Math 221, Section 3.1 and 3.2, Part 2

(1) Examples of determinants of $3 \times 3$ matrices
Plan for Chapter 3

Last time:
- Recall the direct formula of determinant for $2 \times 2$ matrices
- Study the properties for the direct formula
- Derive an alternative formula for computing determinant using Gaussian elimination

This time:
- Use the alternative formula to compute the determinant for $n \times n$ matrices

Next time:
- Provide the direct formula (an algorithm called cofactor expansion) of determinant for $n \times n$ matrices
Remember: Properties of determinants

All properties of determinant for $2 \times 2$ matrices generalize to $3 \times 3$ (and also $n \times n$ matrices)

- Invertibility: $A$ is invertible $\iff \det A \neq 0$.
- If $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$, the volume of the parallelepiped is $|\det A|$.

(Here absolute value is taken because volume is always positive.)
Remember: Properties of determinants

All properties of determinant for $2 \times 2$ matrices generalize to $3 \times 3$ (and also $n \times n$ matrices)

(1) Diagonal, upper/lower triangular matrices.

$$\det \begin{bmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{bmatrix} = \det \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = \det \begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix} = adf.$$  

(2) Unchanged under transpose: $\det A = \det A^T$.

(3) Multiplicative property: $\det(AB) = \det A \det B$.

(4) Linear property

$$\det [u_1 + u_2, v, w] = \det [u_1, v, w] + \det [u_2, v, w],$$  

$$\det [cu, v, w] = c \det [u, v, w],$$  

and similarly for other columns, or rows.

(5) Changes under row operations.
Remember: computing determinant using Gaussian elimination

- If in the process of reducing $A$ into the REF of $A$, you have done in total:
  - $r$ row interchangings,
  - $k$ scalings, every step dividing a scalar $c_1, \ldots, c_k$,

then

$$\det A = \begin{cases} 
0 & \text{if } A \text{ is not invertible}, \\
(-1)^r c_1 \cdots c_k & \text{if } A \text{ is invertible}.
\end{cases}$$

- Note it is regardless of how many row additions/subtractions in the process.
Example

(Sec 3.2, Ex 6) Compute \( \det \begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix} \).

Using Gaussian elimination

\[
A = \begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix} \xrightarrow{R_2 \rightarrow 3R_1} \begin{bmatrix} 1 & 5 & -3 \\ 0 & -18 & R & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

There are row interchanges, and a scalar is divided.

Therefore,

\( \det A = \)
Example

(Sec 3.2, Eg 1) Compute \( \det \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix} \).

Using Gaussian elimination

\[
A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}
\]

\[
\begin{array}{c}
R_2 + 2R_1 \\
R_3 + R_1
\end{array} \Rightarrow
\begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 
\end{bmatrix}
\]

\[
R_2 \leftrightarrow R_3
\Rightarrow
\begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 
\end{bmatrix}
\]

\[
\begin{array}{c}
\frac{1}{3}R_2 \\
\frac{1}{5}R_3
\end{array} \Rightarrow
\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 
\end{bmatrix}^{REF}
\]

There are

\[
\begin{array}{c}
\text{row interchanges, and} \\
\text{scalars are divided.}
\end{array}
\]

Therefore,

\[
\det A = \frac{7}{8}
\]
Diagonal method for $3 \times 3$ matrices

- Some textbook compute the determinant of a $3 \times 3$ matrix as follows:
  - Write a copy of the first two columns to the right of the matrix.
  - Multiply the entries on the six diagonals.
  - Add the downward diagonal products and subtract the upward products.

- **Warning:** you are not encouraged to use this method.
- Because this method generalizes to $4 \times 4$ or larger matrices in an unreasonable and complicated way.