Putnam Practice Problems #1
September 22, 2015
(in no particular order)

(1) Calculate, with proof, the number of ordered 2015-tuples \((y_1, \ldots, y_{2015})\) of real numbers that satisfy
\[ y_1^2 = y_2^2 = \cdots = y_{2015}^2 = y_1 \times y_2 \times \cdots \times y_{2015}. \]

(2) Let \(a_1, a_2, \ldots\) be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer \(n\), the numbers \(a_1, \ldots, a_n\) leave \(n\) different remainders upon division by \(n\). Prove that every integer occurs exactly once in the sequence \(a_1, a_2, \ldots\).

(Remark: a remainder must always be nonnegative, even if the number being divided is negative; for example, \(-7\) divided by 4 has a quotient of \(-2\) and leaves a remainder of 1.)

(3) Let \(f\) be a continuous function defined on the real numbers. Suppose that there exist real numbers \(x_1, x_2, \ldots, x_{2015}\) such that
\[ f(x_1) + f(x_2) + \cdots + f(x_{2015}) = x_1 + x_2 + \cdots + x_{2015}. \]
Prove that \(f\) has a fixed point, that is, a real number \(y\) such that \(f(y) = y\).

(4) Suppose that \(P\) is a convex polyhedron. A new polyhedron \(Q\) is obtained from \(P\) by cutting off a small tip from each vertex of \(P\). The new polyhedron \(Q\) has either 2015 edges, 2015 faces, or 2015 vertices. How many edges did the original polyhedron \(P\) have?

(5) Let \(p(x)\) be a polynomial of degree 2015 with integer coefficients. Suppose that \(p(x)^2 = 1\) for 2015 different integer values of \(x\). Show that \(p(x)\) cannot be factored as the product of two nonconstant polynomials with integer coefficients.

(6) Find all continuous functions \(h(t)\), defined for all real numbers \(t\), such that
\[ \left( \int_{-\infty}^{x} h(t) \, dt \right)^2 = \int_{-\infty}^{x} h(t)^2 \, dt \]
for all real numbers \(x\).

(7) Find the largest positive integer \(N\) with the following property: for any positive numbers \(m_1, \ldots, m_{2015}\) each dividing \(N!^{10}\), there exist two indices \(1 \leq j < k \leq 2015\) such that \(m_j m_k\) is a perfect square.

(8) Let \([y]\) denote the largest integer less than or equal to \(y\), so that \([\pi] = 3\) and \([-\pi] = -4\), for example. Calculate, with proof,
\[ \int_{-\pi/2}^{\pi/2} \left[ 2015x^2 \sin(2015x) \right] \, dx. \]