Math 221 - Sec 205 - Winter 2017 - Quiz 7

Solution

Name: Student #: 

Instructions

1. Put all your personal belongings at the front of the exam room.
2. Bring your student ID card with you and put it on the table.
3. You are prohibited from bringing any electronic devices into exam table or in the pockets, whether the devices are turned off or not. Violators will not only be asked to leave the exam room, but will also be suspected of attempted cheating.
4. You are prohibited from bringing any course materials, scrap papers, or calculators.
5. When time is up, stop writing and hand in the paper immediately. Your information should be filled in at the beginning of the test.
6. Violators of any of the above instructions will be immediately disqualified and reported to the department.
7. If you observe any misconduct or attempted cheating, report to the instructor or the TA immediately.
Total 5 marks.

(i) (4 marks) Compute all eigenvalues of

\[
A = \begin{bmatrix}
10 & -18 \\
6 & -11
\end{bmatrix}
\]

and find the corresponding eigenvectors.

**Solution:**
Compute the eigenvalues using the characteristic polynomial.

\[
\det(A - tI) = \det \begin{bmatrix}
10 - t & -18 \\
6 & -11 - t
\end{bmatrix} = (10 - t)(-11 - t) - (-18)(6) \\
= t^2 + t - 2 = (t + 2)(t - 1)
\]
the eigenvalues are \( \lambda_1 = 1 \) and \( \lambda_2 = -2 \).

For \( \lambda_1 = 1 \), we solve for an eigenvector by row reduction.

\[
A - I = \begin{bmatrix}
9 & -18 \\
6 & -12
\end{bmatrix} \xrightarrow{\frac{1}{3}R_1, \frac{1}{2}R_2, R_2 - R_1} \begin{bmatrix}
1 & -2 \\
0 & 0
\end{bmatrix}_{RREF} \Rightarrow x - 2y = 0
\]

The solution is \( \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix} \). We can take eigenvector

\( \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) (or any scalar multiple of it).

For \( \lambda_2 = -2 \), we solve for an eigenvector by row reduction.

\[
A + 2I = \begin{bmatrix}
12 & -18 \\
6 & -9
\end{bmatrix} \xrightarrow{\frac{1}{3}R_1, \frac{1}{3}R_2, R_2 - R_1} \begin{bmatrix}
2 & -3 \\
0 & 0
\end{bmatrix}_{RREF} \Rightarrow 2x - 3y = 0
\]

The solution is \( \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2}y \\ y \end{bmatrix} = \frac{1}{2} y \begin{bmatrix} 3 \\ 2 \end{bmatrix} \). We can take eigenvector

\( \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \) (or any scalar multiple of it).
No work is required for the following questions.

(ii) (0.5 marks) Is $\lambda = 1$ an eigenvalue of \[
\begin{bmatrix}
2 & 3 \\
100 & 301
\end{bmatrix}
\]?

**Solution:** Yes. $A - \lambda I = \begin{bmatrix}
2 - 1 & 3 \\
100 & 301 - 1
\end{bmatrix} = \begin{bmatrix}
1 & 3 \\
100 & 300
\end{bmatrix}$ is clearly not invertible (say, R2 is a multiple of R1). Any non-trivial solution of $(A - \lambda I)x = 0$ is an eigenvector of $A$ with eigenvalue $\lambda = 1$.

(iii) (0.5 marks) Is \[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\] an eigenvector of \[
\begin{bmatrix}
100 & 1 \\
0 & 100
\end{bmatrix}
\]?

**Solution:** Yes, because \[
\begin{bmatrix}
100 & 1 \\
0 & 100
\end{bmatrix} \begin{bmatrix}
1 \\
0
\end{bmatrix} = \begin{bmatrix}
100 \\
0
\end{bmatrix} = 100 \begin{bmatrix}
1 \\
0
\end{bmatrix}.
\]

Hence \[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\] an eigenvector of \[
\begin{bmatrix}
100 & 1 \\
0 & 100
\end{bmatrix}
\] with eigenvalue $\lambda = 100$. 