for all \( j \geq 5 \).

\[
\alpha_k - \alpha = \alpha_k \left( 1 - \prod_{j=k+1}^{\infty} \left( 1 - \frac{1}{d_j} \right) \right) < \alpha_k \sum_{j=k+1}^{\infty} \frac{1}{d_j} < \frac{2}{d_{k+1}} < \frac{2}{d_k^{d_{k-1}}}
\]

- These are rational approximations of \( \alpha \) by fractions with denominators \( d_k \).
- Since \( d_{k-1} \to \infty \), the number \( \alpha \) is a Liouville number, hence irrational (transcendental even).
\( \alpha \) is irrational

**Definition (reminder)**

- \( d_j = j^{d_{j-1}/(j-1)} \)
- \( \alpha = \prod_{j=2}^{\infty} \left( 1 - \frac{1}{d_j} \right) \)

\{d_j\} grows ridiculously quickly

One can show that \( d_{j+1} > d_j^{d_{j-1}} \) for all \( j \geq 5 \).

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\alpha_k - \alpha = \alpha_k \left( 1 - \prod_{j=k+1}^{\infty} \left( 1 - \frac{1}{d_j} \right) \right) < \alpha_k \sum_{j=k+1}^{\infty} \frac{1}{d_j} < \frac{2}{d_{k+1}} < \frac{2}{d_k^{d_k-1}}
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Abnormal for one moment at least

Illustrating example: \( b = 4 \)

- \( \alpha_4 = (0.222)_{\text{base } 4} \)
- \( \alpha_4 - \alpha = (0.000000000000000000102322210110 \ldots)_{\text{base } 4} \)
- \( \alpha = (0.221333333333333333231011123223 \ldots)_{\text{base } 4} \)

- \( \alpha_b - \alpha < 2/d_b^{d_b-1} \) for all \( b \geq 5 \)
- \( \alpha_b \) terminates in base \( b \), after \( D \) digits say
- \( \alpha \) is a tiny bit less than \( \alpha_b \): the difference starts with about \( d_b-1D \) base-\( b \) digits equaling 0
- So \( \alpha \) has a long string of base-\( b \) digits equaling \( b - 1 \).

How long?

Among the first \( d_b-1D \) base-\( b \) digits of \( \alpha \), at least a proportion \( 1 - C/d_b-1 \) of them equal \( b - 1 \) (for some absolute constant \( C \)).
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Lots of abnormal moments

Among the first $\alpha_b$ base-$b$ digits of $\alpha$, at least a proportion $1 - C/d_{b-1}$ of them equal $b - 1$ (for some absolute constant $C$).

Now change $b$ to $b^2$:
- Among the first $\alpha_{b^2}$ base-$b^2$ digits of $\alpha$, at least a proportion $1 - C/d_{b^2-1}$ of them equal $b^2 - 1$.
- Thus among the first $\alpha_{b^2}$ base-$b$ digits of $\alpha$, at least a proportion $1 - C/d_{b^2-1}$ of them equal $b - 1$.

We find that $\alpha$ has lots of (disjoint) long strings of base-$b$ digits equaling $b - 1$, coming from changing $b$ to $b^2$, $b^3$, $b^4$, . . . .

(and sometimes more)

Such strings also come from $k$ if all prime factors of $k$ divide $b$: $k$-adic fractions are also $b$-adic fractions. For example, in base 10 we already saw long strings of 9s coming from $\alpha_4$ and $\alpha_5$. 
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Absolutely abnormal numbers

Greg Martin
Lots of abnormal moments

Among the first \( ?_b \) base-\( b \) digits of \( \alpha \), at least a proportion \( 1 - \frac{C}{d_b - 1} \) of them equal \( b - 1 \) (for some absolute constant \( C \)).

Now change \( b \) to \( b^2 \):
- Among the first \( ?_{b^2} \) base-\( b^2 \) digits of \( \alpha \), at least a proportion \( 1 - \frac{C}{d_{b^2} - 1} \) of them equal \( b^2 - 1 \).
- Thus among the first \( 2 ?_{b^2} \) base-\( b \) digits of \( \alpha \), at least a proportion \( 1 - \frac{C}{d_{b^2} - 1} \) of them equal \( b - 1 \).

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\( \alpha \) is absolutely abnormal

**Definition (counting base-\( b \) digits equaling \( a \))**

\[
N(\alpha; b, a, x) = \# \{ 1 \leq n \leq x : \text{the } n\text{th digit in the base-}b\text{ expansion of } \alpha \text{ is } a \}
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- We’ve found a sequence \( \{x_1, x_2, \ldots \} = \{?_b, 2?_b^2, \ldots \} \) such that \( N(\alpha; b, b-1, x_j) > (1 - C/d_{b^j-1})x_j \).
- So \( \limsup_{x \to \infty} \frac{N(\alpha; b, b-1, x)}{x} \geq \limsup_{j \to \infty} (1 - C/d_{b^j-1}) = 1 \).
- This conflicts with \( \lim_{x \to \infty} \frac{N(\alpha; b, b-1, x)}{x} = \frac{1}{b} \).

**Theorem (M., 2001)**

\( \alpha \) is an irrational number that fails to be (simply) normal to any base \( b \geq 2 \).
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\( \alpha \) is an irrational number that fails to be (simply) normal to any base \( b \geq 2 \).
Different parameters

The original construction

- $d_2 = 2^2$
- $\alpha_2 = 3/d_2$
- $d_j = j^{d_{j-1}/(j-1)}$
- $\alpha_k = \alpha_2 \prod_{j=3}^{k} \left(1 - 1/d_j\right)$

We can generalize the construction in two ways:

- Start with any dyadic fraction $\alpha_2 = n_1/2^{n_2}$ in place of $3/4$.
- Insert positive integer multiples $n_j$ in the recursion for $d_j$.

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### The generalized construction

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- $d_j = j^{n_j d_{j-1}/(j-1)}$
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- The numbers $n_3, n_4, \ldots$ just accelerate the convergence of $\alpha$, and all the inequalities are still satisfied and more.
- In particular, $\alpha_2 > \alpha > \alpha_2 - 2/d_2$. So by choosing $n_1$ and $n_2$ suitably, we can ensure that $\alpha$ ends up in any prescribed interval.
- Each $\alpha_k$ is a $k$-adic fraction because the key divisibility still holds: $d_k \mid (k+1)^{d_k/k} - 1 \mid (k+1)^{n_k d_k/k} - 1$
- By varying the $n_j$, we obtain uncountably many (distinct) irrational, absolutely abnormal numbers.
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Particular parameters

The generalized construction

- \( d_2 = 2^{n_2} \)
- \( \alpha_2 = n_1 / d_2 \)
- \( d_j = j^{n_j d_{j-1}} / (j-1) \)
- \( \alpha_k = \alpha_2 \prod_{j=3}^{k} (1 - 1 / d_j) \)

One neat example: \( n_1 = n_2 = 1 \) and \( n_j = \phi(j - 1) \) for \( j \geq 3 \)

- The result is that \( d_j = j^{\phi(d_{j-1})} \) for \( j \geq 3 \).
- The key divisibility \( d_k | ((k + 1)^{\phi(d_k)} - 1) \) just follows from Euler’s theorem.

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The end

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