Math 221, Section 2.5 and 2.6, Part 1

(1) Subspaces (2) Basis and dimension
Subspace

Recall from Sec 1.3: for example in $\mathbb{R}^3$,

- Take $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ then
  \[
  \text{Span}\{u\} = \{s \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} | \text{where } s \text{ is a scalar}\}
  \]
  is a 1-dimensional subspace (a line).

- Take $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, then $u, v$ are linearly independent
  \[
  \text{Span}\{u, v\} = \{s \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} | \text{where } s, t \text{ are scalars}\}
  \]
  is a 2-dimensional subspace (a plane).

- Take $w = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$, then
  \[
  \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 7 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{RREF},
  \]
  implies that $w = 1u + 2v$, and $\text{Span}\{u, v, w\} = \text{Span}\{u, v\}$. 
Defining properties of subspace

- A **subspace** is a subset $H$ of $\mathbb{R}^n$ that has three properties.
  - The zero vector $\mathbf{0}$ is in $H$.
  - If $\mathbf{u}, \mathbf{v}$ are in $H$, then the sum $\mathbf{u} + \mathbf{v}$ is in $H$.
  - If $\mathbf{u}$ is in $H$ and $c$ is a scalar, then the scalar multiple $c\mathbf{u}$ is in $H$.

In words, a subspace is **closed** under addition and scalar multiplication.
Verify that: $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a subspace of $\mathbb{R}^n$.

- $0$ is in $H$, because $0 = 0\mathbf{v}_1 + 0\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$.
- If $\mathbf{u} = a\mathbf{v}_1 + b\mathbf{v}_2$ and $\mathbf{v} = c\mathbf{v}_1 + d\mathbf{v}_2$, then
  \[ \mathbf{u} + \mathbf{v} = (a + c)\mathbf{v}_1 + (b + d)\mathbf{v}_2 \]
  which shows that $\mathbf{u} + \mathbf{v}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$.
- If $\mathbf{u} = a\mathbf{v}_1 + b\mathbf{v}_2$, then
  \[ c\mathbf{u} = c(a\mathbf{v}_1 + b\mathbf{v}_2) = c a\mathbf{v}_1 + c b\mathbf{v}_2 \]
  which shows that $c\mathbf{u}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$. 

![Diagram showing vectors $v_1$, $v_2$, and $0$ in $\mathbb{R}^3$](image)
Examples

- $\mathbb{R}^n$ itself is a subspace because it has the three properties required for a subspace.
- The **zero subspace** $\{0\}$, consisting of only the zero vector in $\mathbb{R}^n$, is a subspace.
- The **subspace spanned by** $v_1, \ldots, v_r$,
  
  $\text{Span}\{v_1, \ldots, v_r\} = \{\text{all linear combinations of } v_1, \ldots, v_r\}$

  is a subspace of $\mathbb{R}^n$.

  (The verification of this statement is similar to the argument given on the previous page.)
Non-examples

- A line $L$ *not* passing through the origin is not a subspace, because it does not contain the origin, as required.
- Fig. 2 shows that $L$ is not closed under addition or scalar multiplication.

![Figure 2](image-url)
Non-examples

- In general, a translation of a subspace away from the origin by a non-zero vector $p$, which is

$$p + \text{Span}\{v_1, \ldots, v_r\}$$

is not a subspace of $\mathbb{R}^n$. 
Basis: the smallest spanning subset

- Because a subspace contains an infinite number of vectors, some problems involving a subspace are handled best by working with a small finite subset of vectors that span the subspace.
- The fewer the spanning vectors, the better.
- The smallest possible spanning set must be linearly independent.
  - (Recall from Sec 1.7, if we put the vector together to form a matrix $A$, then all columns of $\text{REF}(A)$ contains a pivot.)
- Example: again take $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $w = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$.
  We checked that $w = u + 2v$, so

$$H = \text{Span}\{u, v, w\} = \text{Span}\{u, v\}$$
Basis and dimension

- A **basis** for a subspace $H$ of $\mathbb{R}^n$ is a linearly independent subset in $H$ that spans $H$.
  - Note: Given a subspace, there are more than one choices of bases.
- The **dimension** of a subspace $H$ is the number of vectors in any basis of $H$.
  - Although the choices of bases are different, but they all have the same number, which must be the dimension.
Example: again take the vectors in Sec 1.3.

