Math 221, Section 4.2

Compute eigenvalues by characteristic polynomial
Plan for Chapter 4

- Sec 4.1: Introduction about the idea of eigenvectors  
  (Week 10, Monday)
- Sec 4.2: Calculation the eigenvalues and eigenvectors  
  (Week 10, Wednesday)
- Sec 4.3, 4.4: Relation with changing of coordinates  
  (Week 11, Monday)
- Sec 4.6: Applications  
  (Week 11, Wednesday and Friday)
  - Matrix powers
  - Dynamical system
  - Fibonacci sequence
  - Predator-Prey system
Recall: determinant and invertibility

Remember the Invertible Matrix Theorem asserts that the following statements are logically equivalent.

- **a.** $A$ is invertible.
- **d.** $Ax = 0$ has only trivial solution.
- **t.** $\det A \neq 0$.

Therefore, the falsehood of these statements are also logically equivalent.

- **a’.** $A$ is non-invertible (singular).
- **d’.** $Ax = 0$ has non-trivial (non-zero) solutions.
- **t’.** $\det A = 0$.

We will proceed with statements $d’ \Leftrightarrow t’$. 

Method to compute eigenvalues

If $\lambda$ is an eigenvalue of $A$, then there exists a non-zero vector $\mathbf{x}$ (an eigenvector of $A$) such that

$$A\mathbf{x} = \lambda \mathbf{x}.$$ 

That means, using statements $d' \Leftrightarrow t'$

- $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has non-trivial solutions.
- $\det(A - \lambda I) = 0$.

In other words,

- $\lambda$ is a solution of the **characteristic polynomial** of $A$:

$$\det(A - tI) = 0,$$

(here $t$ is a variable).

- Solving for eigenvalues of $A$ amounts to solving for the roots of the above equation.
Characteristic polynomials

Example

- (Sec 4.1, E.g. 3) \( A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \).

\[
\det(A - tI) = \det \left( \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \right) = \det \begin{bmatrix} 1 - t & 6 \\ 5 & 2 - t \end{bmatrix}
\]

- Use either Gaussian elimination or cofactor expansion to compute the determinant.

- We obtain a polynomial

\[ t^2 \]

- The roots of this polynomial are \( \frac{5}{8} \).
Characteristic polynomials

Example

\[ A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} \]

\[ \det(A - tl) = \det \begin{bmatrix} 4 - t & -1 & 6 \\ 2 & 1 - t & 6 \\ 2 & -1 & 8 - t \end{bmatrix} \]

\[ \begin{enumerate}
\item We obtain a polynomial \[-t^3\]
\item The roots of this polynomial are \[\frac{6}{8}\]
\end{enumerate} \]
Solve for all eigenvalues and eigenvectors

(Sec 4.2, Ex 2) \( A = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix} \).

- characteristic polynomial
- eigenvalues
- eigenvectors for each eigenvalue
Finally, think about this question:
Suppose $\lambda$ is **not** an eigenvalue of $A$.

- We still form $A - \lambda I$, and solve for $(A - \lambda I)x = 0$.
- What will we get for the solution?