Math 221 - Sec 205 - Winter 2017 - Quiz 9

April 1, 2017

Name: Solution Student #: 

Instructions

(1) Put all your personal belongings at the front of the exam room.

(2) Bring your student ID card with you and put it on the table.

(3) You are prohibited from bringing any electronic devices into exam table or in the pockets, whether the devices are turned off or not. Violators will not only be asked to leave the exam room, but will also be suspected of attempted cheating.

(4) You are prohibited from bringing any course materials, scrap papers, or calculators.

(5) When time is up, stop writing and hand in the paper immediately. Your information should be filled in at the beginning of the test.

(6) Violators of any of the above instructions will be immediately disqualified and reported to the department.

(7) If you observe any misconduct or attempted cheating, report to the instructor or the TA immediately.
Total 6 marks.
The aim of this question is to compute the distance between a point and a plane.

Let \( y = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \) be a point in \( \mathbb{R}^3 \) and \( W \) be a plane spanned by the vectors

\[
\mathbf{u}_1 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}
\]

1. (1 mark) Show that \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) are orthogonal to each other.

**Solution:** Simply take dot product and see if it is 0.

\[
\mathbf{u}_1 \cdot \mathbf{u}_2 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 4 \times 1 + 1 \times (-2) + (-1) \times 2 = 4 - 2 - 2 = 0.
\]

They are orthogonal to each other.

2. (1 mark) Compute the lengths of \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \).

**Solution:** The length is \( \| \mathbf{u} \| = \sqrt{\mathbf{u} \cdot \mathbf{u}} \), so

\[
\| \mathbf{u}_1 \| = \sqrt{\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}} = \sqrt{4^2 + 1^2 + (-1)^2} = \sqrt{18}
\]

and

\[
\| \mathbf{u}_2 \| = \sqrt{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}} = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3
\]
3. (2 marks) Compute the orthogonal projection of $y$ onto $W$.

**Solution:** Apply the orthogonal projection formula directly

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2.$$ 

You may compute the following dot products first.

$$y \cdot u_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} = 2 \times 4 + 3 \times 1 + 1 \times (-1) = 8 + 3 - 1 = 10,$$

and

$$y \cdot u_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 2 \times 1 + 3 \times (-2) + 1 \times 2 = 2 - 6 + 2 = -2.$$

Then

$$\hat{y} = \frac{10}{18} \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} + \frac{-2}{9} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

4. (2 marks) Compute the distance between the point $y$ and the plane $W$.

**Solution:** The distance is the length of the orthogonal complement

$$||y - \hat{y}|| = || \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} || = || \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} || = \sqrt{8}.$$