- Take \( \mathbf{u} = \begin{bmatrix} 1/3 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 3/2 \\ 7 \end{bmatrix} \).
- We checked that \( \mathbf{w} = \mathbf{u} + 2\mathbf{v} \), so
  \[
  H = \text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \text{Span}\{\mathbf{u}, \mathbf{v}\}
  \]
- The subset \( \{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \) is linearly dependent, so this subset is \textit{not} a basis of \( H \).
- The subset \( \{\mathbf{u}, \mathbf{v}\} \) is linearly independent, so this subset is a basis of \( H \), and \( \dim(H) = 2 \).
Basis and dimension

Other examples

- The columns of an $n \times n$ identity matrix are the standard vectors denoted by $\mathbf{e}_1, \ldots, \mathbf{e}_n$.
  - E.g. in $\mathbb{R}^3$, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

- $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linearly independent.
- Hence they form a basis for the subspace
  \[
  \text{Span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mid \text{where } x, y, z \text{ are scalars}\}
  \]
  \[
  = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid \text{where } x, y, z \text{ are scalars}\}
  \]
  It is the whole $\mathbb{R}^3$.
- The set $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is called the standard basis for $\mathbb{R}^n$. 

Basis and dimension

Other subset of $n$ vectors in $\mathbb{R}^n$ form a basis, as long as they are linearly independent.

- Take $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$.

- We checked that $u, v, w$ are linearly independent. So

$$H = \text{Span}\{u, v, w\}$$

is a subspace in $\mathbb{R}^3$.

- $\text{dim}(H)$ is equal to the number of vectors which is 3, so it must be the whole $\mathbb{R}^3$. 
Invertible matrices and basis

- Suppose you have \( r \) linearly independent vectors \( \{v_1, \ldots, v_r\} \) in \( \mathbb{R}^n \).
- If \( r \leq n \), then they form a basis of \( \text{Span}\{v_1, \ldots, v_r\} \), which is an \( r \)-dimensional subspace in \( \mathbb{R}^n \).
- If \( r = n \), then
  - \( \text{Span}\{v_1, \ldots, v_n\} = \mathbb{R}^n \).
  - Combining with the Invertible Matrices Theorem (a.\(\leftrightarrow\) e.), if \( A = [v_1 \ldots v_n] \), then

\[
\text{for } \{v_1, \ldots, v_n\} \text{ forming a basis of } \mathbb{R}^n, \\
A \text{ must be invertible.}
\]
Invertible matrices and basis

Examples

▶ Previous examples

\[
\begin{bmatrix}
1 & 1 & 1 \\
2 & 0 & 2 \\
3 & 2 & 5 \\
1 & 1 & 3 \\
2 & 0 & 2 \\
3 & 2 & 7
\end{bmatrix}
\]

is invertible, so the columns form a basis in \( \mathbb{R}^3 \).

▶ is non-invertible/singular, so the columns do not form a basis in \( \mathbb{R}^3 \).

▶ (Sec 1.8, Ex 15)

Are the columns of \[
\begin{bmatrix}
4 & 16 \\
-2 & -3
\end{bmatrix}
\]
form a basis in \( \mathbb{R}^2 \)?

The REF has pivots, so the columns a basis for \( \mathbb{R}^2 \).
Invertible matrices and basis

Examples

▶ (Sec 1.8, Ex 17)

Are the columns of
\[
\begin{bmatrix}
0 & 5 & 6 \\
0 & 0 & 3 \\
-2 & 4 & 2
\end{bmatrix}
\]
form a basis in \( \mathbb{R}^3 \)?

The REF has pivots, so the columns a basis for \( \mathbb{R}^3 \).

▶ (Sec 1.8, Ex 18)

Are the columns of
\[
\begin{bmatrix}
1 & 3 & 5 \\
1 & -1 & 1 \\
-3 & 2 & -4
\end{bmatrix}
\]
form a basis in \( \mathbb{R}^3 \)?

The REF has pivots, so the columns a basis for \( \mathbb{R}^3 \).

▶ (Sec 1.8, Ex 19)

Are the columns of
\[
\begin{bmatrix}
3 & 6 \\
-8 & 2 \\
1 & -5
\end{bmatrix}
\]
form a basis in \( \mathbb{R}^3 \)?

No, because
Quiz 4

The bold numbers indicate similar types of questions that may appear in Quiz 4.

- Sec 2.2: Practice problem 2, Exercises 1, 3, 5, 8, 9a-d, 10, 13, 17, 29, 31 (use the algorithm I showed in class), 35
- Sec 2.3: Practice problems 1, 2, 3, Exercises 1, 3, 5, 11, 12, 13, 14,

The following questions require the big Invertible Matrix Theorem and some thinking. They will not be in the quiz, but some of them could be in the next midterm and the final. You are strongly suggested to think about them.

- Sec 2.2: 21, 22, 23, 24,
- Sec 2.3: 15, 17, 18, 19, 20, 21, 24, 35, 36, 39

You may check the solution at: