Total 5 marks.

1. In the four sets of vectors below, three of them are linearly dependent and can be found out automatically without any calculation! The remaining one is linearly independent. Please circle this one. (1 mark)

\begin{align*}
\text{(A)} & : & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\text{(B)} & : & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \end{bmatrix}, & \begin{bmatrix} 1 \end{bmatrix} \\
\text{(C)} & : & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \end{bmatrix}, & \begin{bmatrix} 1 \end{bmatrix} \\
\text{(D)} & : & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, & \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, & \begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix}
\end{align*}

(0.5 mark)

\textbf{Answer:} (A) is linearly dependent since there are more vectors than the number of entries. (B) is linearly dependent because it contains a zero vector. (C) is linearly independent. You may compute the RREF of the matrix but not necessary, because the other three sets are linearly dependent. (D) is linearly dependent because the vectors are in multiples of each other.

2. Which one of the following sizes of a matrix must have linearly dependent columns?

\begin{align*}
\text{(A)} & : 3 \times 1 \\
\text{(B)} & : 3 \times 2 \\
\text{(C)} & : 3 \times 3 \\
\text{(D)} & : 3 \times 4
\end{align*}

(0.5 mark)

\textbf{Answer:} (D), because when \( A \) has more columns than rows, then there are more vectors than the number of entries.
3. Suppose that $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ maps the vectors

$\mathbf{u}$ to $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{v}$ to $T(\mathbf{v}) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Compute the following

(i) $T(3\mathbf{u} + 4\mathbf{v})$

(ii) $T(\mathbf{0})$, where $\mathbf{0}$ is the zero-vector.

(2 marks)

**Answer:**

(i) By the linear property, we have

$$T(3\mathbf{u} + 4\mathbf{v}) = T(3\mathbf{u}) + T(4\mathbf{v}) = 3T(\mathbf{u}) + 4T(\mathbf{v}),$$

which is equal to $3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 12 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \\ 10 \end{bmatrix}$.

(ii) There are multiple arguments, for example

$$T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = 0 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Other valid arguments are acceptable.
4. Find the matrix $A$ of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that

(i) first transforms $e_2$ into $e_2 + 3e_1$ (leaving $e_1$ unchanged),

(ii) then reflects along the $y$-axis.

(Hint: as in class, draw the pictures and check where $e_1$ and $e_2$ are transformed under these two operations.) (2 marks)

**Answer:** If you draw the picture correctly, then you find that

\[
e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{\text{Step (i)}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{\text{Step (ii)}} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = Ae_1
\]

and

\[
e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\text{Step (i)}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \xrightarrow{\text{Step (ii)}} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = Ae_2.
\]

Using Theorem 10 of the book, we know that

\[
A = [Ae_1 \ Ae_2] = \begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}.
\]